Open quantum systems in the ultrastrong coupling limit

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Outline

Motivation:

- System $\mathcal{S}$ interacting with a reservoir $\mathcal{R}$ that we cannot control
- Dynamics of $\mathcal{S}$ alone is usually very complicated

$\Rightarrow$ Rigorous derivation of effective dynamics for $\mathcal{S}$ in suitable regimes

- Previous works in different limits:
  - Singular coupling — Hepp, Lieb, *Helv. Phys. Acta* (1973)
  - Low-density limit — Dümke, *Comm. Math. Phys* (1985)
  - Weak-coupling — Davies, *Comm. Math. Phys.* (1974)

    ... refined by Merkli *Ann. of Phys.* (2020)

In this work:

We study yet another limit $\rightarrow$ **ultrastrong (infinite) coupling**!
SM, M. Merkli, arXiv:2411.06817

The model:

Finite-level quantum systems interacting with an infinite bosonic reservoir

- Total Hamiltonian:

$$H = H_{\mathrm{S}} + H_{\mathrm{R}} + \lambda G \otimes \varphi(g).$$

- Reservoir Hamiltonian:

$$H_{\mathrm{R}} = \int_{\mathbb{R}^3} \omega(k) a^\dagger(k) a(k) d^3k, \qquad \text{e.g.} \quad \omega(k) = |k|,$$

- Interaction Hamiltonian:

$$G = \sum_{l=1}^{\nu} \gamma_l P_l \qquad \varphi(g) = \frac{1}{\sqrt{2}} \int_{\mathbb{R}^3} \left(g(k) a^\dagger(k) + \mathrm{h.c.}\right) d^3k,$$

where $g(k) \in L^2(\mathbb{R}^3, d^3k)$ is a complex valued function, called the form factor

Description of states

- Infinite reservoir: H_R has a continuous spectrum, therefore $e^{-\beta H_R}$ is bounded but not trace-class
- In the thermodynamic limit, states are described in terms of expectation values of the observables (algebraic approach[1])

$$\omega_{\mathrm{R},\beta}(W(f)) = e^{-\frac{1}{4}\langle f, \coth(\beta\omega/2) f\rangle}, \quad W(f) = e^{i\varphi(f)}$$

- Initial system-reservoir state factorized $\omega_{\mathrm{SR}} = \omega_{\mathrm{S}} \otimes \omega_{\mathrm{R}}$
- System density matrix in the ultrastrong coupling limit

$$\mathrm{tr}_{\mathrm{S}}\left(\rho_{\mathrm{S}}(t)A\right) = \lim_{|\lambda|\to\infty} \omega_{\mathrm{S}} \otimes \omega_{\mathrm{R}}\left(e^{itH}(A \otimes \mathbb{1}_{\mathrm{R}})e^{-itH}\right),$$

for any $A \in \mathcal{B}(\mathbb{C}^d)$

[1] Bratteli & Robinson *Operator algebras and quantum statistical mechanics*

- Condition on g: $g(k) \neq 0$ for $k \in \mathbb{R}^3$ satisfying $a < |k| < b$ for some $0 \leq a < b$.

Theorem 1

Assume ■. Then, for all $t > 0$, the system density matrix in the ultrastrong coupling limit is given by

$$\rho_{\mathrm{S}}(t) = e^{-itH_{\mathrm{Z}}} \left(\sum_{l=1}^{\nu} P_l \rho_{\mathrm{S}} P_l \right) e^{itH_{\mathrm{Z}}},$$

where H_{Z} is the Zeno Hamiltonian

$$H_{\mathrm{Z}} = \sum_{l=1}^{\nu} P_l H_{\mathrm{S}} P_l.$$

Comments on the result

- If all the projectors P_l are rank-one the dynamics is trivial (there is only the nonselective measurement and no real time evolution)
- If at least one projector P_l is multidimensional, there is unitary evolution inside the respective subspace, unless $H_{\rm S}$ is completely degenerate in that subspace
- This is the same dynamics obtained in the usual quantum Zeno setting (unitary evolution interspersed with repeated measurements)[2]

$$\rho_{\rm S} \mapsto \mathcal{U}_t(\rho_{\rm S}) = e^{-itH_{\rm S}} \rho_{\rm S}\, e^{itH_{\rm S}}.$$

$$\rho_{\rm S}(t,N) = \left(\mathcal{P}\,\mathcal{U}_{t/N}\right)^N \rho_{\rm S}.$$

$$\lim_{N\to\infty} \rho_{\rm S}(t,N) = e^{-itH_{\rm Z}} \Big(\sum_n P_n \rho_{\rm S} P_n\Big) e^{itH_{\rm Z}},$$

[2] see e.g. Misra, Sudarshan JMP 1977 or Facchi et al. PLA 2000

Theorem 2

Assume ■. Let $A \in \mathcal{B}(\mathcal{H}_\mathrm{S})$ and $h \in L^2(\mathbb{R}^3, d^3k)$. We have for all $t > 0$,

$$\lim_{|\lambda|\to\infty} \omega_\mathrm{S} \otimes \omega_\mathrm{R} \Big(e^{itH} \big(e^{\frac{i}{2}\lambda G \mathrm{Re}\langle g, \frac{1-e^{i\omega t}}{\omega} h\rangle} A e^{\frac{i}{2}\lambda G \mathrm{Re}\langle g, \frac{1-e^{i\omega t}}{\omega} h\rangle} \big) \otimes W(h) e^{-itH} \Big)$$

$$= \sum_{l=1}^{\nu} \omega_\mathrm{S} \big(e^{itH_\mathrm{Z}} P_l A P_l e^{-itH_\mathrm{Z}} \big) \omega_\mathrm{R} \big(W(e^{i\omega t} h) \big),$$

where H_Z is the Zeno Hamiltonian

$$H_\mathrm{Z} = \sum_{l=1}^{\nu} P_l H_\mathrm{S} P_l.$$

Generalization to two reservoirs

- Total Hamiltonian

$$H = H_{\mathrm{S}} + H_{\mathrm{R}1} + \lambda_1 G_1 \otimes \varphi_1(g_1) + H_{\mathrm{R}2} + \lambda_2 G_2 \otimes \varphi_2(g_2)$$

- Assumption ($\star$) (valid for Gaussian states):

$$\max_{1\le j\le n} \sup_{0\le t_1,t_2,\dots,t_j\le t} \left| \omega_{\mathrm{R}1}\left(\varphi_1(e^{i\omega t_1} g_1) \cdots \varphi_1(e^{i\omega t_j} g_1) \right) \right| \le B_n,$$

for some numbers $B_n \ge 0$ satisfying

$$\sum_{n\ge 0} \frac{\alpha^n}{n!} B_n < \infty \qquad \text{for any } \alpha > 0.$$

Theorem 3

Assume $(\star)$ and let P_l, $l = 1, \ldots, \nu$ be the spectral projections of G_2. Then, for any observable A of the system and any $t > 0$ we have

$$\lim_{|\lambda_2|\to\infty} \omega_{\mathrm{S}} \otimes \omega_{\mathrm{R1}} \otimes \omega_{\mathrm{R2}} \Big(e^{itH} (A \otimes \mathbb{1}_{\mathrm{R1}} \otimes \mathbb{1}_{\mathrm{R2}}) e^{-itH} \Big)$$

$$= \omega_{\mathrm{S}} \otimes \omega_{\mathrm{R1}} \Big(e^{itH_{\mathrm{Z}}} \big(\sum_{l=1}^{\nu} P_l A P_l \otimes \mathbb{1}_{\mathrm{R1}} \big) e^{-itH_{\mathrm{Z}}} \Big),$$

where H_{Z} is the Zeno Hamiltonian

$$H_{\mathrm{Z}} = \sum_{l=1}^{\nu} P_l \big(H_{\mathrm{S}} + H_{\mathrm{R1}} + \lambda_1 G_1 \otimes \varphi_1(g_1) \big) P_l.$$

Example

- Model: Two qubits interacting with two reservoirs
- Choose $G_2 = \sigma_z \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \sigma_z$
- The spectral projections of G_2 are $P_+ = |00\rangle\langle 00|$, $P_- = |11\rangle\langle 11|$ (both rank one) and $P_0 = |01\rangle\langle 01| + |10\rangle\langle 10|$ (rank two - *effective qubit*)
- Initial density matrix ($-1 \le z \le 1$ and $0 \le |c|^2 \le 1 - z^2$)

$$\rho_{\mathrm{S}} = \frac{1+z}{2}|01\rangle\langle 01| + \frac{1-z}{2}|10\rangle\langle 10| + \frac{c}{2}|01\rangle\langle 10| + \frac{c^*}{2}|10\rangle\langle 01|$$

- Choose $H_{\mathrm{S}} = G_1 = \varepsilon_1\sigma_z \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \varepsilon_2\sigma_z$ so that $z(t) = z$ while defining $\Delta = \varepsilon_1 - \varepsilon_2$ the coherence is

$$c(t) = e^{2it\Delta} D(t) c, \qquad D(t) = \omega_{\mathrm{R}1}\left(W\left(\lambda_1 \Delta \frac{e^{i\omega t}-1}{i\omega} g_1\right)\right),$$

Example

- This is the well-known pure dephasing model
- If ω_{R1} is in thermal equilibrium at inverse temperature β, the decoherence function $D(t)$ has the explicit expression,

$$D_\beta(t) = \exp\Big[-\lambda_1^2 \Delta^2 \int_{\mathbb{R}^3} |g_1(k)|^2 \coth(\beta\omega/2) \frac{\sin^2(\omega t/2)}{\omega^2} d^3k \Big].$$

- depending on $g(k)$ and β the dynamics can display non-Markovianity, i.e. a lack of CP-divisibility (when $D_\beta(t)$ is not monotonic in t)
(see e.g. Haikka et al. PRA 2013)

Sketch of the proof (Theorem 1)

- Define $K = K_\lambda = H_{\mathrm{R}} + \lambda G \otimes \varphi(g)$ and use Dyson series

$$e^{itH} e^{-itK} = \mathbb{1} + \mathcal{D}, \qquad \mathcal{D} = \sum_{n \geq 1} i^n \int_{0 \leq t_n \leq \cdots \leq t_1 \leq t} H_{\mathrm{S}}(t_n) \cdots H_{\mathrm{S}}(t_1),$$

  where $H_{\mathrm{S}}(t) = e^{itK} H_{\mathrm{S}} e^{-itK}$
- Study separately four terms

$$\begin{aligned}
&\omega_{\mathrm{S}} \otimes \omega_{\mathrm{R}} \big(e^{itH} (A \otimes \mathbb{1}_{\mathrm{R}}) e^{-itH} \big) \\
&= \quad \omega_{\mathrm{S}} \otimes \omega_{\mathrm{R}} \big(e^{itK} (A \otimes \mathbb{1}_{\mathrm{R}}) e^{-itK} \big) \\
&\quad + \omega_{\mathrm{S}} \otimes \omega_{\mathrm{R}} \big(\mathcal{D} e^{itK} (A \otimes \mathbb{1}_{\mathrm{R}}) e^{-itK} \big) \\
&\quad + \omega_{\mathrm{S}} \otimes \omega_{\mathrm{R}} \big(e^{itK} (A \otimes \mathbb{1}_{\mathrm{R}}) e^{-itK} \mathcal{D}^* \big) \\
&\quad + \omega_{\mathrm{S}} \otimes \omega_{\mathrm{R}} \big(\mathcal{D} e^{itK} (A \otimes \mathbb{1}_{\mathrm{R}}) e^{-itK} \mathcal{D}^* \big) \\
&\equiv \quad T_1 + T_2 + T_3 + T_4.
\end{aligned}$$

- Start with T_1
- Use that for any operator $A \in \mathcal{B}(\mathbb{C}^d)$,

$$\triangle \quad e^{itK}(A \otimes \mathbb{1}_{\mathrm{R}})e^{-itK} = \sum_{l,r=1}^{\nu} P_l A P_r \, e^{-\frac{i}{2}\mathrm{Im}\langle (e^{i\omega t}-1-i\omega t)(f_l - f_r),(f_l+f_r)\rangle} \times W\big((e^{i\omega t}-1)(f_l - f_r)\big)$$

where $f_l = \frac{\lambda\gamma_l}{i\omega} g, \quad l = 1, \ldots, \nu$

- Therefore

$$\begin{aligned} T_1 =& \omega_{\mathrm{S}} \otimes \omega_{\mathrm{R}}\big(e^{itK}(A \otimes \mathbb{1}_{\mathrm{R}})e^{-itK}\big) \\ =& \sum_{l,r=1}^{\nu} \omega_{\mathrm{S}}\big(P_l A P_r\big) \, e^{-\frac{i}{2}\lambda^2(\gamma_l^2-\gamma_r^2)\mathrm{Im}\langle g, \frac{e^{-i\omega t}+it\omega}{\omega^2} g\rangle} \omega_{\mathrm{R}}\Big(W\big((e^{i\omega t}-1)(f_l - f_r)\big)\Big). \end{aligned}$$

- Explicitely, the expectation of the Weyl reads

$$\omega_{\mathrm{R}}\Big(W\big((e^{i\omega t}-1)(f_l-f_r)\big)\Big)=e^{-\frac{\lambda^2}{4}\left\|\mathcal{C}^{1/2}\left[(\gamma_l-\gamma_r)\frac{e^{i\omega t}-1}{i\omega}g\right]\right\|^2_{L^2}},$$

- In the limit $\lambda\to\infty$ the expression tends to 0 unless the term multiplying λ is vanishing
- This selects the case $l=r$ and therefore

$$\lim_{|\lambda|\to\infty}T_1=\sum_{l=1}^{\nu}\omega_{\mathrm{S}}(P_lAP_l),\qquad t>0.$$

- Now we look at T_2 ... then T_3 and T_4 are treated similarly

- The term T_2 reads

$$T_2 = \omega_{\mathrm{S}} \otimes \omega_{\mathrm{R}} \big(\mathcal{D} e^{itK} (A \otimes \mathbb{1}_{\mathrm{R}}) e^{-itK} \big)$$
$$= \sum_{l,r=1}^{\nu} e^{-\frac{i}{2}\lambda^2(\gamma_l^2 - \gamma_r^2)\mathrm{Im}\langle g, \frac{e^{-i\omega t} + it\omega}{\omega^2} g\rangle} \sum_{n \geq 1} i^n \int_{0 \leq t_n \leq \cdots \leq t_1 \leq t}$$
$$\times\, \omega_{\mathrm{S}} \otimes \omega_{\mathrm{R}} \Big(H_{\mathrm{S}}(t_n) \cdots H_{\mathrm{S}}(t_1) P_l A P_r \; W\big((e^{i\omega t} - 1)(f_l - f_r)\big) \Big).$$

- Use $\triangle$ to rewrite each term $H_{\mathrm{S}}(t_j)$
- Combine together the Weyl ops. appearing after the substitution

$$T_2 = \omega_{\mathrm{S}} \otimes \omega_{\mathrm{R}} \big(\mathcal{D} e^{itK} (A \otimes \mathbb{1}_{\mathrm{R}}) e^{-itK} \big)$$
$$= \sum_{l_0, \ldots, l_n, r_0 = 1}^{\nu} \sum_{n \geq 1} i^n \int_{0 \leq t_n \leq \cdots \leq t_1 \leq t} e^{-i\Phi(\lambda)} \omega_{\mathrm{S}} \big(P_{l_n} H_{\mathrm{S}} P_{l_{n-1}} \cdots P_{l_1} H_{\mathrm{S}} P_{l_0} A P_{r_0} \big)$$
$$\times\, \omega_{\mathrm{R}} \Big(W\big(\sum_{j=1}^{n} (e^{it_j\omega} - 1)(f_{l_j} - f_{l_{j-1}}) \big) W\big((e^{it\omega} - 1)(f_{l_0} - f_{r_0}) \big) \Big),$$

- $\omega_{\mathrm{R}}(W(\lambda\eta)) = e^{-\frac{\lambda^2}{4}\|\mathcal{C}^{1/2}\eta\|_{L^2}^2}$ with a complicated function η
- the Dyson series converges uniformly in λ and we can exchange the limit $\lambda \to \infty$ with the series
- by means of the dominated converge theorem, we can exchange $\lambda \to \infty$ and the multiple integrations
- the limit selects the case when all the projections are equal $r_0 = l_0 = l_1 = \cdots = l_n$ and in this case also the phase vanishes
- after computing analogously T_3 and T_4 the series is resummed

Conclusions

- We computed the reduced dynamics of an open quantum system in the ultrastrong coupling limit $\lambda \to \infty$
- It consists of a nonselective measurement onto the eigenbasis of the coupling G and a unitary evolution with an effective Zeno Hamiltonian (related to the quantum Zeno effect)
- The proof can be generalized to the case of two reservoirs, when λ_1 is finite and $\lambda_2 \to \infty$
- In this case the reduced dynamics of the system is richer and it can be non-Markovian

Outlook

- scaling time to slow down the nonselective measurement
- corrections for finite but large λ

Thank you for your attention!