Large deviations in boundary-driven diffusive systems

A fluctuating hydrodynamic approach

Soumyabrata Saha (TIFR, Mumbai)

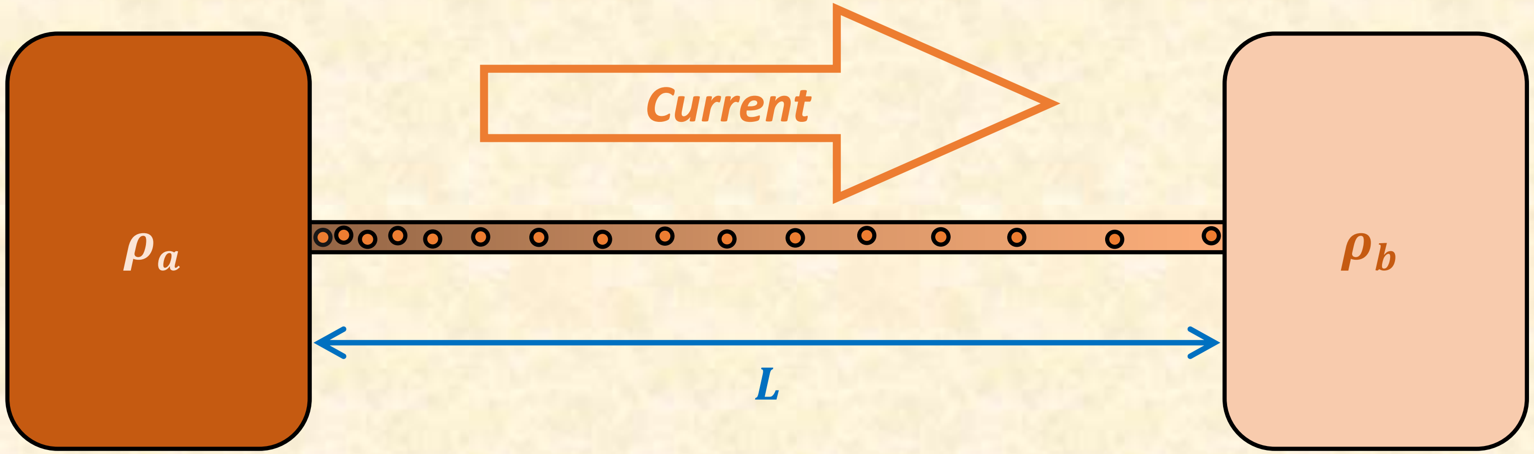
Tridib Sadhu (TIFR, Mumbai)

SS and Tridib Sadhu, arXiv:2501.03164 (2025)

टामूअसं tifr

10th ISPCM, 23-25 April 2025

A diffusive system that locally conserves a quantity is coupled to unequal reservoirs at the boundaries



At long times, the system reaches a *non-equilibrium steady state* (NESS), characterized by a steady flow of the conserved quantity across the system

What are the macroscopic properties of the conserved quantity in the NESS?

The transport of the conserved quantity between the reservoirs …

For a large class of diffusive systems, the complete probability distribution for the current is exactly known

[Bodineau & Derrida, Phys Rev Lett (2004)]

The mean and variance alone determine *all higher-order cumulants* of the current

The spatial distribution of the conserved quantity in the bulk …

What are the statistical properties of the density profile in diffusive systems?

Exact results are quite **limited** and very **difficult** to obtain

[Derrida, Lebowitz & Speer, Phys Rev Lett (2001); J Stat Phys (2002)]

[Bertini, Gabrielli & Lebowitz, J Stat Phys (2005)]

Need a **more general**, but **relatively simpler** approach

A fluctuating hydrodynamic approach …

Macroscopic Fluctuation Theory

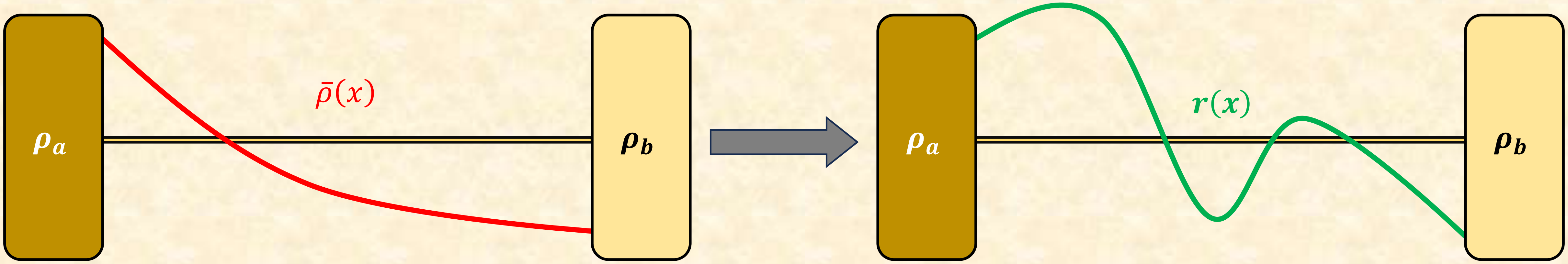
Bertini, De Sole, Gabrielli, Jona-Lasinio & Landim

[Phys Rev Lett (2001)] [J Stat Phys (2002)] … [Phys Rev Lett (2005)] [J Stat Phys (2006)] … [Rev Mod Phys (2015)]

Applicable for **generic** diffusive systems while being analytically **accessible**

- *Reproduces many previously well-known results*
- *New insights into many unsolved problems*

We shall address *two* important questions …

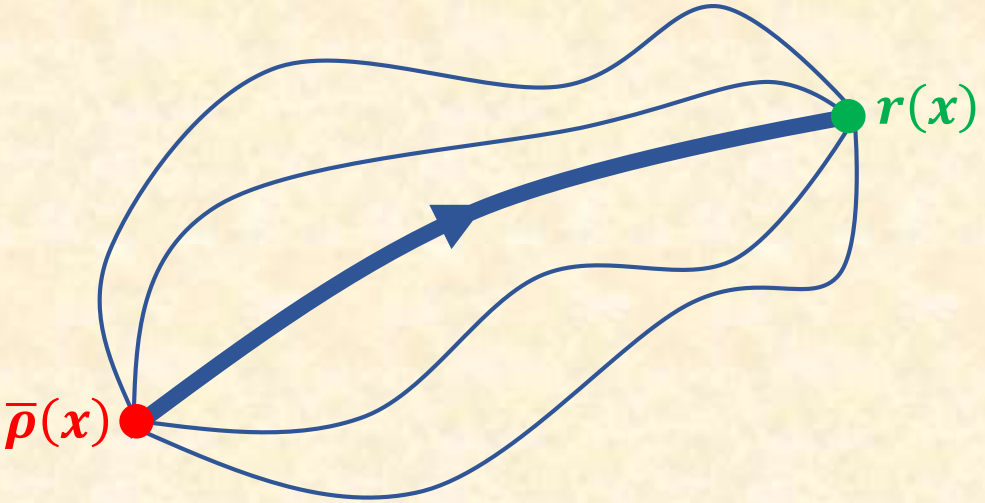


- ***What is the probability of observing a rare fluctuation in the density profile?***

➢ Large deviations functional (*LDF*)

- ***How does the system dynamically evolve to create these rare fluctuations?***

➢ Optimal evolution path



$$\Pr\big(\bar{\rho}(x) \to r(x)\big) = \int_{\bar{\rho}(x)}^{r(x)} [\mathcal{D}\hat{\rho}(x,t)][\mathcal{D}\rho(x,t)]\mathrm{e}^{-LS(\hat{\rho}(x,t),\rho(x,t))}$$

Large system-size limit

$$\Pr\big(\bar{\rho}(x) \to r(x)\big) \approx \mathrm{e}^{-L\psi(r(x))}$$

Response field

Mobility

Diffusivity

$$S\big(\hat{\rho}(x,t),\rho(x,t)\big) = \int_{-\infty}^{0} \mathrm{d}t \int_{0}^{1} \mathrm{d}x \left[\hat{\rho}\partial_t\rho - \left(\frac{\sigma(\rho)}{2}\partial_x\hat{\rho} - D(\rho)\partial_x\rho\right)\partial_x\hat{\rho}\right]$$

Minimal action → LDF

Path of minimal action → Optimal evolution path

Euler-Lagrange equations

$$\partial_t \hat{\rho}(x,t) = -D\big(\rho(x,t)\big)\partial_x^2 \hat{\rho}(x,t) - \frac{\sigma'\big(\rho(x,t)\big)}{2}[\partial_x \hat{\rho}(x,t)]^2$$

$$\partial_t \rho(x,t) = \partial_x\big[D\big(\rho(x,t)\big)\partial_x \rho(x,t)\big] - \partial_x\big[\sigma\big(\rho(x,t)\big)\partial_x \hat{\rho}(x,t)\big]$$

Spatial boundary conditions

$$\hat{\rho}(0,t) = 0, \rho(0,t) = \rho_a$$

$$\hat{\rho}(1,t) = 0, \rho(1,t) = \rho_b$$

Temporal boundary conditions

$$\rho(x,-\infty) = \bar{\rho}(x)$$

$$\rho(x,0) = r(x)$$

A solution to the problem with an arbitrary D and σ is unknown …

We introduce a new ***F* field** via a transformation of the **response field**

$$\hat{\rho}(x,t) = \int_{F(x,t)}^{\rho(x,t)} \mathrm{d}z \, \frac{2D(z)}{\sigma(z)}$$

[**SS** & Sadhu, arXiv (2025)]

The transformation makes the problem more tractable

Provides an exact solution for two large families of diffusive systems …

$\sigma'(\rho) \propto D(\rho)$

Short-range correlations

$$\psi(r(x)) = \psi_{\rm loc}(r(x)|\bar{\rho}(x))$$

$$\partial_t \rho = -\partial_x (D(\rho)\partial_x \rho) + \partial_x \left[\sigma(\rho)\frac{\mathrm{d}}{\mathrm{d}x}\left(f'(\bar{\rho})\right)\right]$$

Constant $D(\rho)$ & Quadratic $\sigma(\rho)$

Long-range correlations

$$\psi(r(x)) = \psi_{\rm loc}(r(x)|F(x)) - \frac{4D}{\sigma''}\int_0^1 \mathrm{d}x \log \frac{F'(x)}{\rho_b - \rho_a}$$

with $r(x) = F(x) + \frac{2}{\sigma''}\frac{\sigma(F)F''(x)}{\left(F'(x)\right)^2}, F(0) = \rho_a, F(1) = \rho_b$

$$\rho = F + \frac{2}{\sigma''}\frac{\sigma(F)\partial_x^2 F}{(\partial_x F)^2} \quad \text{with} \quad \partial_t F = -D\partial_x^2 F$$

Optimal paths of evolution

[**SS** & Sadhu, arXiv (2025)]

Can we solve for generic diffusive systems with an **arbitrary D & σ**?

A perturbation around the equilibrium state *solves* the problem

[SS & Sadhu, arXiv (2025)]

[Bodineau & Derrida, J Stat Phys (2025)]

A highly non-trivial problem …

"… with Thierry Bodineau we tried, so far without success, to calculate $\mathcal{F}\big(\rho(x)\big)$ for general D and σ in powers of $\rho_a - \rho_b$ …"

[Derrida, J Stat Mech (2007)]

The perturbative solution for **any D and σ** …

Non-locality starts appearing at the quadratic order

$$\psi\big(r(x)\big) = \psi_{\mathrm{loc}}\big(r(x)|\bar{\rho}(x)\big) - (\boldsymbol{\rho_a} - \boldsymbol{\rho_b})^2 \frac{2D(\rho_a)}{\sigma^2(\rho_a)} \int_0^1 \mathrm{d}x \int_{-\infty}^{0} \mathrm{d}t \left[g'(\rho_a)\Big(\sigma\big(\boldsymbol{\rho_{eq}}\big) - \sigma(\rho_a)\Big) - \sigma'(\rho_a)\Big(g\big(\boldsymbol{\rho_{eq}}\big) - g(\rho_a)\Big)\right] + \cdots$$

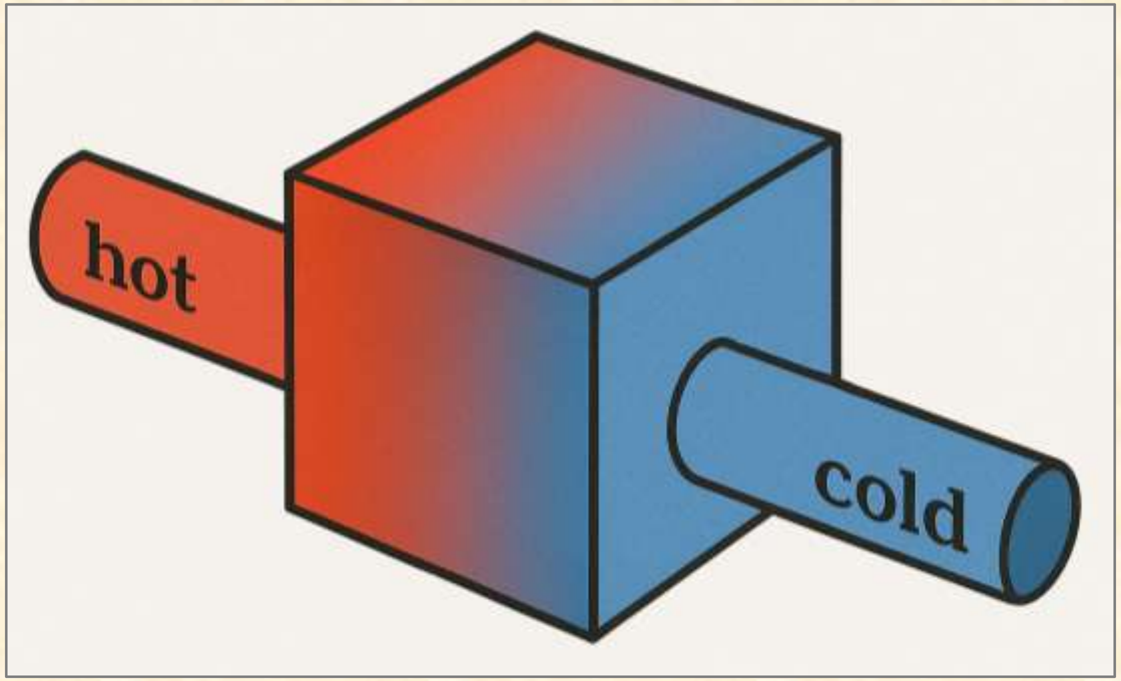
$\boldsymbol{g'(\rho) \equiv D(\rho)}$

Relaxation path in equilibrium determines non-locality

[**SS** & Sadhu, arXiv (2025)]

[Bodineau & Derrida, J Stat Phys (2025)]

What about generalizing to **higher dimensions**?



The perturbation-based approach to the solution still works

[**SS** & Sadhu, arXiv (2025)]

[Bodineau & Derrida, J Stat Phys (2025)]

The take-home message

- A powerful approach that solves the long-standing open problem of **large deviations** in **density profile** for generic diffusive systems and in arbitrary dimensions

- Deriving the **exact fluctuating hydrodynamic description** from the **microscopic dynamics** is a challenging task in itself

- [**SS** and Sadhu, SciPost Phys (2024)]
- [Mukherjee, **SS**, Sadhu, Dhar & Sabhapandit, Phys Rev E (2025)]

Work in progress for a more general derivation …

Thank you for your attention.