Ph.D. synopsis

Out of time ordered effective dynamics of a Brownian particle

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1 Introduction

The effective theory framework is a very useful tool for studying the dynamics of quantum systems. In any such study, it is important to first identify the appropriate set of degrees of freedom for the observables of interest. Then, based on the symmetries of the system, one can try to construct an effective theory which governs the dynamics of these relevant degrees of freedom. Such an effective theory enables one to compute physically interesting quantities in a scenario where a microscopic computation is not feasible.

Traditionally, this effective theory paradigm has been employed to study the evolution of a system given some data on its initial conditions. The physical observables associated with this forward evolution (in time) of the system are determined by the time-ordered correlators of its operators (in the Heisenberg picture).

However, quite recently, a different class of problems have drawn the attention of physicists. These involve the sensitivity of the evolution of a quantum system to small changes in the initial conditions. In such problems, one needs to compare two states of the system which are connected by a succession of backward and forward evolutions in time accompanied with insertions of operators in the middle. Consequently, such delicate comparisons require information of correlation functions where the insertions violate time-ordering.

These Out-of-Time-Order Correlators (OTOCs) have been studied in varied contexts. They have been found to be useful in determining the rate of scrambling of information in quantum systems [1–4], and have been used as diagnostic measures for related phenomena such as chaos [1, 5–8], thermalisation and many-body localisation [9–13]. Parallelly, several experimental protocols [4, 14–16] have been suggested to measure these OTOCs which may lead to a set of new observables that encode hitherto unknown features of quantum systems.

Despite these progresses in the study of OTOCs, we still lack a convenient framework to compute them as many familiar tools of effective theory are yet to be extended to include the information contained in them. In this synopsis, we will summarise the results of some works [17–20] which are aimed towards filling this gap.

To develop the basic ideas behind the construction of an effective theory for OTOCs, we will consider a very simple system viz. a Brownian particle interacting with a large environment. In this setup, we will discuss the OTO dynamics of the particle in a path integral formalism defined on a contour with multiple time-folds[1]. We will see that couplings in this effective dynamics receive contributions from the contour-ordered correlators of the environment. These relations between the particle's effective couplings and the OTOCs of the environment provide a convenient way to extract information of these OTOCs by performing measurements on the particle. Moreover, these relations also allow one to study the effects of symmetries in the environment's microscopic dynamics on the particle's effective theory. We will consider the effects of one such symmetry viz. microscopic time-reversal invariance in the environment's dynamics, and show that it leads to OTO generalisations of the Onsager-Casimir reciprocal relations [26–28] between the particle's effective couplings.

Apart from the generalised Onsager-Casimir relations mentioned above, there are some additional constraints on the effective couplings when the environment is in a thermal state. These constraints follow from the Kubo-Matin-Schwinger relations [29–31] between the thermal correlators of the environment. At the level of the quadratic effective couplings in the high temperature limit, the KMS relations imply the well-known fluctuation-dissipation relation [32–35] which connects the thermal random force experienced by the particle to its damping coefficient. We will show that the cubic and quartic couplings in the effective dynamics satisfy some generalisations of this fluctuation-dissipation relation which connect the non-Gaussianity in the thermal noise experienced by the particle to a thermal jitter in its damping coefficient. We will argue

[1]This is a generalisation [21, 22] of the Schwinger-Keldysh formalism[23–25].

that these generalised fluctuation-dissipation relations arise due to a combined effect of microscopic reversibility in the environment's dynamics and its thermality.

Organisation of the synopsis:

In section 2, we will lay down the structure of a cubic OTO effective action of a Brownian particle interacting with a general environment. We will see that certain cubic couplings in this effective action receive contributions from the environment's OTOCs. We will provide the expressions of these couplings in terms of the environment's OTOCs. Such expressions may be useful in extracting information about the OTOCs of the environment by performing measurements on the particle.

In section 3, we will focus on the case where the environment is a thermal bath and describe some relations between its correlators. These relations are based on microscopic unitarity and thermality of the bath. We will show that these relations can be encapsulated concisely by expressing the bath's correlators in terms of a minimal set of independent spectral functions. Such representations of the bath's correlators are useful for exploring the constraints imposed on the particle's effective dynamics due to thermality of the bath.

In section 4, we will discuss the constraints imposed on the particle's cubic effective dynamics by microscopic reversibility and thermality of the bath. To illustrate these constraints with a concrete model, we will consider a bath comprising of two sets of harmonic oscillators coupled nonlinearly to the particle. We will show that the bath's microscopic reversibility leads to a set of generalised Onsager relations which connect the particle's cubic OTO couplings to its Schwinger-Keldysh (SK) couplings. Moreover, the bath's thermality induces an OTO generalisation of the fluctuation-dissipation relation (FDR) between a cubic OTO coupling and a Schwinger-Keldysh coupling. Combining these two kinds of relations, we will obtain a generalised fluctuation-dissipation relation between two cubic SK couplings. We will show that the SK effective couplings enter as parameters in a dual non-linear Langevin dynamics, where the generalised FDR connects the non-Gaussianity in the noise distribution with a thermal jitter in the particle's damping.

In section 5, we will discuss the quartic terms in the OTO effective action. There, we will extend many of the results obtained in the previous sections to obtain some generalised Onsager relations and a generalised fluctuation-dissipation relation between the quartic couplings.

In section 6, we will conclude with some discussion on future directions.

2 Cubic OTO effective theory

In this section, we will develop a cubic effective theory of a Brownian particle which

(a) incorporates the effects of the 3-point OTOCs of the environment, and

(b) enables one to compute similar OTOCs of the particle.

In order to develop this effective theory, let us first specify the particle's coupling to the environment. Consider the situation where the particle (q) and the environment (X) are initially unentangled. The initial density matrix of their combined system is given by

$$\rho_0 = \rho_{p0} \otimes \rho_{E0} \ , \tag{2.1}$$

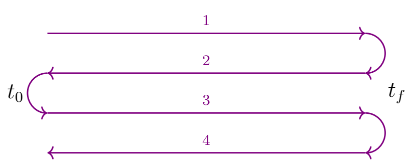
where ρ_{p0} and ρ_{E0} are the initial density matrices of the particle and the environment respectively. Then an interaction between the two is switched on at a time t_0. Let us assume that this interaction is weak and its strength is given by a small parameter λ. The Lagrangian of the (particle+environment) system is given by

$$L[q, X] = \frac{1}{2} m_{p0} \left(\dot{q}^2 - \overline{\mu}_0^2 q^2 \right) + L_E[X] + \lambda \ O \ q \tag{2.2}$$

where m_{p0} and $\overline{\mu}_0$ are the mass and frequency of the particle, $L_E[X]$ is the Lagrangian of the environment, and O is an operator of the environment which couples to the particle.

To get a path integral representation of the particle's OTO correlators, one has to consider a generalised Schwinger-Keldysh contour as shown in Figure 1. For each leg

Figure 1. A contour with 2 time-folds



in this contour, one has to take one copy of the degrees of freedom of both the particle and the environment: $\{q_1, X_1\}, \{q_2, X_2\}, \{q_3, X_3\}$ and $\{q_4, X_4\}$. The OTO action of the (particle+environment) system on this contour is given by

$$S_{\text{2-fold}} = \int_{t_0}^{t_f} dt \Big\{ L[q_1, X_1] - L[-q_2, X_2] + L[q_3, X_3] - L[-q_4, X_4] \Big\}. \tag{2.3}$$

In order to calculate OTOCs of the particle, we can first integrate out the degrees of freedom of the environment to obtain a generalized influence phase [25] for the particle. This generalized influence phase W can be expanded in powers of λ as

$$W = \lambda W_1 + \lambda^2 W_2 + \lambda^3 W_3 + \dots \quad . \tag{2.4}$$

For $n \geq 1$, W_n is given by[2]

$$W_n = i^{n-1} \int_{t_0}^{t_f} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \sum_{i_1, \cdots, i_n = 1}^{4} \langle T_C O_{i_1}(t_1) \cdots O_{i_n}(t_n) \rangle_c \, q_{i_1}(t_1) \cdots q_{i_n}(t_n) \ , \tag{2.5}$$

where $\langle T_C O_{i_1}(t_1) \cdots O_{i_n}(t_n) \rangle_c$ is the cumulant (connected part) of a contour-ordered correlator of the operator O calculated in the initial state ρ_{E0} with the insertion at time t_j being on the i_j^{th} leg.

Using this generalized influence phase, one can calculate the OTOCs of the particle. As one can see from (2.5), the terms in the generalized influence phase are non-local in time. However, if the cumulants of the operator O decay much faster than the particle's evolution, then one can get an approximately local form for this generalized influence phase. Such an approximately local limit of the effective dynamics is often known as the 'Markovian limit'[36].

In such a Markovian limit, it is possible to write down a temporally local 1-particle irreducible effective action of the particle[3]. This local 1-PI effective action should satisfy certain conditions which are based on the following two facts:

a) the particle is a part of a closed system described by a unitary dynamics,

b) the operator q is Hermitian.

[2]Here, we work in units where $\hbar = 1$.

[3]The connected tree level diagrams computed with this 1-PI effective action provide the full perturbative expansions (in λ) of the cumulants of the particle's correlators.

We enumerate these conditions below:

1. Collapse rules: The 1-PI effective action should become independent of $\tilde{q}$ under any of the following identifications:

$$\text{a) } q_1 = -q_2 = \tilde{q}, \ \text{b) } q_2 = -q_3 = \tilde{q}, \ \text{c) } q_3 = -q_4 = \tilde{q}.$$

   Under any of these collapses, the 1-PI effective action reduces to the Schwinger-Keldysh effective action in which the residual degrees of freedom play the role of the right-moving and the left-moving coordinates[31].

2. Reality condition: The 1-PI effective action should become the negative of itself under complex conjugation of all the couplings and the following exchanges:

$$q_1 \leftrightarrow -q_4, \ q_2 \leftrightarrow -q_3 \ .$$

These conditions are straightforward extensions of the conditions imposed on the Schwinger-Keldysh effective action in [37].

We will write down a local 1-PI effective Lagrangian consistent with the above conditions which has the following expansion:

$$L_{1\text{PI}} = L^{(1)}_{1\text{PI}} + L^{(2)}_{1\text{PI}} + L^{(3)}_{1\text{PI}} + \cdots \ , \tag{2.6}$$

where the $L^{(1)}_{1\text{PI}}$, $L^{(2)}_{1\text{PI}}$ and $L^{(3)}_{1\text{PI}}$ are the terms linear, quadratic and cubic in q's respectively. The linear and quadratic terms are given in (2.7) and (2.8) respectively.

$$L^{(1)}_{1\text{PI}} = F(q_1 + q_2 + q_3 + q_4) \ , \tag{2.7}$$

$$\begin{aligned} L^{(2)}_{1\text{-PI}} =& \frac{1}{2}(\dot{q}_1^2 - \dot{q}_2^2 + \dot{q}_3^2 - \dot{q}_4^2) - \frac{\overline{\mu}^2}{2}(q_1^2 - q_2^2 + q_3^2 - q_4^2) \\ &- \frac{\gamma}{2}\Big[(q_1 + q_2)(\dot{q}_1 - \dot{q}_2 - \dot{q}_3 - \dot{q}_4) + (q_3 + q_4)(\dot{q}_1 + \dot{q}_2 + \dot{q}_3 - \dot{q}_4)\Big] \\ &+ \frac{i\langle f^2\rangle}{2}(q_1 + q_2 + q_3 + q_4)^2 - \frac{iZ_I}{2}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3 + \dot{q}_4)^2 \ . \end{aligned} \tag{2.8}$$

Here we consider all quadratic terms up to two derivatives acting on the q's. We have included the double derivative terms to take into account the renormalisation of the kinetic term in the action. Such a renormalisation introduces a correction to the effective mass of the particle on top of the bare mass m_{p0}. After taking into account this correction, we choose to work in units where the renormalised mass of the particle is unity.

The cubic terms can be split into 2 parts: One part which reduces to the terms in the Schwinger-Keldysh 1-PI effective action under any of the collapses mentioned above, and another part which vanishes under such collapses. These 2 sets of terms are given in (2.10) and (2.11).

$$L^{(3)}_{\text{1PI}} = L^{(3)}_{\text{1PI,SK}} + L^{(3)}_{\text{1PI,OTO}} \ , \tag{2.9}$$

where

$$\begin{aligned} L^{(3)}_{\text{1PI,SK}} = &-\frac{\overline{\lambda}_3}{8}\Big[(q_1-q_2-q_3-q_4)^2(q_1+q_2)+(q_1+q_2+q_3-q_4)^2(q_3+q_4)\Big] \\ &-\zeta_N(q_1+q_2+q_3+q_4)^3-\frac{i\zeta_\mu}{2}(q_1+q_2+q_3+q_4)(q_1^2-q_2^2+q_3^2-q_4^2) \\ &-\frac{\overline{\lambda}_3}{8}\Big[(q_1-q_2-q_3-q_4)^2(\dot{q}_1+\dot{q}_2)+(q_1+q_2+q_3-q_4)^2(\dot{q}_3+\dot{q}_4)\Big] \\ &+\frac{i\zeta_\gamma}{2}(\dot{q}_1+\dot{q}_2+\dot{q}_3+\dot{q}_4)(q_1^2-q_2^2+q_3^2-q_4^2), \end{aligned} \tag{2.10}$$

$$\begin{aligned} L^{(3)}_{\text{1PI,OTO}} = &\frac{1}{2}\overline{\kappa}_3(q_1+q_2)(q_2+q_3)(q_3+q_4) \\ &+\frac{1}{2}(q_2+q_3)\Big[\Big(\overline{\kappa}_{3\gamma}-\frac{4}{3}i\widehat{\kappa}_{3\gamma}\Big)(\dot{q}_1+\dot{q}_2)(q_3+q_4) \\ &\qquad\qquad +\Big(\overline{\kappa}_{3\gamma}+\frac{4}{3}i\widehat{\kappa}_{3\gamma}\Big)(q_1+q_2)(\dot{q}_3+\dot{q}_4)\Big] \ . \end{aligned} \tag{2.11}$$

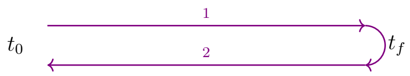
Among the cubic terms, we have kept those with at most a single derivative acting on the q's. The reality condition implies that all the couplings in (2.7), (2.8), (2.10) and (2.11) are real.

The terms given in (2.7), (2.8) and (2.10) are extensions of the terms appearing in the Schwinger-Keldysh(SK) effective action given below:

$$\begin{aligned} L_{SK} = &\frac{1}{2}(\dot{q}_1^2-\dot{q}_2^2)-\frac{\overline{\mu}^2}{2}(q_1^2-q_2^2)-\frac{\gamma}{2}(q_1+q_2)(\dot{q}_1-\dot{q}_2) \\ &+\frac{i\langle f^2\rangle}{2}(q_1+q_2)^2-\frac{iZ_I}{2}(\dot{q}_1+\dot{q}_2)^2+F(q_1+q_2) \\ &-\frac{\overline{\lambda}_3}{8}(q_1-q_2)^2(q_1+q_2)-\zeta_N(q_1+q_2)^3-\frac{i\zeta_\mu}{2}(q_1+q_2)(q_1^2-q_2^2) \\ &-\frac{\overline{\lambda}_3}{8}(q_1-q_2)^2(\dot{q}_1+\dot{q}_2)+\frac{i\zeta_\gamma}{2}(\dot{q}_1+\dot{q}_2)(q_1^2-q_2^2) \ . \end{aligned} \tag{2.12}$$

Here q_1 and q_2 are the coordinates on the Schwinger-Keldysh contour shown in figure 2.

Figure 2. Schwinger-Keldysh contour



The couplings $\overline{\kappa}_3$, $\overline{\kappa}_{3\gamma}$ and $\widehat{\kappa}_{3\gamma}$ are not present in the Schwinger-Keldysh 1-PI effective action. They encode information about the 3-point OTO correlators of the operator $O(t)$ as shown below:

$$\begin{aligned}
\overline{\kappa}_3 &= 2\lambda^3 \lim_{t_1-t_0\to\infty} \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \langle[321]\rangle + \mathrm{O}(\lambda^5) \\
\overline{\kappa}_{3\gamma} &= -\lambda^3 \lim_{t_1-t_0\to\infty} \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \langle[321]\rangle\ (t_{32}+t_{31}) + \mathrm{O}(\lambda^5) \\
\widehat{\kappa}_{3\gamma} &= \frac{3i\lambda^3}{4} \lim_{t_1-t_0\to\infty} \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \Big[\Big(\langle[123_+]\rangle + \langle[321_+]\rangle\Big) t_{12} + \langle[12_+3]\rangle\ t_{13}\Big] + \mathrm{O}(\lambda^5)
\end{aligned} \tag{2.13}$$

where

$$\begin{aligned}
&t_{ij} \equiv t_i - t_j,\ \langle[321]\rangle \equiv \langle[[O(t_3),O(t_2)],O(t_1)]\rangle_c,\ \langle[321_+]\rangle \equiv \langle\{[O(t_3),O(t_2)],O(t_1)\}\rangle_c, \\
&\langle[123_+]\rangle \equiv \langle\{[O(t_1),O(t_2)],O(t_3)\}\rangle_c,\ \langle[12_+3]\rangle \equiv \langle[\{O(t_1),O(t_2)\},O(t_3)]\rangle_c
\end{aligned} \tag{2.14}$$

These relations between the particle's cubic OTO couplings and the 3-point OTOCs of the environment provide a way to extract information about these OTOCs by performing measurements on the particle. Moreover, as we will see in section 4, such relations also allow one to explore the constraints imposed by symmetries in the environment (eg. time-reversal invariance) on the effective dynamics of the particle.

When the environment is a thermal bath [4], there are additional analytic properties of the bath's correlators. We will discuss these analytic properties in the next section. Later, in section 4, we will see that they lead to further constraints on the effective theory of the particle.

[4]In this context, by a thermal bath, we mean a quantum system with a large number of degrees of freedom at thermal equilibrium.

3 Spectral Representation of Thermal OTO Correlators

In the previous section, we saw that the generalised influence phase of the particle receives contributions from the contour-ordered correlators of the environment's operator O. Now, not all of these contour-ordered correlators are independent. The unitarity of the environment's dynamics leads to the equality of any two correlators which can be obtained from one another by sliding an insertion from one leg to the next across a turning point without encountering any other insertion in the process. Moreover, if the environment is in a thermal state (at temperature $\frac{1}{\beta}$) then the correlators satisfy the Kubo-Martin-Schwinger relations [29–31] of the following form:

$$\langle O(t_1)\cdots O(t_{n-1})O(t_n)\rangle = \langle O(t_n - i\beta)O(t_1)\cdots O(t_{n-1})\rangle \tag{3.1}$$

In frequency space, such relations reduce to

$$\langle \widetilde{O}(\omega_1)\cdots \widetilde{O}(\omega_{n-1})\widetilde{O}(\omega_n)\rangle = e^{-\beta\omega_n}\langle \widetilde{O}(\omega_n)\widetilde{O}(\omega_1)\cdots \widetilde{O}(\omega_{n-1})\rangle \tag{3.2}$$

These relations following from the unitarity and thermality of the environment imply that it is redundant to keep track of contributions of all the contour-ordered correlators to the particle's effective dynamics separately. Rather, it is useful to express these contour-ordered correlators in terms of a basis of independent correlators [18] and then look at the contribution of such minimal set of objects to the effective theory of the particle.

To obtain such a basis, first note that in any n-point contour-ordered correlator, one can put each of the n insertions on any one of the four legs. Therefore, one can construct an n-dimensional array out of these correlators, where the index corresponding to each axis takes values over $\{1, 2, 3, 4\}$. Let us look at this array of correlators in frequency space. The elements of this array are as defined below:

$$\int_{-\infty}^{\infty}\frac{d\omega_1}{2\pi}\cdots\int_{-\infty}^{\infty}\frac{d\omega_n}{2\pi}\; M^{(\mathrm{n})}_{i_1,i_2,\cdots,i_n}(\omega_1,\cdots,\omega_n)e^{-i(\omega_1 t_1+\cdots+\omega_n t_n)} \equiv \langle T_C O_{i_1}(t_1)\cdots O_{i_n}(t_n)\rangle_c \tag{3.3}$$

in the domain $t_1 > t_2 > \cdots > t_n$. For each axis in this array, we choose the following basis of column vectors:

$$\begin{aligned}
\bar{e}_{_P}^{(1)}(\omega) &\equiv \{1+\mathfrak{f}(\omega), 1+\mathfrak{f}(\omega), 1+\mathfrak{f}(\omega), 1+\mathfrak{f}(\omega)\}^T,\\
\bar{e}_{_P}^{(2)}(\omega) &\equiv \{\mathfrak{f}(\omega), \mathfrak{f}(\omega), 1+\mathfrak{f}(\omega), 1+\mathfrak{f}(\omega)\}^T,\\
\bar{e}_{_F}^{(1)}(\omega) &\equiv \{\mathfrak{f}(\omega), 1+\mathfrak{f}(\omega), 1+\mathfrak{f}(\omega), 1+\mathfrak{f}(\omega)\}^T,\\
\bar{e}_{_F}^{(2)}(\omega) &\equiv \{\mathfrak{f}(\omega), \mathfrak{f}(\omega), \mathfrak{f}(\omega), 1+\mathfrak{f}(\omega)\}^T \ .
\end{aligned} \tag{3.4}$$

where $\mathfrak{f}(\omega) \equiv \frac{1}{e^{\beta\omega}-1}$. For convenience, we also define

$$\bar{e}_{_P}^{(3)}(\omega) \equiv e^{-\beta\omega}\bar{e}_{_P}^{(1)}(\omega). \tag{3.5}$$

The expressions of the arrays of 2-point, 3-point and 4-point correlators in the basis of tensor products of the above column vectors are given below.

Array of 2-point correlators:

$$M^{(2)}(\omega_1,\omega_2) = \rho[12]\sum_{r=1}^{2}\Big(\bar{e}_{_P}^{(r+1)}(\omega_1) - \bar{e}_{_P}^{(r)}(\omega_1)\Big)\otimes \bar{e}_{_F}^{(r)}(\omega_2), \tag{3.6}$$

where

$$\int_{-\infty}^{\infty}\frac{d\omega_1}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}\rho[12]e^{-i(\omega_1 t_1+\omega_2 t_2)} \equiv \langle[O(t_1),O(t_2)]\rangle_c \ . \tag{3.7}$$

Array of 3-point correlators:

$$\begin{aligned}
M^{(3)}(\omega_1,\omega_2,\omega_3) = &-\rho[123]\sum_{r=1}^{2}\Big(\bar{e}_{_P}^{(r+1)}(\omega_1) - \bar{e}_{_P}^{(r)}(\omega_1)\Big)\otimes \bar{e}_{_F}^{(r)}(\omega_2)\otimes \bar{e}_{_F}^{(r)}(\omega_3)\\
&+\rho[321]\sum_{r=1}^{2}\Big[\bar{e}_{_P}^{(r+1)}(\omega_1)\otimes\bar{e}_{_P}^{(r+1)}(\omega_2)\otimes\bar{e}_{_F}^{(r)}(\omega_3)\\
&\qquad -\bar{e}_{_P}^{(r)}(\omega_1)\otimes\bar{e}_{_P}^{(r)}(\omega_2)\otimes\bar{e}_{_F}^{(r)}(\omega_3)\Big],
\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{d\omega_1}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega_3}{2\pi}\rho[123]e^{-i(\omega_1 t_1+\omega_2 t_2+\omega_3 t_3)} &\equiv \langle[[O(t_1),O(t_2)],O(t_3)]\rangle_c \ ,\\
\int_{-\infty}^{\infty}\frac{d\omega_1}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega_3}{2\pi}\rho[321]e^{-i(\omega_1 t_1+\omega_2 t_2+\omega_3 t_3)} &\equiv \langle[[O(t_3),O(t_2)],O(t_1)]\rangle_c,
\end{aligned} \tag{3.9}$$

Array of 4-point correlators:

$$M^{(4)}(\omega_1,\omega_2,\omega_3,\omega_4) = M^{(4)}_{PPPF} + M^{(4)}_{PFFF} + M^{(4)}_{PFPF} + M^{(4)}_{PPFF} \tag{3.10}$$

$$M^{(4)}_{PPPF} = -\rho[4321] \sum_{r=1}^{k} \left(\bar{e}^{(r+1)}_{P} \otimes \bar{e}^{(r+1)}_{P} \otimes \bar{e}^{(r+1)}_{P} \otimes \bar{e}^{(r)}_{F} - \bar{e}^{(r)}_{P} \otimes \bar{e}^{(r)}_{P} \otimes \bar{e}^{(r)}_{P} \otimes \bar{e}^{(r)}_{F} \right) , \tag{3.11}$$

$$M^{(4)}_{PFFF} = \rho[1234] \sum_{r=1}^{k} \left(\bar{e}^{(r+1)}_{P} - \bar{e}^{(r)}_{P} \right) \otimes \bar{e}^{(r)}_{F} \otimes \bar{e}^{(r)}_{F} \otimes \bar{e}^{(r)}_{F} , \tag{3.12}$$

$$\begin{aligned} M^{(4)}_{PFPF} = &\sum_{r,s=1}^{k} \Big(\theta_{r>s}\ \rho[12][34] + \theta_{r\leq s}\ \rho[34][12] \Big) \left(\bar{e}^{(r+1)}_{P} - \bar{e}^{(r)}_{P} \right) \otimes \bar{e}^{(r)}_{F} \otimes \bar{e}^{(s+1)}_{P} \otimes \bar{e}^{(s)}_{F} \\ &- \sum_{r,s=1}^{k} \Big(\theta_{r\geq s} \rho[12][34] + \theta_{r<s}\ \rho[34][12] \Big) \left(\bar{e}^{(r+1)}_{P} - \bar{e}^{(r)}_{P} \right) \otimes \bar{e}^{(r)}_{F} \otimes \bar{e}^{(s)}_{P} \otimes \bar{e}^{(s)}_{F} , \end{aligned} \tag{3.13}$$

$$\begin{aligned} M^{(4)}_{PPFF} = &\sum_{r,s=1}^{k} \Big(\theta_{r>s}\ \rho[13][24] + \theta_{r\leq s}\ \rho[24][13] \Big) \left(\bar{e}^{(r+1)}_{P} - \bar{e}^{(r)}_{P} \right) \otimes \bar{e}^{(s+1)}_{P} \otimes \bar{e}^{(r)}_{F} \otimes \bar{e}^{(s)}_{F} \\ &- \sum_{r,s=1}^{k} \Big(\theta_{r\geq s}\ \rho[13][24] + \theta_{r<s}\ \rho[24][13] \Big) \left(\bar{e}^{(r+1)}_{P} - \bar{e}^{(r)}_{P} \right) \otimes \bar{e}^{(s)}_{P} \otimes \bar{e}^{(r)}_{F} \otimes \bar{e}^{(s)}_{F} \\ &+ \sum_{r,s=1}^{k} \Big(\theta_{r\geq s}\ \rho[14][23] + \theta_{r<s}\ \rho[23][14] \Big) \bar{e}^{(r+1)}_{P} \otimes \left(\bar{e}^{(s+1)}_{P} - \bar{e}^{(s)}_{P} \right) \otimes \bar{e}^{(s)}_{F} \otimes \bar{e}^{(r)}_{F} \\ &- \sum_{r,s=1}^{k} \Big(\theta_{r>s}\ \rho[14][23] + \theta_{r\leq s}\ \rho[23][14] \Big) \bar{e}^{(r)}_{P} \otimes \left(\bar{e}^{(s+1)}_{P} - \bar{e}^{(s)}_{P} \right) \otimes \bar{e}^{(s)}_{F} \otimes \bar{e}^{(r)}_{F} \\ &+ \rho[2314] \sum_{r=1}^{k} \left(\bar{e}^{(r+1)}_{P} \otimes \bar{e}^{(r+1)}_{P} - \bar{e}^{(r)}_{P} \otimes \bar{e}^{(r)}_{P} \right) \otimes \bar{e}^{(r)}_{F} \otimes \bar{e}^{(r)}_{F} . \end{aligned} \tag{3.14}$$

The spectral functions $\rho[ijkl]$, $\rho[ij][kl]$ in the above expressions are defined as follows:

$$\begin{aligned} &\int \frac{d^4\omega}{(2\pi)^4} \rho[ijkl] e^{-i(\omega_1 t_1 + \omega_2 t_2 + \omega_3 t_3 + \omega_4 t_4)} \equiv \langle [[[O(t_i), O(t_j)], O(t_k)], O(t_l)] \rangle_c , \\ &\int \frac{d^4\omega}{(2\pi)^4} \rho[ij][kl] e^{-i(\omega_1 t_1 + \omega_2 t_2 + \omega_3 t_3 + \omega_4 t_4)} \equiv \langle [O(t_i), O(t_j)][O(t_k)], O(t_l)] \rangle_c , \end{aligned} \tag{3.15}$$

in the domain $t_1 > t_2 > t_3 > t_4$. Here, the measure $\frac{d^4\omega}{(2\pi)^4}$ is defined as follows:

$$\int \frac{d^4\omega}{(2\pi)^4} \equiv \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_3}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_4}{2\pi} . \tag{3.16}$$

These spectral representations of the bath's OTOCs are generalisations of similar representations of thermal Schwinger-Keldysh correlators discussed in [38]. They provide a convenient way to explore the constraints imposed on the effective theory of the particle due to the KMS relations between the bath's correlators. We will discuss these constraints in the following section.

4 Cubic effective theory for a particle coupled to a thermal bath

In this section, we will look at some special features of the cubic effective theory introduced in section 2 when the environment is in a thermal state [19]. To study these features, we will introduce a toy model which is a simple extension of the well-known Caldeira-Leggett model[39]. For this model, we will state the effective couplings in the high temperature limit. We will find that these effective couplings satisfy some relations due to microscopic reversibility and thermality of the bath. On combining these relations, we will obtain a generalised fluctuation-dissipation relation between two Schwinger-Keldysh couplings. To understand this relation better, we will demonstrate a duality between the Schwinger-Keldysh effective theory and a non-linear Langevin dynamics with a non-Gaussian noise distribution. We will see that, from the point of view of this non-linear Langevin dynamics, the generalised fluctuation-dissipation relation connects the non-Gaussianity in the noise distribution to a thermal jitter in the damping coefficient of the particle.

4.1 Description of the qXY model

Consider the situation where the bath comprises of two sets of harmonic oscillators denoted by X and Y. The particle couples to these oscillators via Caldeira-Leggett-like bilinear interactions[39]. On top of such bilinear interactions, there are small cubic interactions which couple the particle to pairs of bath oscillators, where every pair consists of one oscillator from each set. The Lagrangian of the (particle+bath) combined system is given by

$$\begin{aligned} L[q,X,Y] =& \frac{1}{2}m_{p0}(\dot{q}^2-\overline{\mu}_0^2 q^2)+\frac{1}{2}\sum_i m_{x,i}(\dot{X}_i^2-\mu_{x,i}^2X_i^2)+\frac{1}{2}\sum_j m_{y,j}(\dot{Y}_j^2-\mu_{y,j}^2Y_j^2) \\ &+\lambda\left(\sum_i g_{x,i}X_i+\sum_j g_{y,j}Y_j+\sum_{i,j} g_{xy,ij}X_iY_j\right)q, \end{aligned} \tag{4.1}$$

where $m_{p0}, m_{x,i}, m_{y,j}$ are the masses of the particle(q) and the bath oscilattors X_i and Y_j respectivey, and $\overline{\mu}_0, \mu_{x,i}, \mu_{y,j}$ are their respective frequencies. The bath operator that couples to the particle is given by

$$\lambda O \equiv \lambda\Big(\sum_i g_{x,i}X_i + \sum_j g_{y,j}Y_j + \sum_{i,j} g_{xy,ij}X_iY_j\Big). \tag{4.2}$$

While calculating the correlators of this operator, we assume that there is a large number of bath oscillators and the frequencies of these oscillators are distributed densely. Hence one can go to the continuum limit and replace the following sums over the oscillator frequencies by integrals as shown below:

$$\begin{aligned}
&\sum_i \frac{g_{x,i}^2}{m_{x,i}} \to \int_0^\infty \frac{d\mu_x}{2\pi}\Big\langle\Big\langle \frac{g_x^2}{m_x}\Big\rangle\Big\rangle, \ \sum_j \frac{g_{y,j}^2}{m_{y,j}} \to \int_0^\infty \frac{d\mu_y}{2\pi}\Big\langle\Big\langle \frac{g_y^2}{m_y}\Big\rangle\Big\rangle,\\
&\sum_{ij} \frac{g_{xy,ij}^2}{m_{x,i}m_{y,j}} \to \int_0^\infty \frac{d\mu_x}{2\pi}\int_0^\infty \frac{d\mu_y}{2\pi}\Big\langle\Big\langle \frac{g_{xy}^2}{m_x m_y}\Big\rangle\Big\rangle,\\
&\sum_{ij} \frac{g_{x,i}g_{y,j}g_{xy,ij}}{m_{x,i}m_{y,j}} \to \int_0^\infty \frac{d\mu_x}{2\pi}\int_0^\infty \frac{d\mu_y}{2\pi}\Big\langle\Big\langle \frac{g_x g_y g_{xy}}{m_x m_y}\Big\rangle\Big\rangle.
\end{aligned} \tag{4.3}$$

Now, we want to choose the distribution of the couplings in such a way that the cumulants of the operator O decay rapidly. One can then tune the parameters of this distribution so that the particle's evolution is much slower than the decay rate of the bath cumulants. This would then lead to an approximately local dynamics of the particle. Keeping these in mind, we choose a distribution of the couplings whose moments have the following forms:

$$\begin{aligned}
&\lambda^2\Big\langle\Big\langle \frac{g_x^2}{m_x}\Big\rangle\Big\rangle = \gamma_x\frac{4\mu_x^2\Omega^2}{\mu_x^2+\Omega^2}, \quad \lambda^2\Big\langle\Big\langle \frac{g_y^2}{m_y}\Big\rangle\Big\rangle = \gamma_y\frac{4\mu_y^2\Omega^2}{\mu_y^2+\Omega^2},\\
&\lambda^2\Big\langle\Big\langle \frac{g_{xy}^2}{m_x m_y}\Big\rangle\Big\rangle = \Gamma_{xy}\ \frac{4\mu_x^2\Omega^2}{\mu_x^2+\Omega^2}\ \frac{4\mu_y^2\Omega^2}{\mu_y^2+\Omega^2}, \quad \lambda^3\Big\langle\Big\langle \frac{g_x g_y g_{xy}}{m_x m_y}\Big\rangle\Big\rangle = \frac{\Gamma_3}{4}\ \frac{4\mu_x^2\Omega^2}{\mu_x^2+\Omega^2}\ \frac{4\mu_y^2\Omega^2}{\mu_y^2+\Omega^2},
\end{aligned} \tag{4.4}$$

where Ω is a UV-regulator. For this distribution, the bath's 2-point and 3-point cumulants decay at rates which are of the order of Ω [19]. By choosing the parameters in this distribution to lie in the following regime, one can ensure that the decay rates of the bath's cumulants are much larger than the rate at which the particle evolves.

$$\beta\Omega \ll 1, \overline{\mu}_0 \ll \Omega, \gamma_x \ll \Omega, \gamma_y \ll \Omega, \Gamma_{xy} \ll \frac{\beta(\gamma_x+\gamma_y)}{\Omega}, \Gamma_3 \ll \beta(\gamma_x+\gamma_y)\ . \tag{4.5}$$

Hence, the Markov approximation for the particle's effective dynamics is justified in this regime. Working in this regime, one can compute the effective couplings in terms of the parameters appearing in (4.4). We provide the values of the couplings up to leading order in λ and β below.

Linear coupling:

$$F = 0. \tag{4.6}$$

Quadratic couplings:

$$\begin{aligned} Z_I &= \frac{1}{\beta\Omega^2}\left(2\gamma_x + 2\gamma_y + \frac{\Gamma_{xy}\Omega}{4\beta}\right) , \quad \overline{\mu}^2 = \overline{\mu}_0^2 - \Omega\left(\gamma_x + \gamma_y + \frac{\Gamma_{xy}\Omega}{\beta}\right) , \\ \langle f^2\rangle &= \frac{2}{\beta}\left(\gamma_x + \gamma_y + \frac{\Gamma_{xy}\Omega}{2\beta}\right) , \quad \gamma = \gamma_x + \gamma_y + \frac{\Gamma_{xy}\Omega}{2\beta} . \end{aligned} \tag{4.7}$$

Cubic couplings:

$$\begin{aligned} \overline{\lambda}_3 = \overline{\kappa}_3 = -\frac{3}{2}\Gamma_3\Omega^2 , \quad \overline{\lambda}_{3\gamma} = \overline{\kappa}_{3\gamma} = -2\Gamma_3\Omega , \quad \zeta_\gamma = \frac{1}{2}\widehat{\kappa}_{3\gamma} = \frac{3\Gamma_3}{2\beta} , \\ \zeta_\mu = -\frac{2\Gamma_3\Omega}{\beta} , \quad \zeta_N = \frac{\Gamma_3}{\beta^2} . \end{aligned} \tag{4.8}$$

From these leading order forms of the effective couplings, one can see that there are several relations between them. As we will discuss next, these relations arise from the microscopic reversibility and the thermality of the bath.

4.2 Consequences of microscopic reversibility and thermality of the bath

In the qXY model, the bath's dynamics is symmetric under the following transformation:

$$\mathbf{T}X_i(t)\mathbf{T}^{-1} = X_i(-t), \ \ \mathbf{T}Y_j(t)\mathbf{T}^{-1} = Y_j(-t), \tag{4.9}$$

where $\mathbf{T}$ is an anti-linear, anti-unitary operator which corresponds to time-reversal [40]. Moreover, the bath operator O that couples to the particle has an even parity under this transformation which implies that

$$\mathbf{T}O(t)\mathbf{T}^{-1} = O(-t). \tag{4.10}$$

This leads to the following kind of relations between its n-point correlators

$$\begin{aligned} \langle O(t_1)\cdots O(t_n)\rangle &= \Big(\langle \mathbf{T}O(t_1)\mathbf{T}^{-1}\cdots \mathbf{T}O(t_n)\mathbf{T}^{-1}\rangle\Big)^* = \Big(\langle O(-t_1)\cdots O(-t_n)\rangle\Big)^* \\ &= \langle O(-t_n)\cdots O(-t_1)\rangle. \end{aligned} \tag{4.11}$$

for $t_1 > \cdots > t_n$. The last step in the above equation follows from the Hermiticity of the operator O. As discussed in [19], such relations between the bath's correlators can connect OTO correlators of the bath to its Schwinger-Keldysh correlators. These relations between the OTOCs and Schwinger-Keldysh correlators of the bath lead to the following relations between the effective couplings of the particle [19]:

$$\overline{\kappa}_3 = \overline{\lambda}_3,\ \overline{\kappa}_{3\gamma} = \overline{\lambda}_{3\gamma},\ \widehat{\kappa}_{3\gamma} = 2\zeta_\gamma. \tag{4.12}$$

Since these relations are based on the microscopic reversibility of the bath, they can be interpreted as OTO generalisations of the well-known Onsager-Casimir reciprocal relations [26–28].

Apart from the microscopic reversibility of the bath, there is another source of relations between the effective couplings viz. the KMS relations between the thermal correlators of the bath which were discussed in the previous section. Such KMS relations between the 2-point functions of the bath lead to the well-known fluctuation-dissipation relation (FDR) [32–35] between two quadratic couplings in the high temperature limit. This fluctuation-dissipation relation is given below:

$$\langle f^2 \rangle = \frac{2}{\beta}\gamma. \tag{4.13}$$

In case of higher point functions of the bath, the KMS relations can connect OTOCs to Schwinger-Keldysh correlators by analytic continuation [41]. Such KMS relations between the bath's 3-point correlators lead to the following OTO generalisation of the FDR between two cubic effective couplings of the particle:

$$\widehat{\kappa}_{3\gamma} = 3\beta\zeta_N \tag{4.14}$$

up to leading order in λ and β. Combining this with the relations given in (4.12), one gets a generalised fluctuation-dissipation relation between two cubic Schwinger-Keldysh couplings:

$$\zeta_\gamma = \frac{3}{2}\beta\zeta_N\ . \tag{4.15}$$

In order to better understand this generalised FDR, let us now demonstrate a duality between the Schwinger-Keldysh effective theory and a stochastic dynamics governed by a non-linear Langevin equation. In the next subsection, we will see that, from the point of view of this Langevin dynamics, the generalised FDR connects a thermal jitter in the damping coefficient of the particle to the non-Gaussianity in the thermal noise distribution.

4.3 Duality with a stochastic dynamics

In this subsection we will outline the argument for a duality between the cubic Schwinger-Keldysh effective theory described by the Lagrangian in (2.12) and a stochastic theory governed by a non-linear Langevin dynamics. It will be an extension of a similar duality between the quadratic effective theory of a Brownian particle and a linear Langevin dynamics [42]. We will first propose the form this stochastic theory and then briefly present the proof of the duality. This proof will be based on the techniques developed by Martin-Siggia-Rose[43], De Dominicis-Peliti[44] and Janssen[45].

The dual non-linear Langevin dynamics:

Consider the following non-linear Langevin equation

$$\mathcal{E}[q] \equiv \frac{d^2q}{dt^2} + (\gamma + \zeta_\gamma \mathcal{N})\frac{dq}{dt} + (\bar{\mu}^2 + \zeta_\mu \mathcal{N})\, q + \left(\overline{\lambda}_3 - \overline{\lambda}_{3\gamma}\,\frac{d}{dt}\right)\frac{q^2}{2!} - F = \langle f^2\rangle \mathcal{N} \quad , \tag{4.16}$$

where $\mathcal{N}$ is a noise drawn from the non-Gaussian distribution given below

$$P[\mathcal{N}] \propto \exp\left\{-\frac{1}{2\langle f^2\rangle}\int dt\, \left(\langle f^2\rangle \mathcal{N} - \zeta_N \mathcal{N}^2\right)^2 - \frac{1}{2} Z_I \int dt\, \dot{\mathcal{N}}^2\right\} \; . \tag{4.17}$$

The non-linearities in this dynamics as well as the non-Gaussianity in the noise are fixed by the following parameters: $\zeta_N, \zeta_\gamma, \zeta_\mu, \overline{\lambda}_3, \overline{\lambda}_{3\gamma}$. All these parameters are $O(\lambda^3)$ in the particle-bath coupling λ. If we ignore these $O(\lambda^3)$ contributions, then the dynamics satisfies a linear Langevin equation of the following form:

$$\ddot{q} + \gamma\dot{q} + \overline{\mu}^2 q = \langle f^2\rangle \mathcal{N}, \tag{4.18}$$

where the noise is drawn from the Gaussian probability distribution given below

$$P[\mathcal{N}] \sim \exp\left[-\int dt \Big(\frac{\langle f^2\rangle}{2}\mathcal{N}^2 + \frac{Z_I}{2}\dot{\mathcal{N}}^2\Big)\right] \; . \tag{4.19}$$

- <u>**Parameters in the linear Langevin dynamics**</u>

  The parameters appearing in the linear Langevin dynamics given in (4.18) and (4.19) can be interpreted in the following manner:

  1. $\langle f^2\rangle$ is the strength of an additive noise in the dynamics.
  2. Z_I introduces nonzero correlations between the noise at two different times.

3. $\overline{\mu}$ is the renormalised frequency.

4. γ is the coefficient of damping.

- **Additional parameters in the non-linear dynamics**

  If we include the contribution of the $O(\lambda^3)$ parameters in the dynamics, then these additional parameters can be interpreted as follows:

  1. ζ_μ is a jitter in the renormalised frequency due to the thermal noise.

  2. ζ_γ is a jitter in the damping coefficient due to the thermal noise.

  3. ζ_N is the strength of non-Gaussianity in the noise distribution.

  4. $\overline{\lambda}_3$ and $\overline{\lambda}_{3\gamma}$ are the strengths of anharmonic terms in the equation of motion.

Now, let us demonstrate the duality between this non-linear Langevin dynamics and the quartic effective theory that we introduced earlier.

We start by considering the functional integral[5] over noise realisations along with the imposition of the non-linear Langevin equation on a variable $q_a(t)$:

$$\begin{aligned}\mathcal{Z} &= \int [dq_a][d\mathcal{N}]\ P[\mathcal{N}]\ \delta\Big[\langle f^2\rangle\mathcal{N} - \mathcal{E}[q_a]\Big] \\ &= \int [dq_a][dq_d][d\mathcal{N}]\ P[\mathcal{N}]\ \exp\left\{i\int dt\ q_d\Big[\langle f^2\rangle\mathcal{N} - \mathcal{E}[q_a]\Big]\right\}\ ,\end{aligned} \tag{4.20}$$

where we have given the standard functional integral representation of the delta function. We can now discretise the noise integral, add appropriate counterterms and perform the path integral perturbatively in the small parameters $\{\zeta_\gamma, \zeta_\mu, \overline{\lambda}_3, \overline{\lambda}_{3\gamma}, \zeta_N, Z_I\}$.

[5] In this integral, we ignore the Jacobian det $\left[\frac{\delta\mathcal{E}[q(t)]}{\delta q(t')}\right]$ as it does not correct the coefficients of the terms obtained in (4.21) up to leading order in the particle-bath coupling.

This exercise yields

$$
\begin{aligned}
\mathcal{Z} &= \lim_{\delta t\to 0}\int [dq_a][dq_d][d\mathcal{N}]e^{-i\frac{3\zeta_N}{\langle f^2\rangle\delta t}\int dt\ q_d}\ P[\mathcal{N}]\exp\left\{i\int dt\ q_d\Big(\langle f^2\rangle\mathcal{N}-\mathcal{E}[q_a]\Big)\right\}\\
&\approx \int [dq_a][dq_d]\exp\left\{i\int dt\ \left[\frac{i}{2}\langle f^2\rangle q_d^2-\frac{i}{2}Z_I\dot{q}_d^2-\zeta_N q_d^3-q_d\mathcal{E}[q_a]_{\mathcal{N}=0}-iq_d^2\frac{\partial\mathcal{E}[q_a]}{\partial\mathcal{N}}\right]\right\}\\
&= \int [dq_a][dq_d]\exp\left\{i\int dt\ \left[\dot{q}_a\dot{q}_d+Fq_d-\overline{\mu}^2q_aq_d-\gamma q_d\dot{q}_a+\frac{i\langle f^2\rangle}{2}q_d^2-\frac{iZ_I}{2}\dot{q}_d^2\right.\right.\\
&\qquad -\frac{\overline{\lambda}_3}{2}q_a^2q_d-\zeta_N q_d^3-i\zeta_\mu q_aq_d^2-\frac{\overline{\lambda}_{3\gamma}}{2}q_a^2\dot{q}_d+i\zeta_\gamma q_aq_d\dot{q}_d\\
&\qquad \left.\left.+\,(\text{total derivatives})\right]\right\}
\end{aligned}
\tag{4.21}
$$

The Lagrangian in the above integral is the same as the cubic Schwinger Keldysh effective Lagrangian given in (2.12) under the following identification [6]:

$$
q_d = q_1+q_2,\ \ q_a=\frac{1}{2}(q_1-q_2). \tag{4.22}
$$

This concludes our argument for the duality between the cubic effective theory and the non-linear Langevin dynamics. We refer the reader to [42] for a more detailed discussion on such dualities between stochastic and Schwinger-Keldysh path integrals.

In this dual non-linear Langevin dynamics, the fluctuation-dissipation relation given in (4.13) implies that the strength of the additive noise ($\langle f^2\rangle$) is connected to the damping coefficient (γ) of the particle. The generalised FDR given in (4.15) connects the non-Gaussianity (ζ_N) in the noise distribution and the thermal jitter (ζ_γ) in the damping coefficient of the particle. In the next section, we will show that a similar relation holds between two couplings in the quartic effective theory of a Brownian particle as well.

5 Quartic OTO effective theory

In this section we will discuss an extension of the OTO dynamics of a Brownian particle which includes the quartic terms in the effective action[20]. To illustrate the features of this quartic effective theory we will slightly modify the qXY model introduced in

[6] The basis $\{q_a, q_d\}$ for the Schwinger-Keldysh degrees of freedom is known as the Keldysh basis [24, 42].

section 4.1. For this modified qXY model, we will first enumerate all the terms in the quartic OTO effective action. This effective action will reduce to a quartic Schwinger-Keldysh (SK) effective action under any of the collapses mentioned in section 2. We will see that, just like the cubic SK effective theory, this quartic effective dynamics is also dual to a non-linear Langevin dynamics.

Analogous to the cubic effective theory, the quartic effective theory of the particle will also be constrained by the microscopic reversibility and thermality of the bath. These constraints manifest in the form of quartic generalisations of Onsager reciprocal relations and fluctuation-dissipation relation. Combining these relations, we will get a generalised fluctuation-dissipation relation between two SK effective couplings. We will see that, just like the case discussed in the previous section, this generalised fluctuation-dissipation relation connects the non-Gaussianity in the noise distribution to a thermal jitter in the damping coefficient of the particle.

5.1 Quartic effective action for a modifed qXY model

To study the quartic effective theory of the particle in the qXY model, we will switch off the Caldeira-Leggett-like bilinear interactions between the particle and the bath oscillators. The Lagrangian of the model is then given by

$$\begin{aligned} L[q,X,Y] = \frac{m_{p0}}{2}(\dot{q}^2 - \overline{\mu}_0^2 q^2) + \sum_i \frac{m_{x,i}}{2}(\dot{X}^{(i)2} - \mu_{x,i}^2 X^{(i)2}) \\ + \sum_j \frac{m_{y,j}}{2}(\dot{Y}^{(j)2} - \mu_{y,j}^2 Y^{(j)2}) + \lambda \sum_{i,j} g_{xy,ij} X^{(i)} Y^{(j)} q. \end{aligned} \tag{5.1}$$

The bath operator that couples to the particle is

$$\lambda O \equiv \lambda \sum_{i,j} g_{xy,ij} X^{(i)} Y^{(j)}. \tag{5.2}$$

Notice that all odd point correlators of this operator vanish in the thermal state. This leads to the vanishing of all odd degree terms in the effective action of the particle. Among the remaining terms, we restrict our attention here to only the quadratic and the quartic ones. From the form of the generalised influence given in (2.5), one can see that the quadratic and the quartic effective couplings receive contributions from the connected parts of 2-point and 4-point correlators of O respectively at leading order in λ. While computing these correlators, we assume that there is a large number of oscillators in the bath, and the frequencies of these oscillators are densely distributed. As earlier, one can go to the continuum limit of this distribution, and replace the sum over the frequencies by integrals in the following way:

$$\sum_{i,j} \frac{g^2_{xy,ij}}{m_{x,i} m_{y,j}} \to \int_0^\infty \frac{d\mu_x}{2\pi} \int_0^\infty \frac{d\mu_y}{2\pi} \Big\langle\Big\langle \frac{g^2_{xy}(\mu_x, \mu_y)}{m_x m_y} \Big\rangle\Big\rangle, \tag{5.3}$$

$$\begin{aligned} &\sum_{i_1,j_1} \sum_{i_2,j_2} \frac{g_{xy,i_1j_1} g_{xy,i_1j_2} g_{xy,i_2j_1} g_{xy,i_2j_2}}{m_{x,i_1} m_{y,j_1} m_{x,i_2} m_{y,j_2}} \\ &\to \int_0^\infty \frac{d\mu_x}{2\pi} \int_0^\infty \frac{d\mu_y}{2\pi} \int_0^\infty \frac{d\mu'_x}{2\pi} \int_0^\infty \frac{d\mu'_y}{2\pi} \Big\langle\Big\langle \frac{g_{xy}(\mu_x, \mu_y) g_{xy}(\mu_x, \mu'_y) g_{xy}(\mu'_x, \mu_y) g_{xy}(\mu'_x, \mu'_y)}{m_x m_y m'_x m'_y} \Big\rangle\Big\rangle, \end{aligned} \tag{5.4}$$

where we choose

$$\lambda^2 \Big\langle\Big\langle \frac{g^2_{xy}(\mu_x, \mu_y)}{m_x m_y} \Big\rangle\Big\rangle = \Gamma_2 \frac{4\mu_x^2 \Omega^2}{\mu_x^2 + \Omega^2} \frac{4\mu_y^2 \Omega^2}{\mu_y^2 + \Omega^2}, \tag{5.5}$$

$$\begin{aligned} &\lambda^4 \Big\langle\Big\langle \frac{g_{xy}(\mu_x, \mu_y) g_{xy}(\mu_x, \mu'_y) g_{xy}(\mu'_x, \mu_y) g_{xy}(\mu'_x, \mu'_y)}{m_x m_y m'_x m'_y} \Big\rangle\Big\rangle \\ &= \Gamma_4 \Big(\frac{4\mu_x^2 \Omega^2}{\mu_x^2 + \Omega^2} \Big) \Big(\frac{4\mu_y^2 \Omega^2}{\mu_y^2 + \Omega^2} \Big) \Big(\frac{4\mu_x'^2 \Omega^2}{\mu_x'^2 + \Omega^2} \Big) \Big(\frac{4\mu_y'^2 \Omega^2}{\mu_y'^2 + \Omega^2} \Big). \end{aligned} \tag{5.6}$$

The 2-point and 4-point cumulants of O again decay at rates which are of the order of Ω [20]. To get an approximately local dynamics for the particle, we work in the following Markovian regime:

$$\beta\Omega \ll 1,\ \overline{\mu}_0 \ll \Omega,\ \Gamma_2 \ll \beta(\beta\Omega),\ \Gamma_4 \ll (\Gamma_2)^2\ . \tag{5.7}$$

In this Markovian limit, one can determine the 1-PI effective action of the particle from the conditions given in section 2. The quadratic terms in this effective action are as given in (2.8). As in case of the cubic terms discussed in section 2, the quartic terms can be split into two parts:

1. terms which reduce to their SK counterparts under any of the collapses,

2. terms which go to zero under these collapses.

Accordingly, the quartic effective Lagrangian can be written as

$$L^{(4)}_{\text{1-PI}} = L^{(4)}_{\text{1-PI,SK}} + L^{(4)}_{\text{1-PI,OTO}}, \tag{5.8}$$

where $L^{(4)}_{\text{1-PI,SK}}$ and $L^{(4)}_{\text{1-PI,OTO}}$ are the two sets of terms mentioned above. Each of these terms can be further split into terms with different number of time derivatives acting on the q's:

$$\begin{aligned} L^{(4)}_{\text{1-PI,SK}} &= L^{(4,0)}_{\text{1-PI,SK}} + L^{(4,1)}_{\text{1-PI,SK}} + \cdots \ , \\ L^{(4)}_{\text{1-PI,OTO}} &= L^{(4,0)}_{\text{1-PI,OTO}} + L^{(4,1)}_{\text{1-PI,OTO}} + \cdots \ , \end{aligned} \tag{5.9}$$

where

$$\begin{aligned} &L^{(4,0)}_{\text{1-PI,SK}} \\ &= -\frac{\overline{\lambda}_4}{48}\Big[(q_1+q_2)(q_1-q_2-q_3-q_4)^3 + (q_3+q_4)(q_1+q_2+q_3-q_4)^3\Big] \\ &\quad + \frac{\zeta^{(2)}_\mu}{2}(q_1+q_2+q_3+q_4)^2(q_1^2-q_2^2+q_3^2-q_4^2) - \frac{i\overline{\zeta}_3}{8}(q_1^2-q_2^2+q_3^2-q_4^2)^2 \\ &\quad + \frac{i\zeta^{(4)}_N}{24}(q_1+q_2+q_3+q_4)^4 \ , \end{aligned} \tag{5.10}$$

$$\begin{aligned} &L^{(4,1)}_{\text{1-PI,SK}} \\ &= -\frac{\overline{\lambda}_{4\gamma}}{48}\Big[(\dot{q}_1+\dot{q}_2)(q_1-q_2-q_3-q_4)^3 + (\dot{q}_3+\dot{q}_4)(q_1+q_2+q_3-q_4)^3\Big] \\ &\quad + \frac{\zeta^{(2)}_\gamma}{2}(q_1+q_2+q_3+q_4)^2\Big[(q_1+q_2)(\dot{q}_1-\dot{q}_2-\dot{q}_3-\dot{q}_4) + (q_3+q_4)(\dot{q}_1+\dot{q}_2+\dot{q}_3-\dot{q}_4)\Big] \\ &\quad + \frac{i\overline{\zeta}_{3\gamma}}{4}(q_1+q_2+q_3+q_4)\Big[(q_1+q_2)(q_1-q_2-q_3-q_4)(\dot{q}_1-\dot{q}_2-\dot{q}_3-\dot{q}_4) \\ &\qquad + (q_3+q_4)(q_1+q_2+q_3-q_4)(\dot{q}_1+\dot{q}_2+\dot{q}_3-\dot{q}_4)\Big] \ , \end{aligned} \tag{5.11}$$

$$\begin{aligned} &L^{(4,0)}_{\text{1-PI,OTO}} \\ &= (q_1+q_2)(q_2+q_3)(q_3+q_4)\Big[A_1(q_1+q_2) + A_2(q_3+q_4) + A_3(q_1-q_4) + A_4(q_2+q_3)\Big] \ , \end{aligned} \tag{5.12}$$

$$\begin{aligned} L^{(4,1)}_{\text{1-PI,OTO}} =& (q_1+q_2)(q_2+q_3)(\dot{q}_3+\dot{q}_4)\Big[B_1(q_1-q_4) + B_2(q_1+q_2) + B_3(q_2+q_3)\Big] \\ &+ (q_1+q_2)(\dot{q}_2+\dot{q}_3)(q_3+q_4)\Big[B_4(q_1-q_4) + B_5(q_1+q_2) + B_6(q_3+q_4)\Big] \\ &+ (\dot{q}_1+\dot{q}_2)(q_2+q_3)(q_3+q_4)\Big[B_7(q_1-q_4) + B_8(q_3+q_4) + B_9(q_2+q_3)\Big] \ . \end{aligned} \tag{5.13}$$

These quartic terms in the effective Lagrangian reduce to their Schwinger-Keldysh (SK) counterparts under any of the collapses mentioned in section 2. We provide the form of this SK effective Lagrangian (up to quartic terms) below:

$$\begin{aligned} L_{\rm SK} = \dot{q}_d\dot{q}_a - \frac{i}{2}\ Z_I\dot{q}_d^2 - \overline{\mu}^2 q_d q_a + \frac{i}{2}\ \langle f^2\rangle q_d^2 - \gamma q_d\dot{q}_a + \frac{i\zeta_N^{(4)}}{4!}\ q_d^4 + \zeta_\mu^{(2)}\ q_d^3 q_a \\ - \frac{i\overline{\zeta}_3}{2!}q_d^2 q_a^2 - \frac{\overline{\lambda}_4}{3!}q_d q_a^3 + \zeta_\gamma^{(2)} q_d^3\dot{q}_a - \frac{\overline{\lambda}_{4\gamma}}{3!}\dot{q}_d q_a^3 + i\overline{\zeta}_{3\gamma}q_d^2 q_a\dot{q}_a\ . \end{aligned} \tag{5.14}$$

The reality conditions mentioned in section 2 imply that all the SK effective couplings given in (5.14) are real. Moreover, they impose the following constraints on the quartic OTO couplings:

$$\begin{aligned} &A_1 = -A_2^*,\ A_3 = A_3^*,\ A_4 = -A_4^*, \\ &B_1 = B_7^*,\ B_2 = -B_8^*,\ B_3 = -B_9^*,\ B_4 = B_4^*,\ B_5 = -B_6^*\ . \end{aligned} \tag{5.15}$$

Just like the cubic effective couplings discussed in the previous section, the quartic couplings also satisfy some constraints due to microscopic reversibility and thermality of the bath. We will now discuss these constraints.

5.2 Generalised Onsager relations and fluctuation-dissipation relation

To analyse the constraints on the effective couplings, we find it convenient to re-express the quartic OTO couplings in terms of some new real parameters as shown below:

$$A_1 = -A_2^* = \frac{1}{12}\Big[(-\overline{\lambda}_4 + \widetilde{\kappa}_4) - 6i(\varrho_4 - \widetilde{\varrho}_4)\Big]\ ,\ A_3 = A_3^* = \kappa_4\ ,\ A_4 = -A_4^* = \frac{i}{2}(\varrho_4 + \widetilde{\varrho}_4)\ . \tag{5.16}$$

$$\begin{aligned} B_1 = B_7^* &= \frac{1}{4}\Big[(2\kappa_{4\gamma}^{II} + 4\kappa_{4\gamma}^{III} + \widetilde{\kappa}_{4\gamma}^{II}) + i(\varrho_{4\gamma}^{II} + \widetilde{\varrho}_{4\gamma}^{I} - \widetilde{\varrho}_{4\gamma}^{II})\Big], \\ B_2 = -B_8^* &= \frac{1}{16}\Big[(-\overline{\lambda}_{4\gamma} + 24\zeta_\gamma^{(2)} + 12\kappa_{4\gamma}^{I} - 4\widetilde{\kappa}_{4\gamma}^{I} - 6\widetilde{\kappa}_{4\gamma}^{II}) \\ &\qquad + i(4\overline{\zeta}_{3\gamma} - 2\varrho_{4\gamma}^{I} + 2\varrho_{4\gamma}^{II} - 6\widetilde{\varrho}_{4\gamma}^{I} + 2\widetilde{\varrho}_{4\gamma}^{II})\Big]\ , \\ B_3 = -B_9^* &= \frac{1}{4}\Big[(-2\kappa_{4\gamma}^{II} + \widetilde{\kappa}_{4\gamma}^{II}) + i\varrho_{4\gamma}^{I}\Big]\ ,\ B_4 = B_4^* = \frac{1}{2}\Big(2\kappa_{4\gamma}^{II} + \widetilde{\kappa}_{4\gamma}^{II}\Big)\ , \\ B_5 = -B_6^* &= \frac{1}{16}\Big[(\overline{\lambda}_{4\gamma} + 8\zeta_\gamma^{(2)} + 4\kappa_{4\gamma}^{I} + 4\widetilde{\kappa}_{4\gamma}^{I} - 2\widetilde{\kappa}_{4\gamma}^{II}) \\ &\qquad + i(-4\overline{\zeta}_{3\gamma} + 2\varrho_{4\gamma}^{I} + 6\varrho_{4\gamma}^{II} - 2\widetilde{\varrho}_{4\gamma}^{I} + 6\widetilde{\varrho}_{4\gamma}^{II})\Big]. \end{aligned} \tag{5.17}$$

These new parameters are chosen so that they have definite parities under time-reversal. The presence or absence of tilde over any coupling indicates whether it has odd or even parity respectively. The symbols κ and ϱ are used to represent the couplings which get multiplied to real and imaginary terms in the effective action respectively.

All the couplings with odd parity under time reversal must go to zero due to the microscopic reversibility in the bath's dynamics. This imposes the following generalised Onsager relations on the effective couplings:

$$\widetilde{\kappa}_4 = \widetilde{\varrho}_4 = \widetilde{\kappa}^I_{4\gamma} = \widetilde{\kappa}^{II}_{4\gamma} = \widetilde{\varrho}^I_{4\gamma} = \widetilde{\varrho}^{II}_{4\gamma} = 0, \tag{5.18}$$

$$\overline{\lambda}_{4\gamma} - 24\zeta^{(2)}_\gamma - 8(\kappa^I_{4\gamma} - \kappa^{II}_{4\gamma}) = 0. \tag{5.19}$$

We provide the values (in the high temperature limit) of the non-zero effective couplings of the particle for the modified qXY model in (5.20), (5.21), and (5.22).

Quadratic couplings:

$$Z_I = \frac{\Gamma_2}{4\beta^2\Omega} \ , \ \Delta\overline{\mu}^2 = -\frac{\Gamma_2\Omega^2}{\beta} \ , \ \langle f^2\rangle = \frac{\Gamma_2\Omega}{\beta^2} \ , \ \gamma = \frac{\Gamma_2\Omega}{2\beta} \ . \tag{5.20}$$

Quartic couplings:

A) **Schwinger-Keldysh couplings:**

$$\begin{aligned} &\zeta^{(4)}_N = -\frac{15\Gamma_4\Omega}{\beta^4}, \ \overline{\lambda}_4 = -\frac{6\Gamma_4\Omega^4}{\beta}, \ \overline{\zeta}_3 = -\frac{6\Gamma_4\Omega^3}{\beta^2}, \ \zeta^{(2)}_\mu = -\frac{5\Gamma_4\Omega^2}{2\beta^3}, \\ &\overline{\lambda}_{4\gamma} = -\frac{6\Gamma_4\Omega^3}{\beta}, \ \overline{\zeta}_{3\gamma} = -\frac{15\Gamma_4\Omega^2}{4\beta^2}, \ \zeta^{(2)}_\gamma = \frac{5\Gamma_4\Omega}{4\beta^3}. \end{aligned} \tag{5.21}$$

B) **OTO couplings:**

$$\begin{aligned} &\kappa_4 = -\frac{3\Gamma_4\Omega^4}{2\beta}, \ \varrho_4 = \frac{7\Gamma_4\Omega^3}{2\beta^2}, \ \kappa^I_{4\gamma} = -\frac{15\Gamma_4\Omega}{4\beta^3}, \ \kappa^{II}_{4\gamma} = \frac{5\Gamma_4\Omega^3}{8\beta}, \\ &\kappa^{III}_{4\gamma} = -\frac{19\Gamma_4\Omega^3}{16\beta}, \ \varrho^I_{4\gamma} = -\frac{15\Gamma_4\Omega^2}{4\beta^2}, \ \varrho^{II}_{4\gamma} = \frac{5\Gamma_4\Omega^2}{2\beta^2}. \end{aligned} \tag{5.22}$$

From the values of these couplings, one can see that $\overline{\lambda}_{4\gamma}$ and $\kappa^{II}_{4\gamma}$ are suppressed by a factor of $(\beta\Omega)^2$ compared to $\kappa^I_{4\gamma}$ and $\zeta^{(2)}_\gamma$. Therefore, in the high temperature limit, one can ignore $\overline{\lambda}_{4\gamma}$ and $\kappa^{II}_{4\gamma}$ in (5.19) and obtain

$$\kappa^I_{4\gamma} = -3\zeta^{(2)}_\gamma. \tag{5.23}$$

In addition, there is a relation (in the high temperature limit) involving the OTO coupling $\kappa^I_{4\gamma}$ and the SK coupling $\zeta^{(4)}_N$ which follows from the KMS relations between the bath's 4-point correlators. This is an OTO generalisation of the fluctuation-dissipation relation which holds even when the the bath's dynamics lacks microscopic reversibiltiy. We provide this relation below:

$$\kappa^I_{4\gamma} = \frac{\beta}{4}\zeta^{(4)}_N. \tag{5.24}$$

Combining (5.23) and (5.24), we get the following generalised fluctuation-dissipation relation between two SK effective couplings:

$$\zeta^{(2)}_\gamma = -\frac{\beta}{12}\zeta^{(4)}_N. \tag{5.25}$$

In the following subsection, we will see that, analogous to the cubic effective theory, the quartic SK effective theory is also dual to a non-linear Langevin dynamics. In this non-linear Langevin dynamics, the generalised FDR given in (5.25) connects a thermal jitter in the damping to the non-Gaussianity in the noise.

5.3 Duality with a non-linear Langevin dynamics

Following the arguments given in section 4.3, one can show that the quartic SK effective dynamics (see the Lagrangian given in (5.14)) is also dual to a non-linear Langevin equation [20]. The form of this Langevin equation is given below:

$$\begin{aligned}\mathcal{E}[q] \equiv \ddot{q} + \Big(\gamma + \zeta^{(2)}_\gamma \mathcal{N}^2\Big)\dot{q} + \Big(\overline{\mu}^2 + \zeta^{(2)}_\mu \mathcal{N}^2\Big)q + \mathcal{N}\Big(\overline{\zeta}_3 - \overline{\zeta}_{3\gamma}\frac{d}{dt}\Big)\frac{q^2}{2!} \\ + \Big(\overline{\lambda}_4 - \overline{\lambda}_{4\gamma}\frac{d}{dt}\Big)\frac{q^3}{3!} - \langle f^2\rangle\mathcal{N} = 0,\end{aligned} \tag{5.26}$$

where $\mathcal{N}$ is a noise drawn from the following non-Gaussian probability distribution

$$P[\mathcal{N}] \propto \exp\Big[-\int dt \Big(\frac{\langle f^2\rangle}{2}\mathcal{N}^2 + \frac{Z_I}{2}\dot{\mathcal{N}}^2 + \frac{\zeta^{(4)}_N}{4!}\mathcal{N}^4\Big)\Big]\ . \tag{5.27}$$

The non-linearities in this dynamics as well as the non-Gaussianity in the noise are fixed by the following parameters: $\zeta^{(4)}_N, \zeta^{(2)}_\gamma, \zeta^{(2)}_\mu, \overline{\zeta}_3, \overline{\zeta}_{3\gamma}, \overline{\lambda}_4, \overline{\lambda}_{4\gamma}$. All these parameters are $O(\lambda^4)$ in the particle-bath coupling λ. If we ignore these $O(\lambda^4)$ contributions, then the dynamics satisfies the linear Langevin equation given in (4.18). The additional non-linear parameters can be interpreted as follows:

1. $\zeta^{(2)}_\mu$ is a jitter in the renormalised frequency due to the thermal noise.

2. $\zeta^{(2)}_\gamma$ is a jitter in the damping coefficient due to the thermal noise.

3. $\zeta_N^{(4)}$ is the strength of non-Gaussianity in the noise distribution.

4. $\overline{\lambda}_4$ and $\overline{\lambda}_{4\gamma}$ are the strengths of anharmonic terms in the equation of motion.

5. $\overline{\zeta}_3$ and $\overline{\zeta}_{3\gamma}$ are the strengths of anharmonic terms which couple to the noise.

Therefore, one can see that,as in the case of the cubic effective theory, the generalised FDR given in (5.25) relates the thermal jitter in the damping coefficient to the non-Gaussianity in the noise distribution.

The similarity between the generalised FDRs obtained for the cubic and the quartic effective theories suggests that such relations probably hold for even higher degree terms in the effective action when the bath's microscopic dynamics is reversible. It would be interesting to identify the general form of these relations.

6 Conclusion and discussion

In this synopsis, we have discussed the out of time ordered effective dynamics of a Brownian particle weakly interacting with an environment. Most of our discussion has revolved around the scenario where the environment is a thermal bath. To illustrate the features of the effective dynamics, we have introduced simple toy models (the qXY model and its modified version) where the bath comprises of two sets of harmonic oscillators coupled to the particle through cubic interactions.

For these models, we have identified appropriate Markovian regimes where the particle's effective dynamics is approximately local in time. Working in these regimes, we have constructed cubic and quartic effective actions of the particle in a generalised Schwinger-Keldysh formalism. These effective actions allow one to compute OTO correlators of the particle.

The couplings in these effective actions receive contributions from the OTOCs of the bath. The relations between the particle's effective couplings and the bath's OTOCs lead to the possibility of extracting information about these OTOCs by performing measurements on the particle. Moreover, these relations also allow one to study the constraints imposed on the particle's effective dynamics due to microscopic reversibility and thermality of the bath. We have shown that these constraints manifest in the form of OTO generalisations of the Onsager-Casimir reciprocal relations and the fluctuation-dissipation relation. By combining these relations we have obtained some generalised fluctuation-dissipation relations between the Schwinger-Keldysh effective couplings of the particle. We have demonstrated that these Schwinger-Keldysh effective couplings appear as parameters in dual stochastic theories governed by non-linear Langevin equations. In these non-linear Langevin dynamics, the generalised fluctuation-dissipation relations connect the thermal jitter in the damping of the particle to the non-Gaussianity in the noise distribution.

The particle's OTO effective dynamics leads one to wonder whether the particle's OTOCs would eventually thermalise or not. To address this question, one needs to look at the behaviour of these OTOCs when the separation between any two insertions is very large compared to the time-scales set by the parameters in the effective dynamics. This requires going beyond the perturbative analysis employed here by resumming the perturbation series expansions of the correlators.

Studying the question of thermalisation of the particle's OTOCs can provide useful insight into similar questions for more general open quantum systems. In particular, it can shed light on the circumstances in which a system with a large number of degrees of freedom can be brought to thermal equilibrium with an environment of even larger size. Experimentally, this may lead to a specification of the conditions in which a thermal bath can be prepared by making a quantum system interact with an environment at thermal equilibrium. Hence, the OTO effective theory developed here can be seen as a step towards understanding when quantum mechanical systems can serve as ideal thermal baths.

Although the construction of the effective theories have been discussed here in the context of some simple toy models, the analysis mostly relies on the validity of the Markov approximation for the particle's dynamics. Hence, it may be employed to study the effective theory of the particle when it interacts with more complicated baths. For instance, the bath may even be a strongly coupled system [7] in which case a microscopic analysis of the particle's dynamics is very difficult. In such a scenario, the OTO effective theory of the particle would allow one to determine the particle's

[7]Notice that we have assumed a weak coupling only between the particle and the bath. The couplings between the internal degrees of freedom of the bath may be strong.

OTOCs in terms of the effective couplings.

Moreover, this effective theory can be valid even in case of a chaotic bath [46–49] as long as the bath's OTO cumulants saturate to sufficiently small values much faster than the particle's evolution [17]. This opens up the possibility of probing the Lyapunov exponents [5] in such chaotic baths by measuring the OTO effective couplings of the particle [17].

The applicability of this effective theory framework to the scenario where the bath is chaotic and strongly coupled implies that it may be possible to construct a holographic dual description [50–54] of the particle's non-linear dynamics. The OTO extension of this non-linear dynamics may be useful in determining a holographic prescription for computing the particle's OTOCs [53].

It will be useful to formulate a Wilsonian counterpart of the out of time ordered 1-PI effective action discussed here. Such a Wilsonian effective theory can be extended to open quantum field theories [37, 55–62] which show up in the study of quantum cosmology and heavy ion physics. It will be interesting to determine the RG flow [37] of the OTO couplings in this Wilsonian framework to estimate their relative importance at different energy scales.

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