Critical Scaling through Gini index

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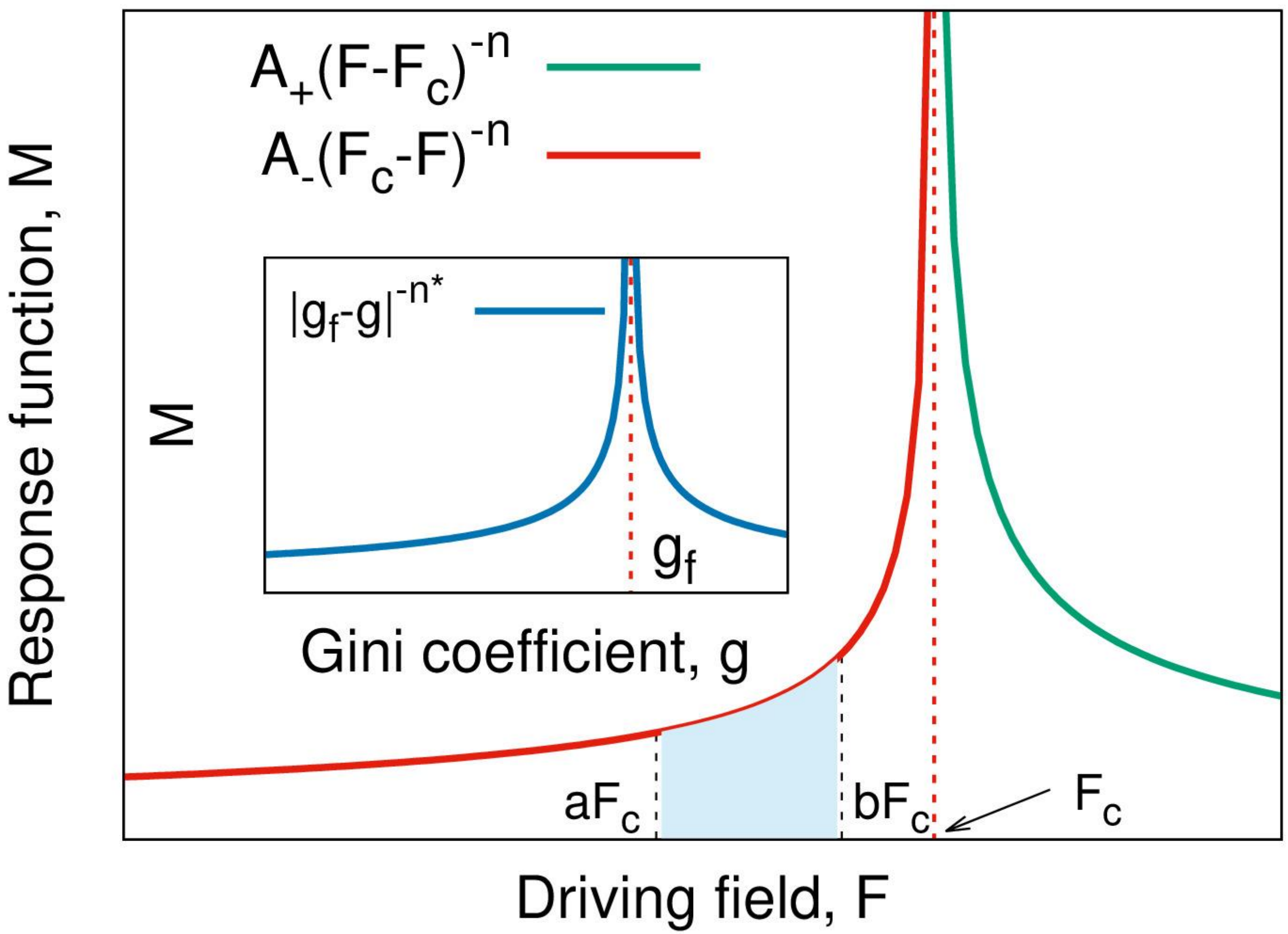
Critical Scaling

$$M \propto |F - F_c|^{-n}$$

n Universal critical exponent

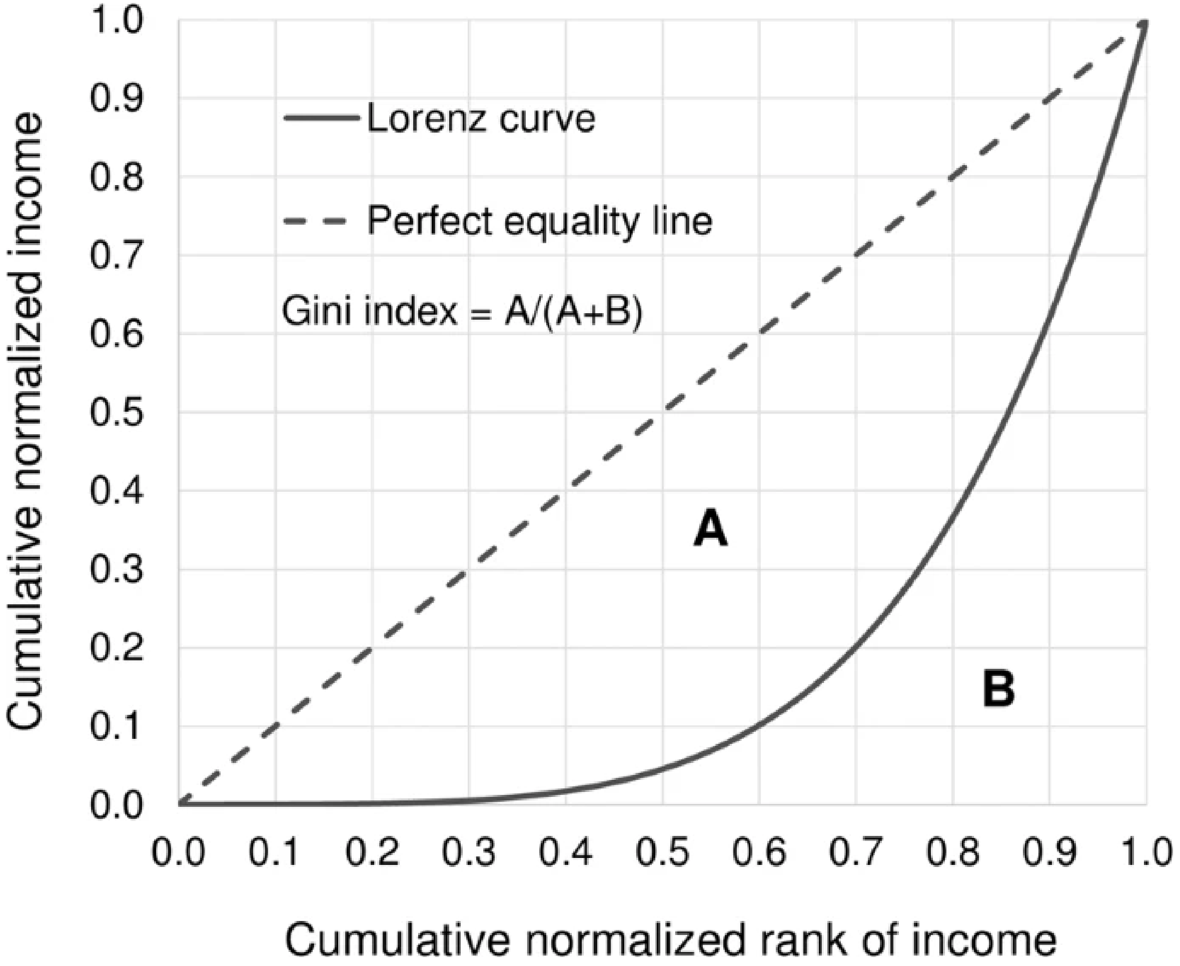
$\frac{A_+}{A_-}$ Universal amplitude ratio

F_c Non-universal critical point



Inequality & Gini index

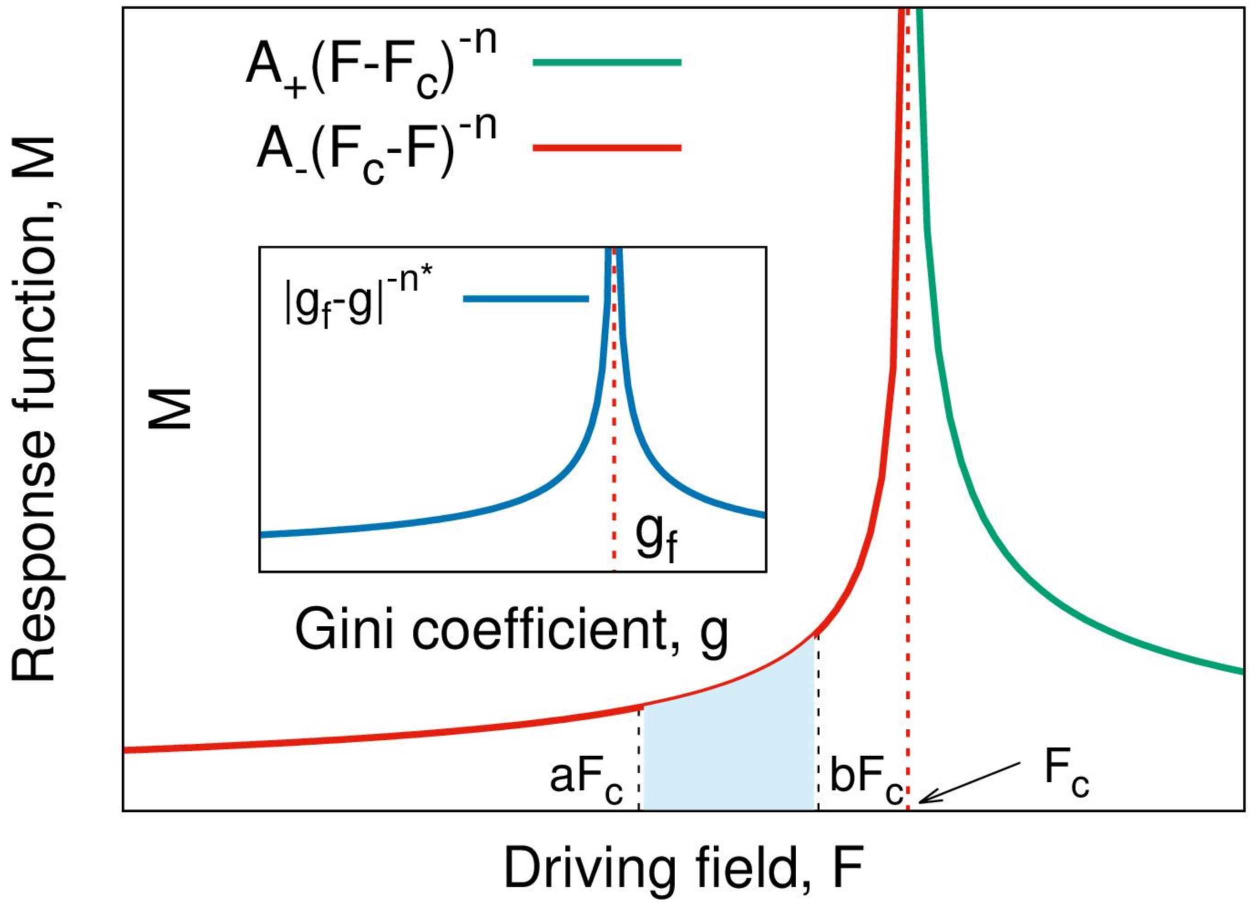
Responses of a system near a critical point are very much unequal



C. Gini, *Measurement of inequality of incomes*, Economics Journal 31, 124126 (1921).

Sitthiyot, Holasut, Palgrave Communication, 6, 112 (2020)

Gini index of response functions

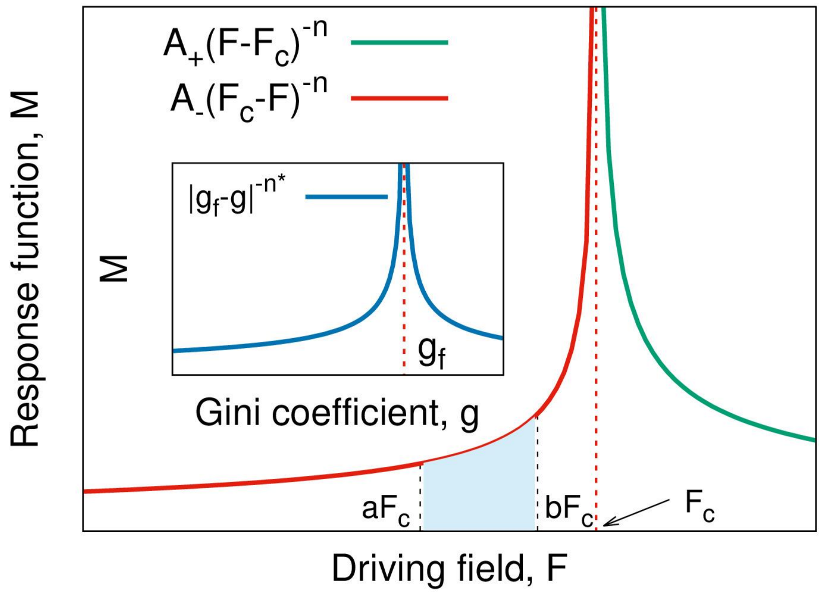


$$L(p,n,a,b) = \frac{\int_{aF_c}^{aF_c+p(b-a)F_c} M(F,F_c)\,dF}{\int_{aF_c}^{bF_c} M(F,F_c)\,dF},$$

$$g = 1 - 2\int_0^1 L(p)dp$$

S. Das, SB, arxiv:2211.01281 (2022).

Gini index of response functions



$$L(p,q,a,b) = \frac{(1-a)^{1-n} - (1-a-p(b-a))^{1-n}}{(1-a)^{1-n} - (1-b)^{1-n}},$$

$$g(n,a,b) = 1-2\left[\frac{(1-a)^{1-n}}{(1-a)^{1-n}-(1-b)^{1-n}} + \frac{(b-a)^{1-n}}{(2-n)\left((1-a)^{1-n}-(1-b)^{1-n}\right)}\left\{\left(\frac{1-b}{b-a}\right)^{2-n} - \left(\frac{1-a}{b-a}\right)^{2-n}\right\}\right]$$

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Gini index of response functions

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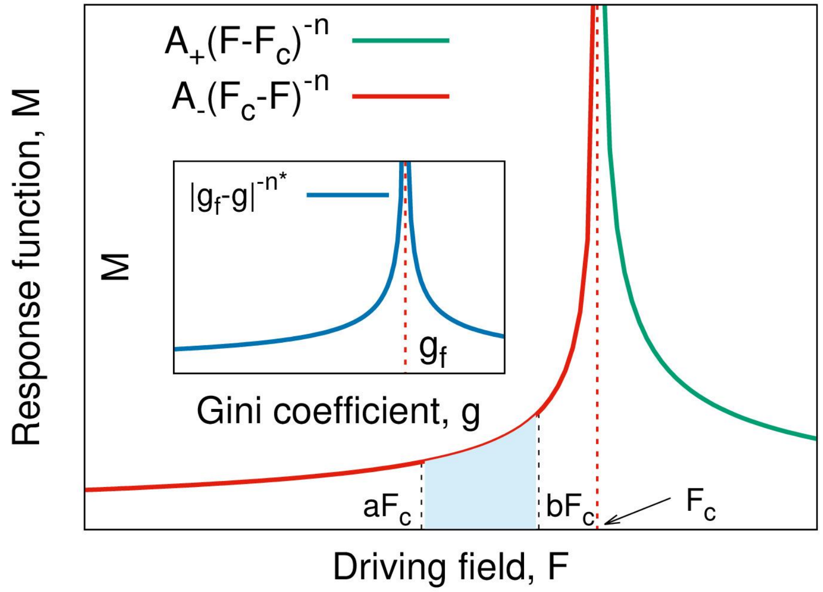


Case : 1

$$\lim_{b\to 1} g(n,a) = g_f = \frac{n}{2-n} \; for \;\; 0 < n < 1$$

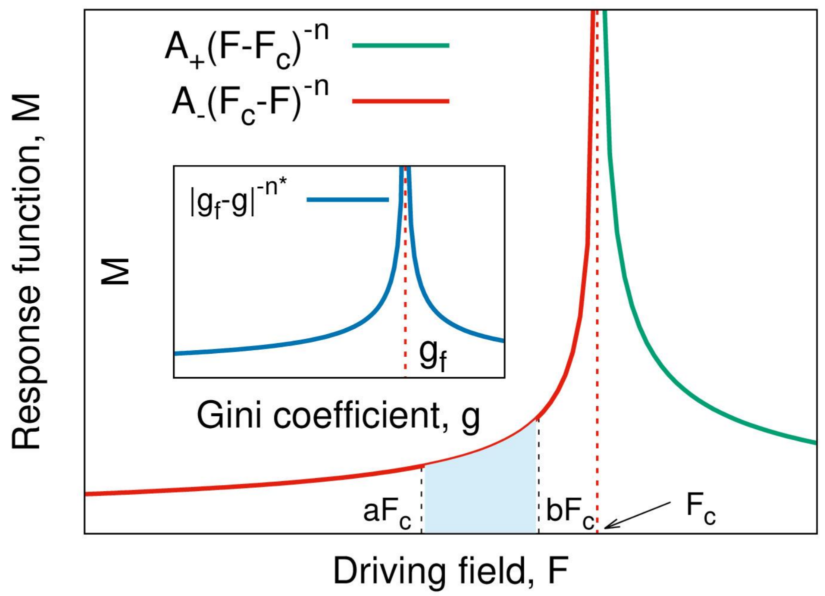
So, for small values of $(1-b)$

$$g(n,a,b) \approx \frac{n}{2-n} - 2(1-b)^{1-n}(1-a)^{n-1}$$

So, $g - g_f \propto |F - F_c|^{1-n}$, giving $M \propto \left|g - g_f\right|^{-\frac{n}{1-n}}$

Gini index of response functions

$$g(n,a,b) = 1-2\left[\frac{(1-a)^{1-n}}{(1-a)^{1-n}-(1-b)^{1-n}} + \frac{(b-a)^{1-n}}{(2-n)\left((1-a)^{1-n}-(1-b)^{1-n}\right)}\left\{\left(\frac{1-b}{b-a}\right)^{2-n} - \left(\frac{1-a}{b-a}\right)^{2-n}\right\}\right]$$



Case : 2

$$\lim_{b\to 1} g(n,a) = g_f \to 1 \;\; for \;\; 1 < n < 2$$

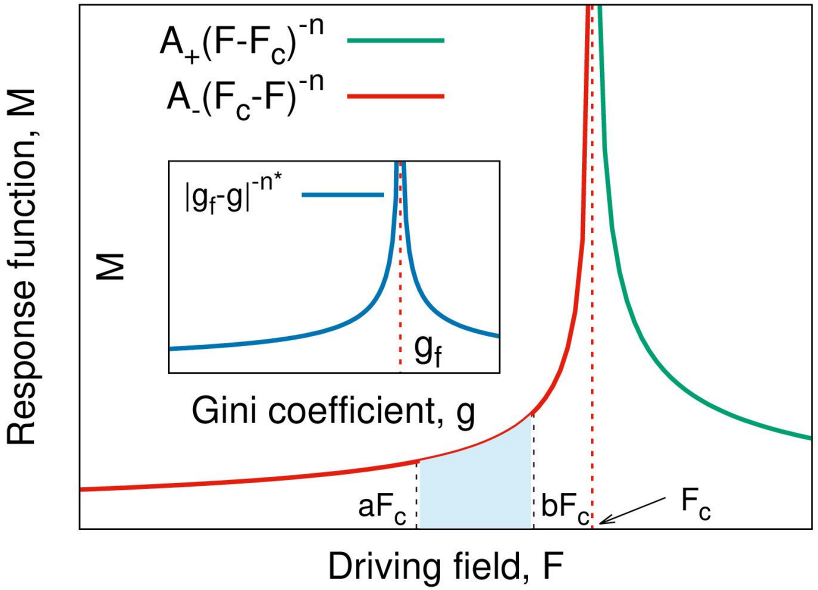
So, for small values of $(1-b)$

$$g(n,a,b) \approx 1-(1-b)^{n-1}(1-a)^{1-n}$$

So, $g - g_f \propto |F - F_c|^{n-1}$, giving $M \propto \left|g - g_f\right|^{-\frac{n}{n-1}}$

Gini index of response functions

$$g(n,a,b) = 1-2\left[\frac{(1-a)^{1-n}}{(1-a)^{1-n}-(1-b)^{1-n}} + \frac{(b-a)^{1-n}}{(2-n)\left((1-a)^{1-n}-(1-b)^{1-n}\right)}\left\{\left(\frac{1-b}{b-a}\right)^{2-n} - \left(\frac{1-a}{b-a}\right)^{2-n}\right\}\right]$$



Case : 3

$$\lim_{b\to 1} g(n,a) = g_f \to 1 \ for \ n > 2$$

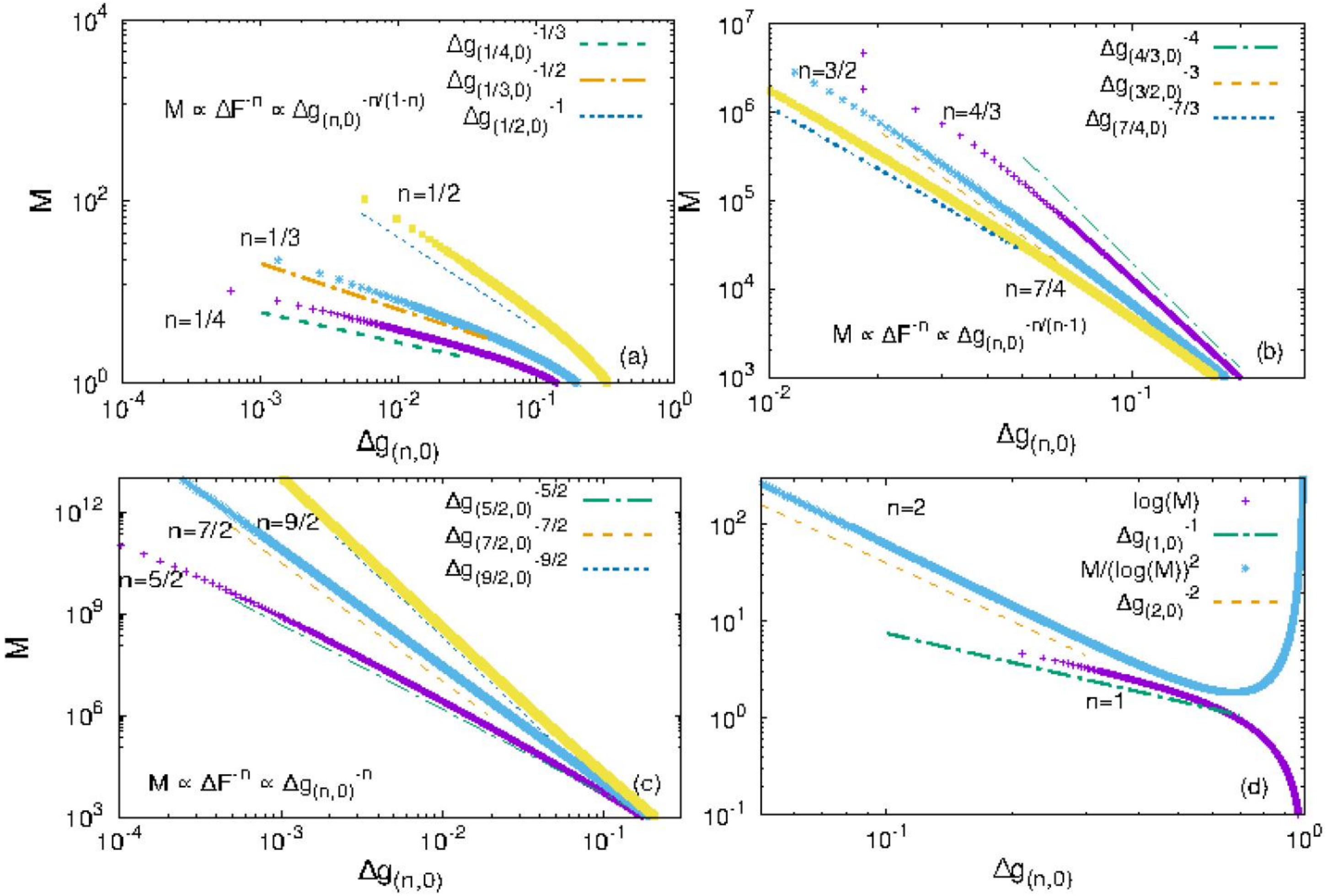
So, for small values of $(1-b)$

$$g(n,a,b) \approx 1-(1-b)/(1-a)$$

So, $g - g_f \propto |F - F_c|$, giving $M \propto |g-g_f|^{-n}$

Gini index of response functions

$$g(n,a,b) = 1 - 2\left[\frac{(1-a)^{1-n}}{(1-a)^{1-n}-(1-b)^{1-n}} + \frac{(b-a)^{1-n}}{(2-n)\left((1-a)^{1-n}-(1-b)^{1-n}\right)}\left\{\left(\frac{1-b}{b-a}\right)^{2-n} - \left(\frac{1-a}{b-a}\right)^{2-n}\right\}\right]$$



$$M \propto |F - F_c|^{-n}$$

$$0 < n < 1, \quad M \propto |g - g_f|^{-\frac{n}{1-n}}$$

$$g_f = n/(2-n)$$

$$1 < n < 2, \quad M \propto |g - g_f|^{-\frac{n}{n-1}}$$

$$g_f = 1$$

$$n > 2, \quad M \propto |g - g_f|^{-n}$$

$$g_f = 1$$

S. Das, SB, arxiv:2211.01281 (2022).

Example 1: Ising Model in the Mean Field limit

- Magnetization: $m = \tanh(\frac{mT_c}{T})$,
- In the critical regime where T $\approx T_c$ and |m| $\ll$ 1

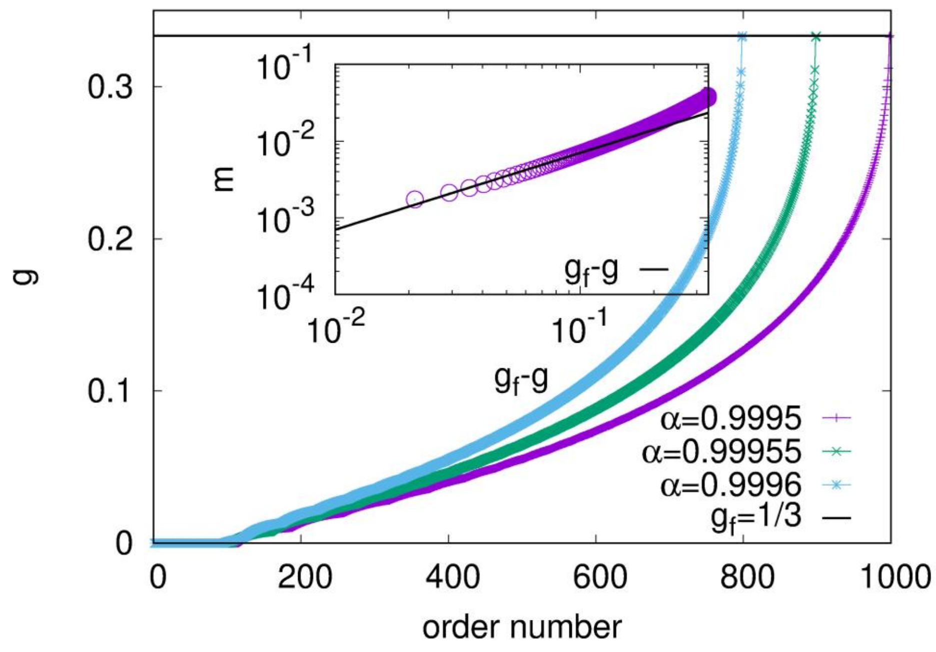
$$m^2 = 3\left(\frac{T}{T_c}\right)^3\left(\frac{T_c - T}{T}\right) \quad \rightarrow \quad m \sim (T_c - T)^{1/2}$$

- Consider

$$M = \frac{dm}{dT} \sim |T_c - T|^{-\frac{1}{2}}$$

meaning $n = \frac{1}{2}$, $and\ g_f = \frac{n}{2-n} = \frac{1}{3}$

$$m \sim |g - g_f|$$



True for order parameter scaling in **any** system.

Example 1: Ising Model in the Mean Field limit

- Magnetization: $m = \tanh(\frac{mT_c}{T})$,
- In the critical regime where T ≈ T_c and |m| ≪ 1

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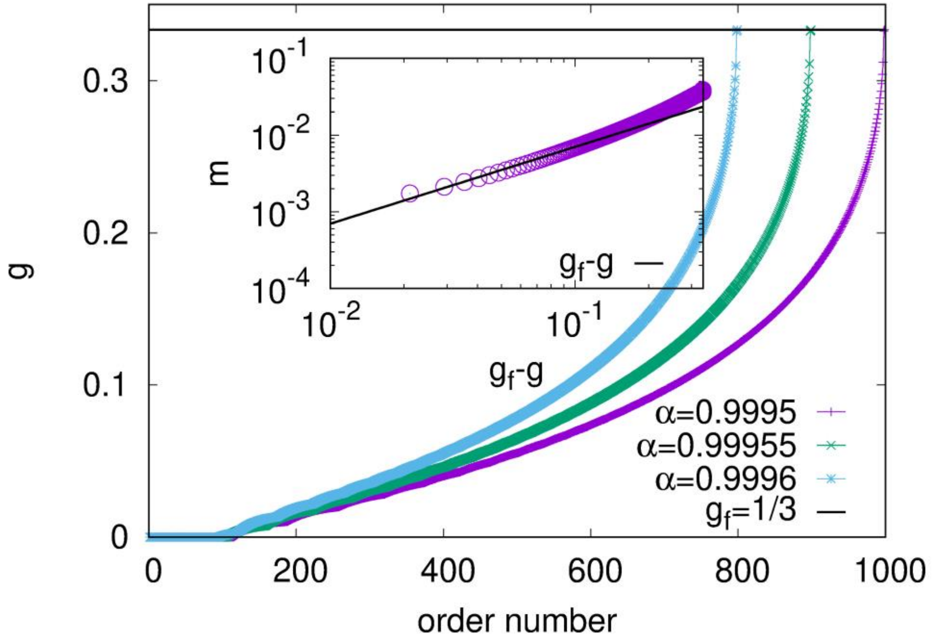
True for order parameter scaling in **any** system.



$$m \sim |T - T_c|^{\beta}$$

$$M \sim \frac{dm}{dT} = |T - T_c|^{\beta - 1}$$

So, $n = 1 - \beta$

But, $|T - T_c| \sim |g - g_f|^{\frac{1}{1-n}}$

Hence, $m \sim |T - T_c|^{\beta} \sim |g - g_f|^{\frac{\beta}{1-n}} \sim |g - g_f|$

Example 2: Fiber Bundle Model (FBM) of Fracture

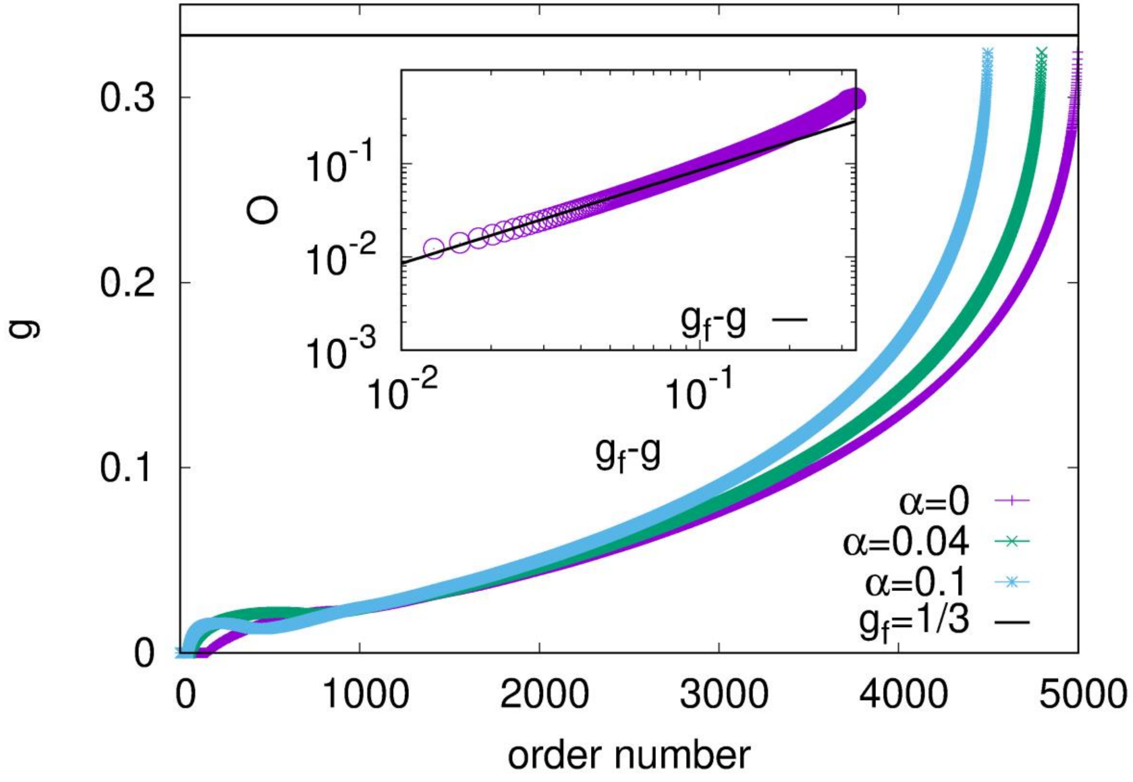
- Order parameter: $O(\sigma) \sim [\sigma_c - \sigma]^{\frac{1}{2}}$, σ = applied stress
- $M = |\frac{dO}{d\sigma}| = \frac{1}{2}[\sigma_c - \sigma]^{-\frac{1}{2}}$

M here means the **avalanches**

(acoustic emissions)

Again,

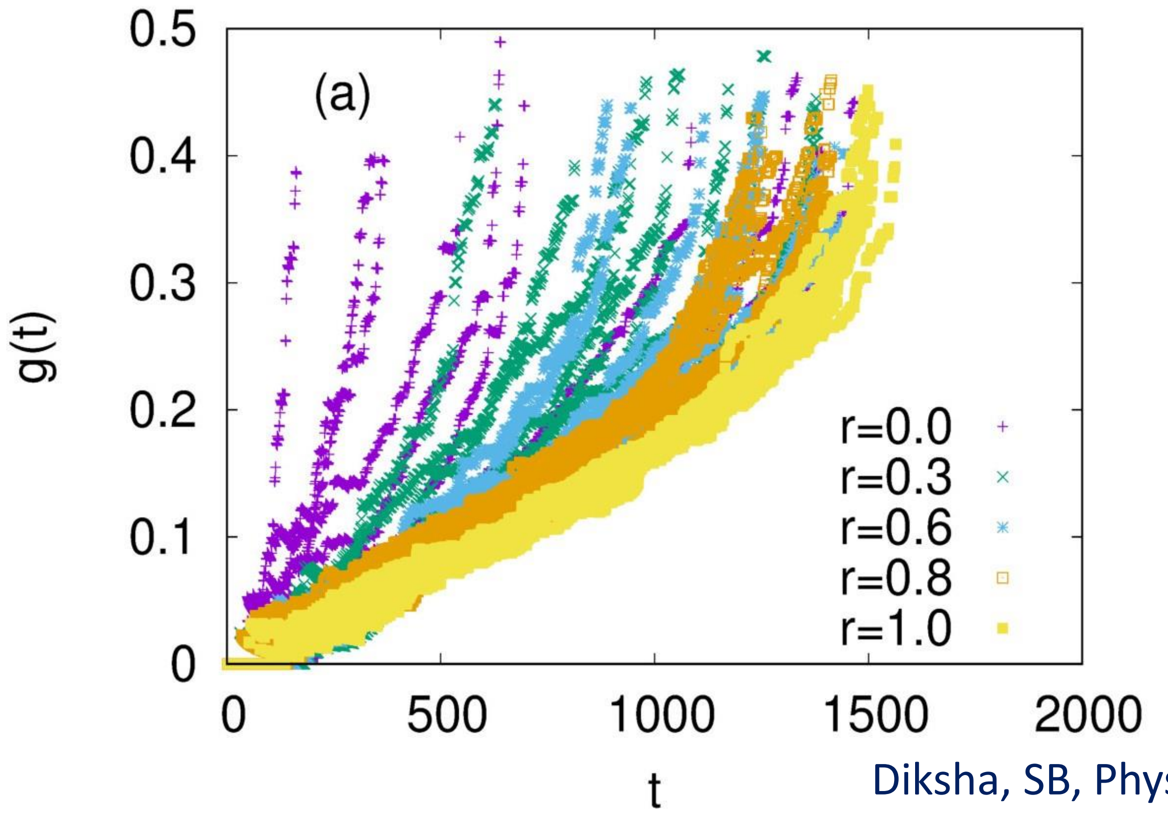
order parameter, $O \sim |g - g_f|$



Example 2: Fiber Bundle Model (FBM) of Fracture

- order parameter, O$\sim |g - g_f|$

- More importantly, Gini index can be used for predicting failure point:



Diksha, SB, Phys. Rev. E 106, 025003 (2021).

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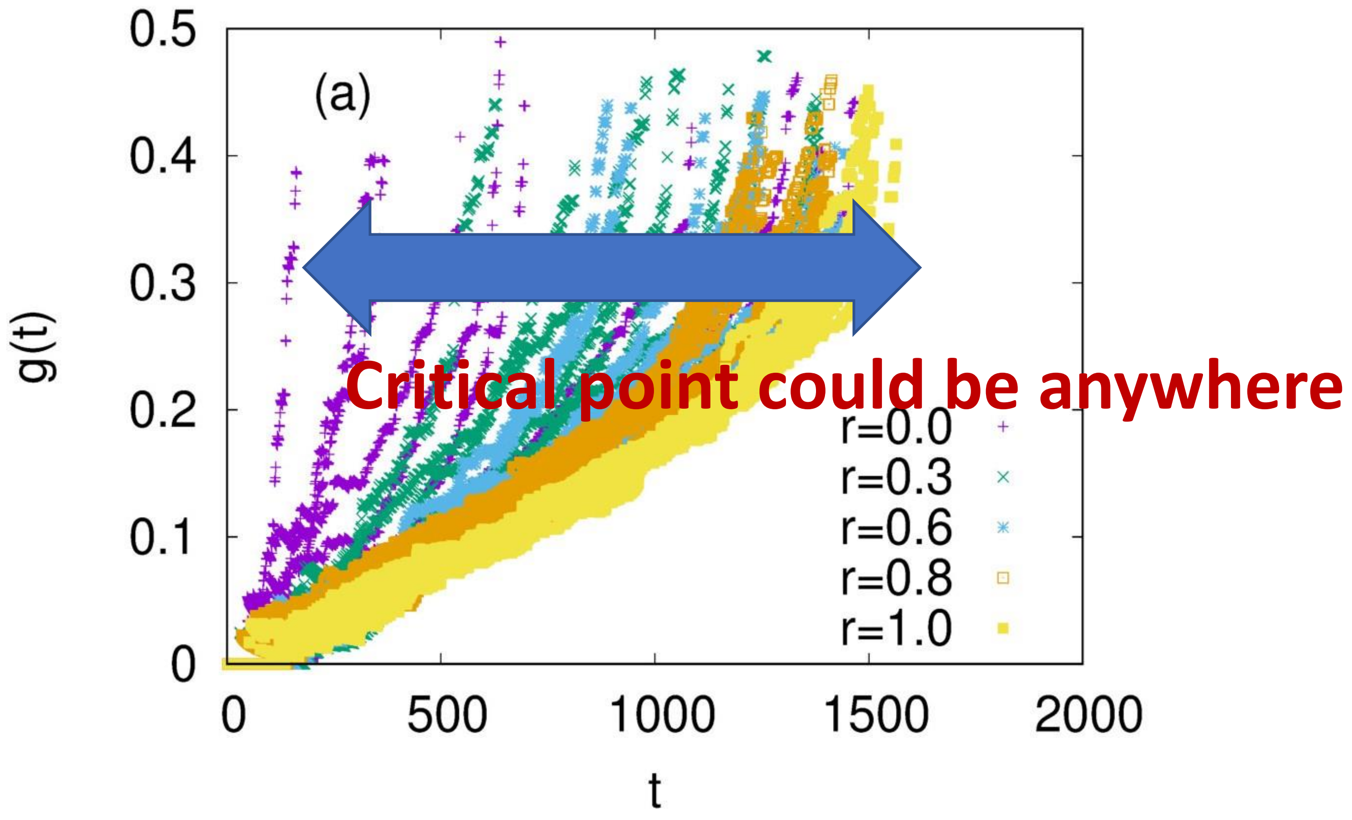
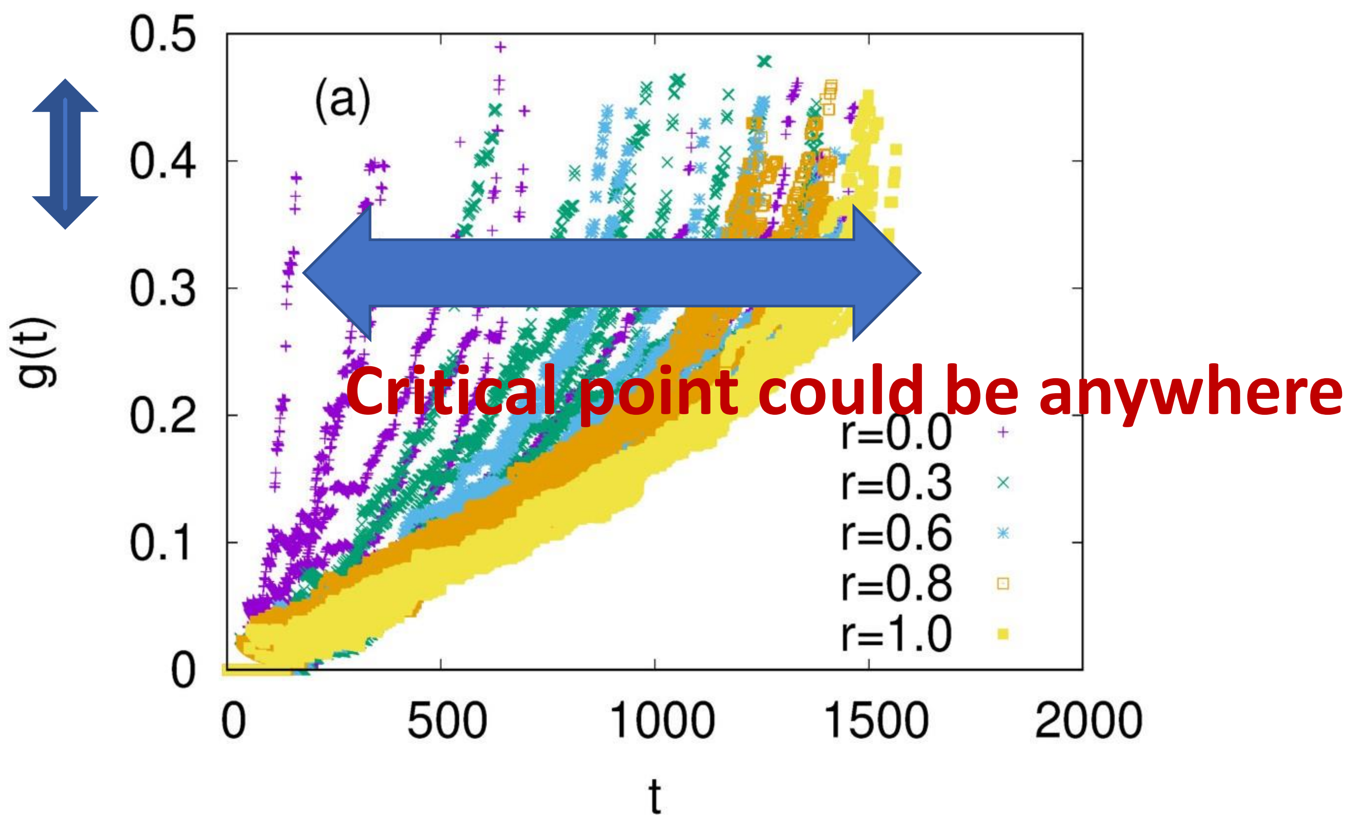


Example 2: Fiber Bundle Model (FBM) of Fracture

- order parameter, O~ $|g - g_f|$

- More importantly, Gini index can be used for predicting failure point:

g_f is well defined!



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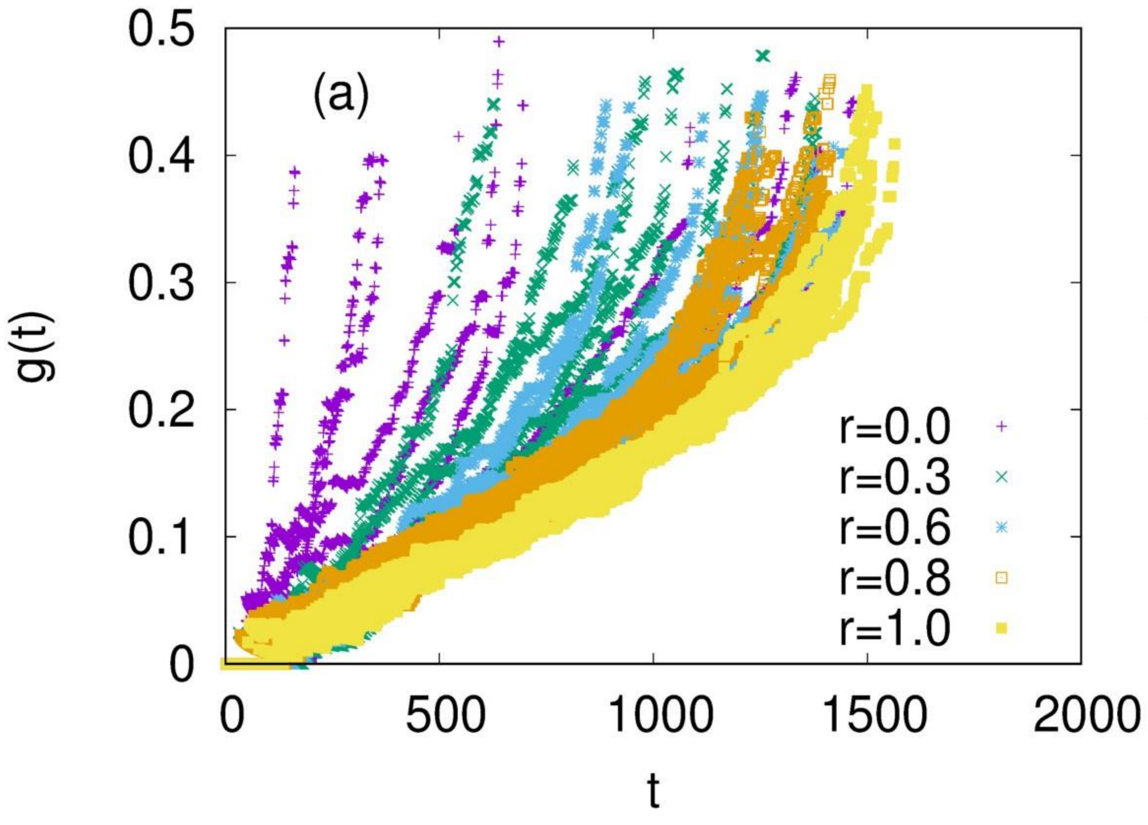

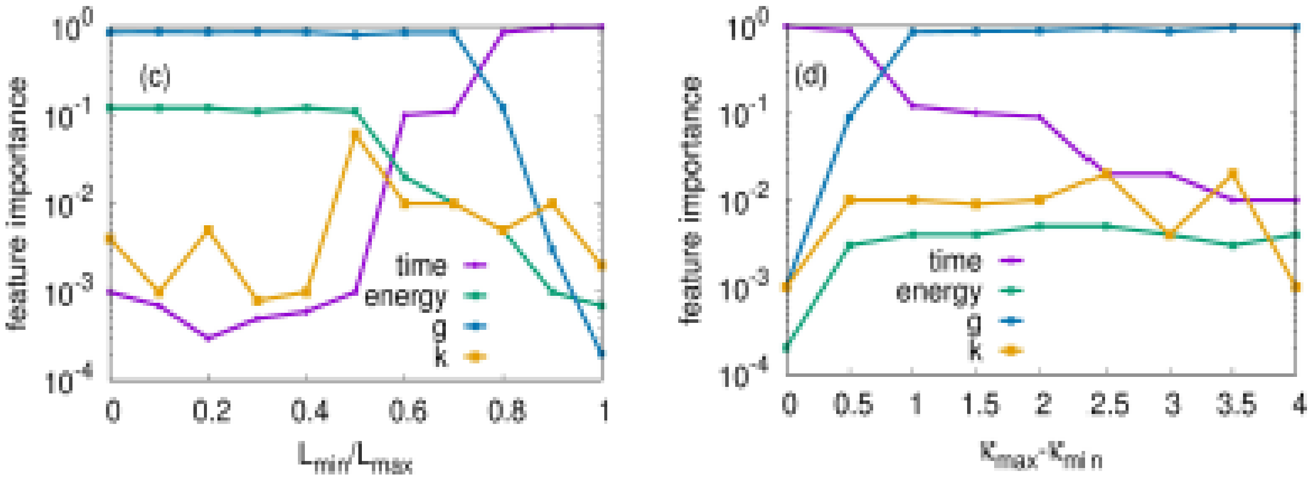

Feature importance in Machine Learning

Diksha, SB, Phys. Rev. E 106, 025003 (2021).

Summary

If $M \sim |F - F_c|^{-n}$, then one can always write it as

$$M \sim |g - g_f|^{-n^*}$$

Where g_f and n^* are at least as universal as the critical exponent n.

Could be useful for predicting imminent criticality.

S. Das, SB, arxiv:2211.01281 (2022).

Thank You!