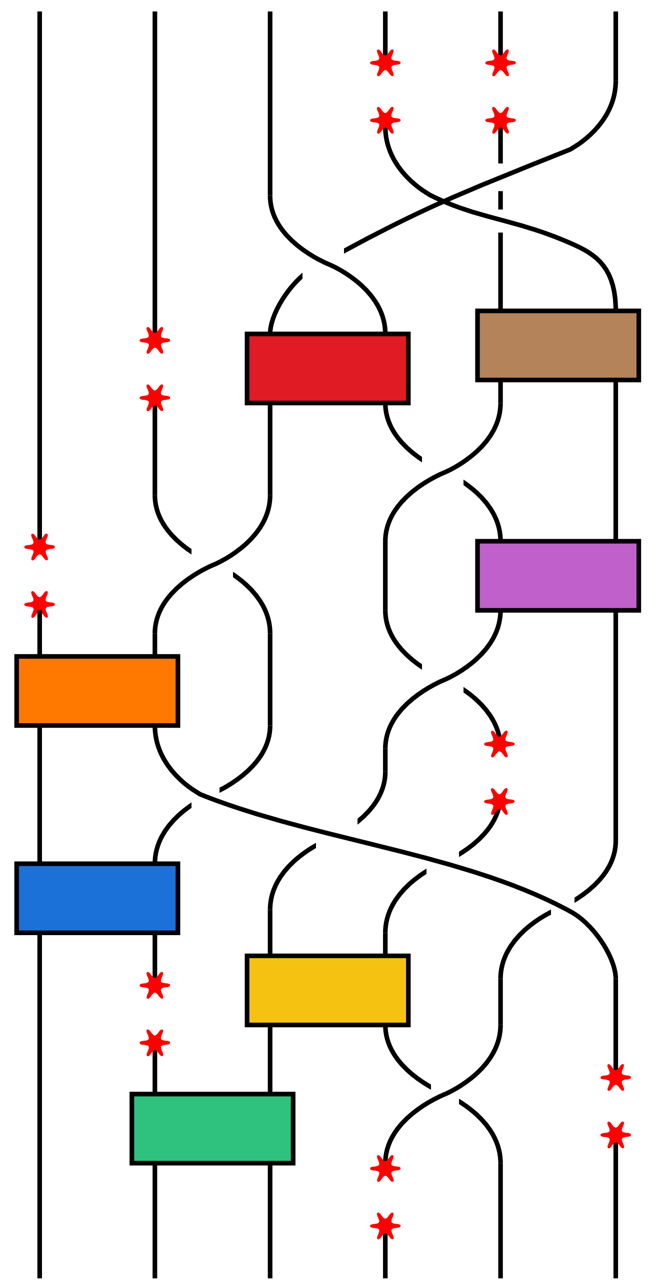
Measurement-induced entanglement transitions in certain tensor networks



PRX Quantum **2**, 010352 (2021)

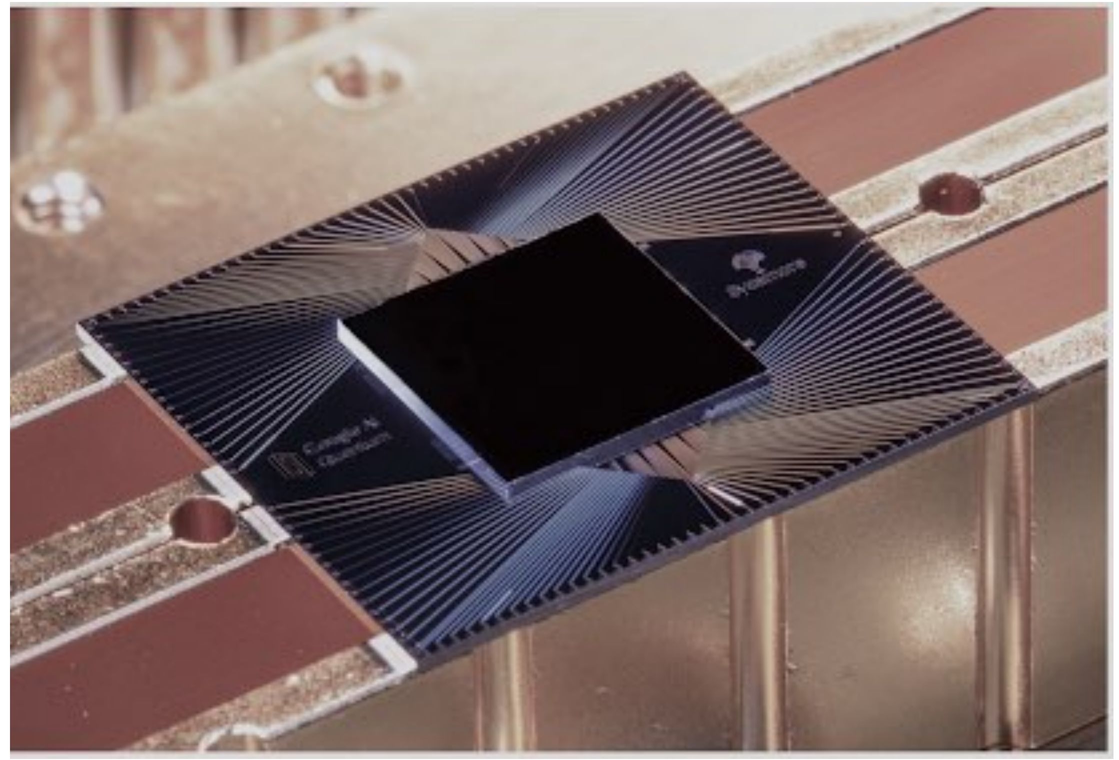
Sthitadhi Roy
ICTS-TIFR, Bengaluru

work with A. Nahum (Oxford → ENS, Paris), B. Skinner (OSU), and J. Ruhman (Bar-Ilan)

Dynamics of quantum entanglement

Why quantum dynamics?

- Quantum mechanics is operationally a theory of unitary dynamics and non-unitary measurements
- How do quantum correlations and quantum information propagate in the system?
- We would like to be able to manipulate quantum states and store/retrieve information using them… in real time



Google's Sycamore processor

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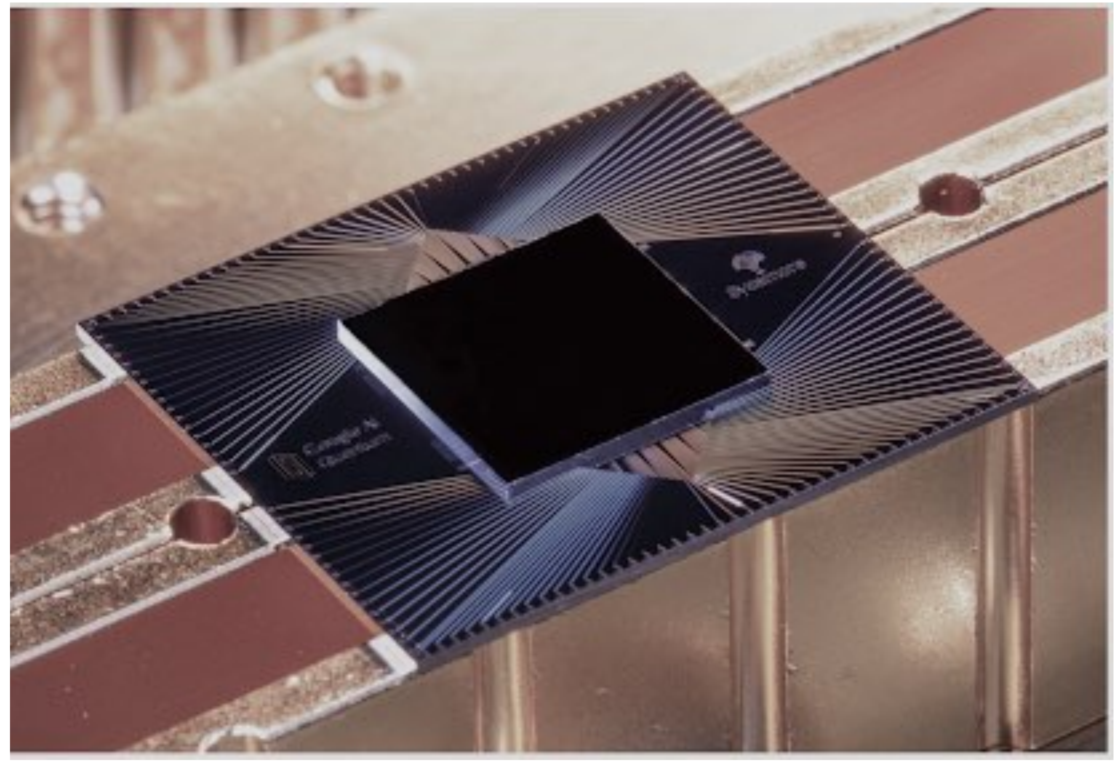

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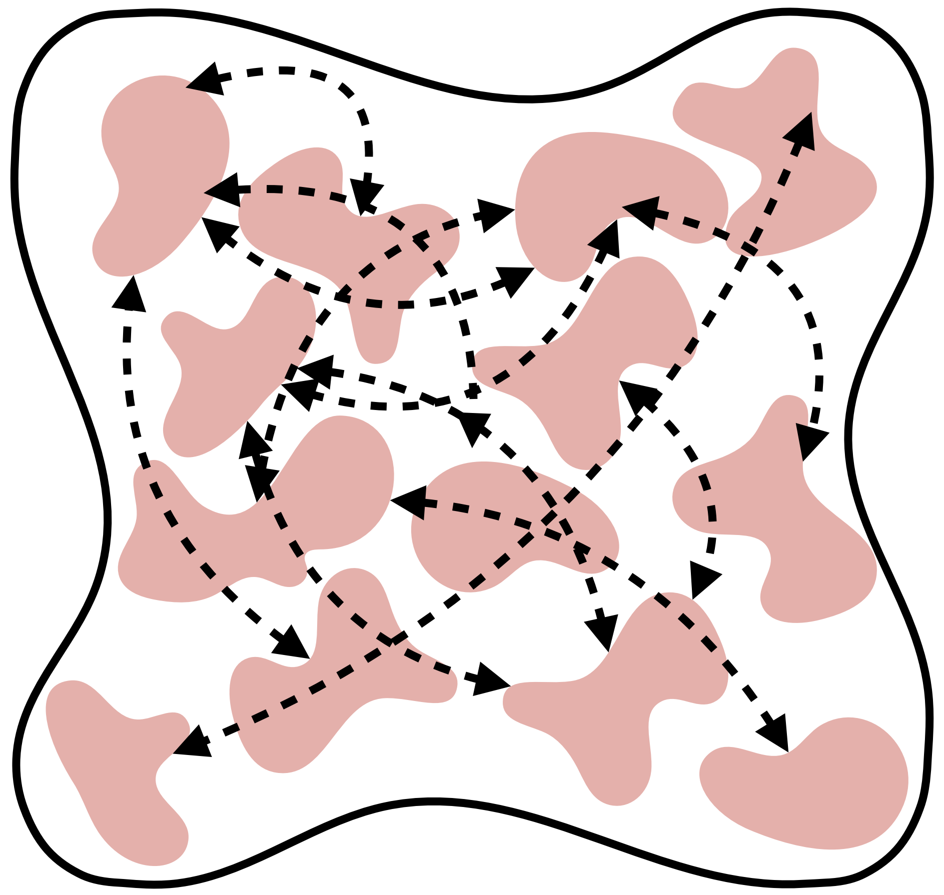
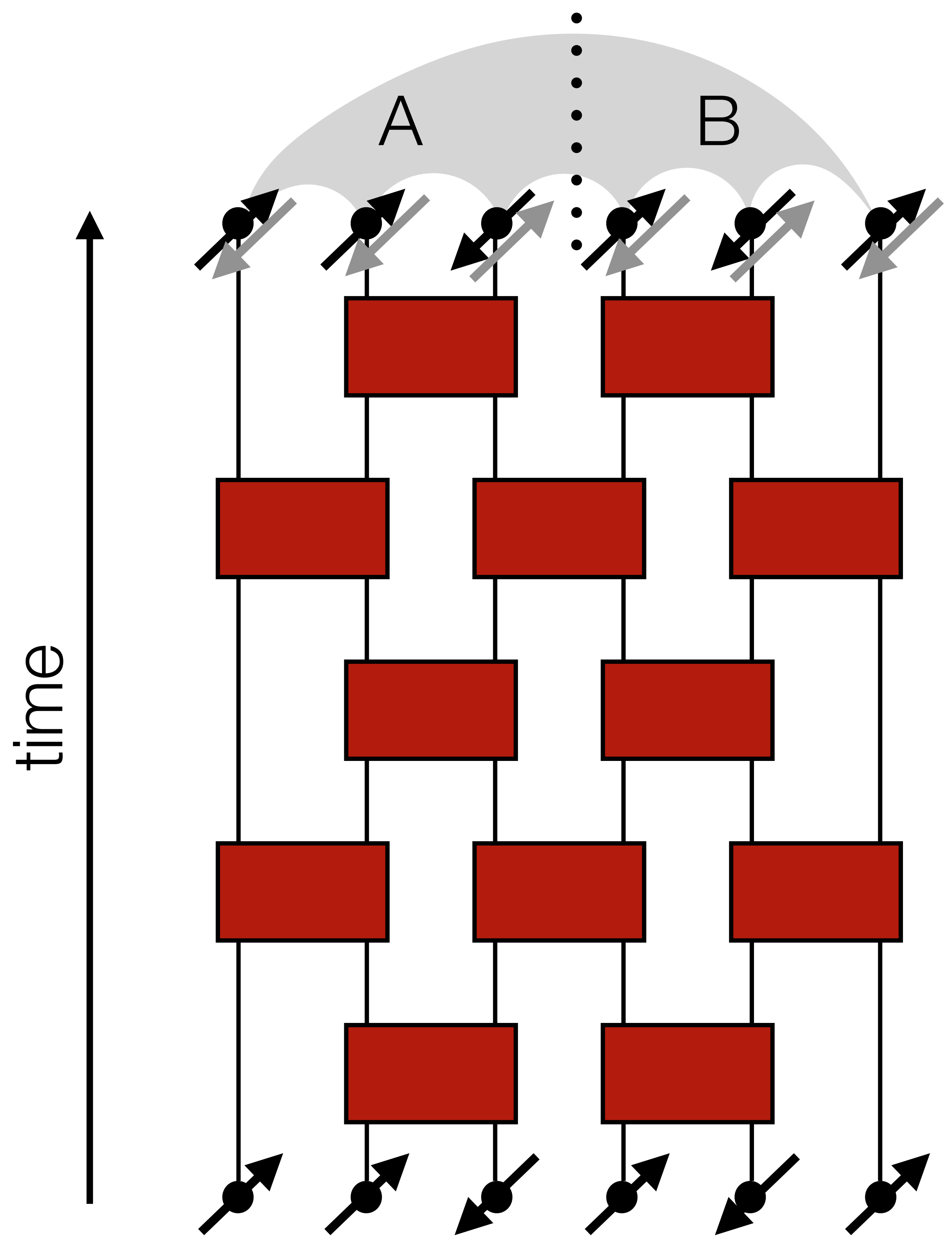
Why entanglement?

Quantum Entanglement: web of non-local correlations throughout a quantum system



- Entanglement *can* be used to distinguish *phases* of quantum matter
- How strongly is information shared between different parts of a system?
- How robustly is the information encoded in a quantum state
  - can it be destroyed by local perturbations/errors?
- How easy is to prepare/manipulate a state and retrieve quantum information from it?

Dynamics of quantum entanglement

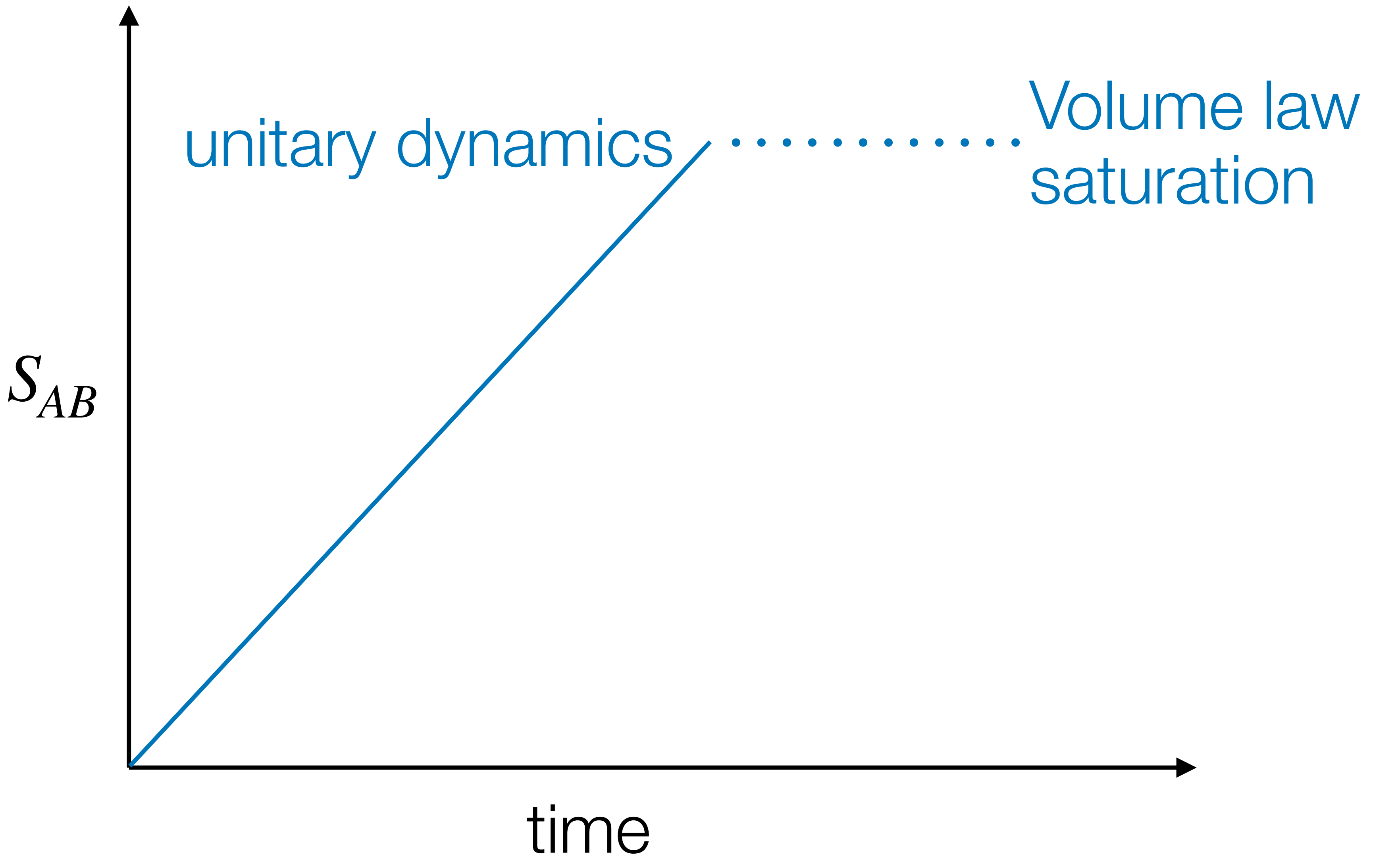


$$S_{AB}^{vN} = -\,\mathrm{Tr}_A[\rho_A \ln \rho_A] \quad \text{where} \quad \rho_A = \mathrm{Tr}_B[\rho]$$

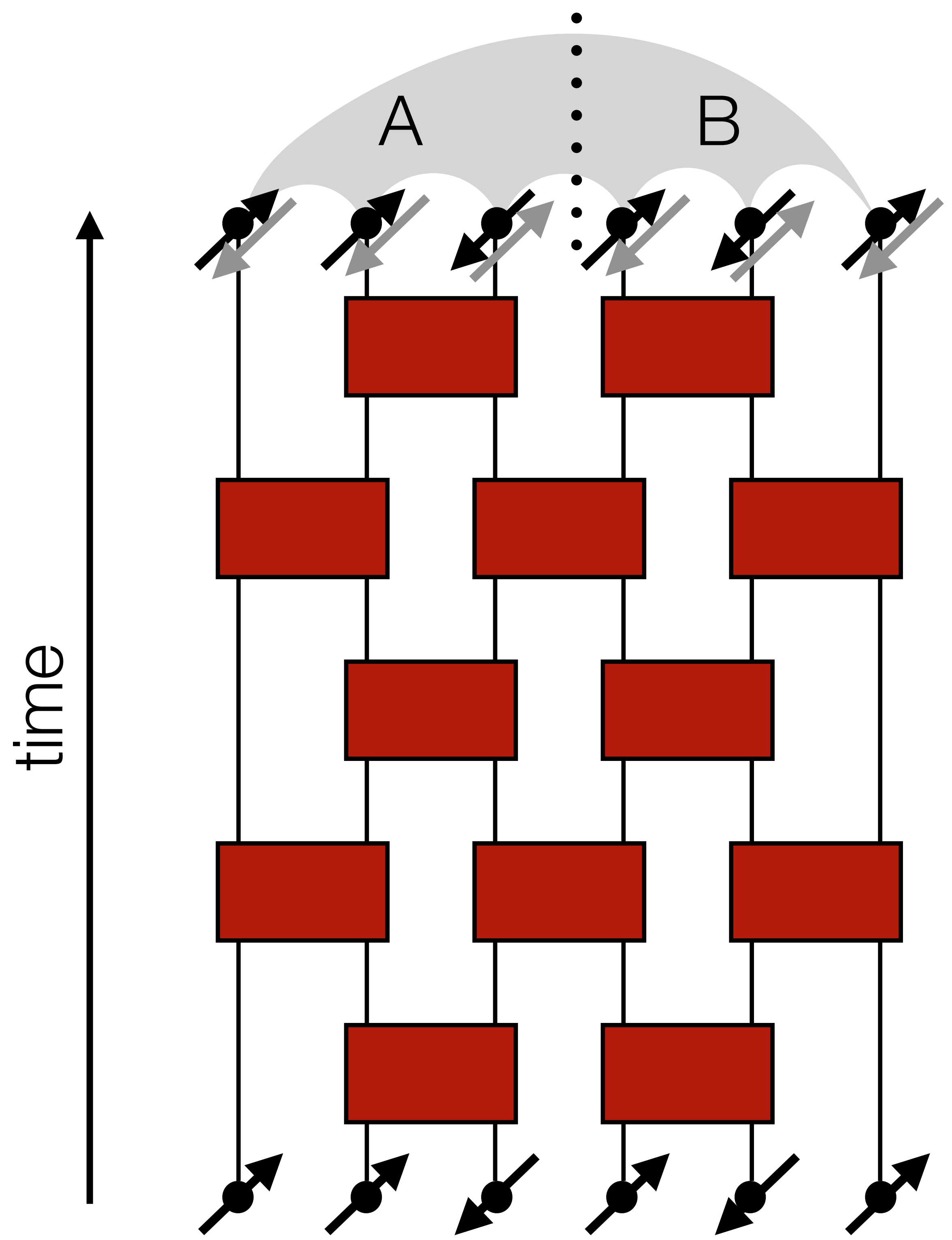
— von Neumann entropy of entanglement

$$S_{AB}^{n} = \frac{-1}{n-1} \ln \mathrm{Tr}_A[\rho_A^n]$$

— n^{th} Rényi entropy of entanglement



Dynamics of quantum entanglement



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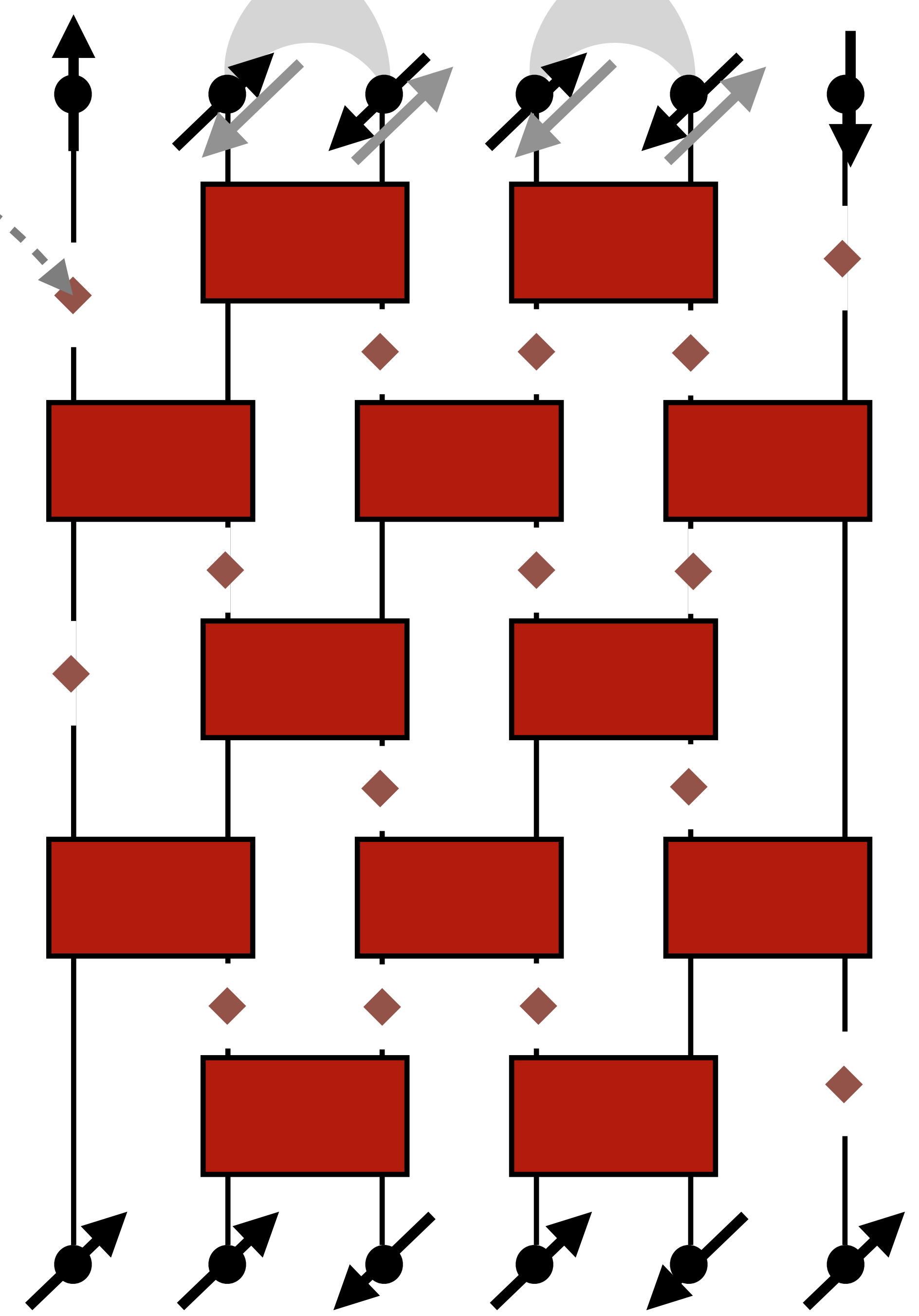
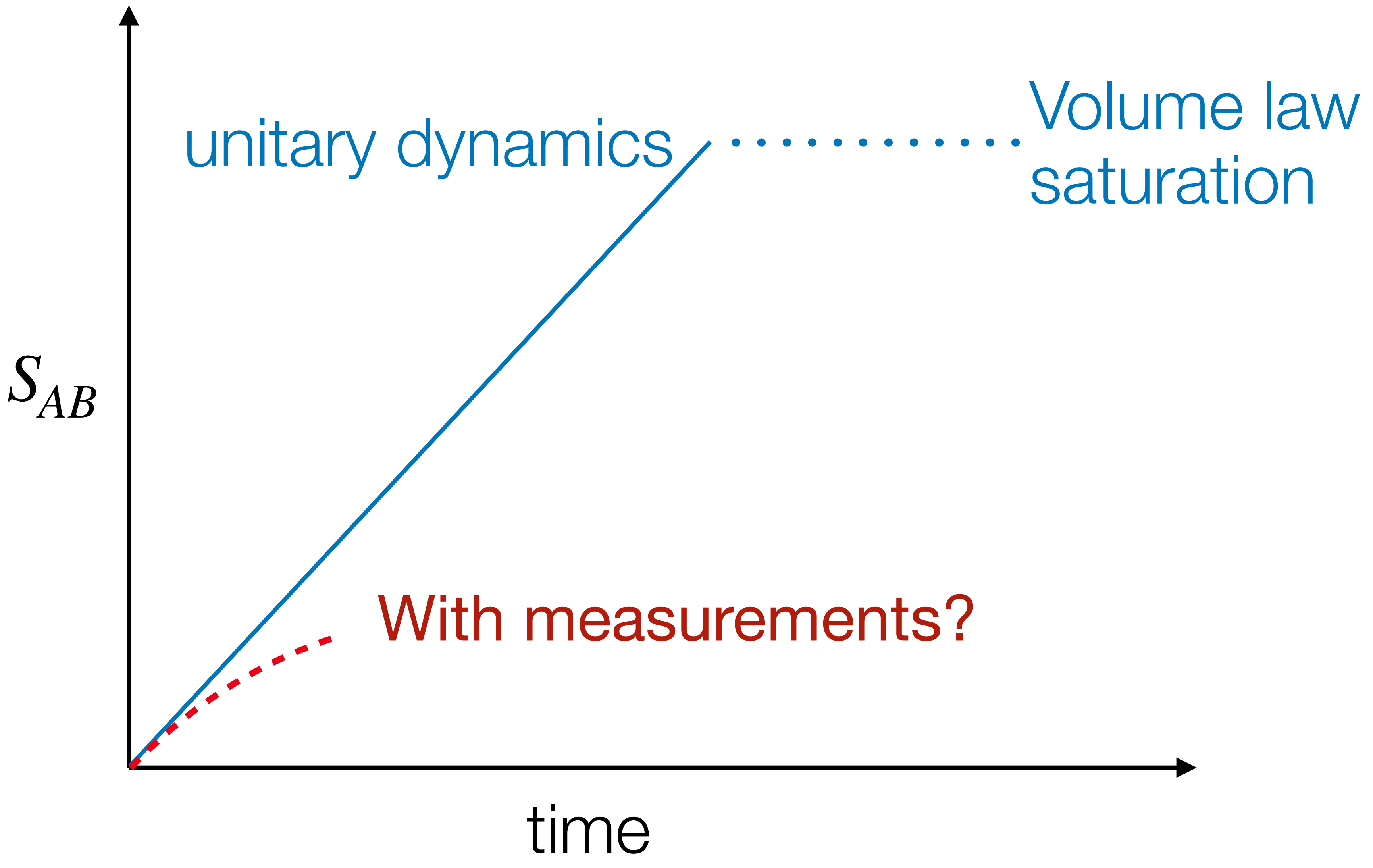
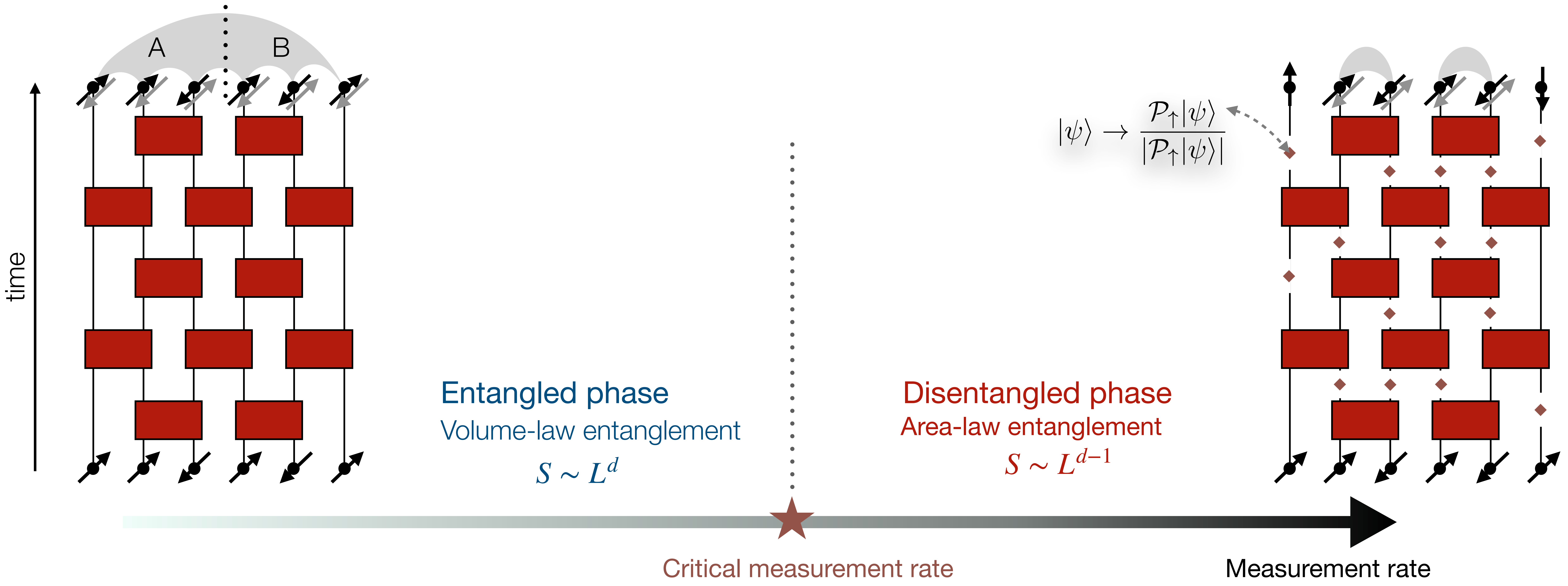
$$|\psi\rangle \to \frac{\mathcal{P}_\uparrow|\psi\rangle}{|\mathcal{P}_\uparrow|\psi\rangle|}$$





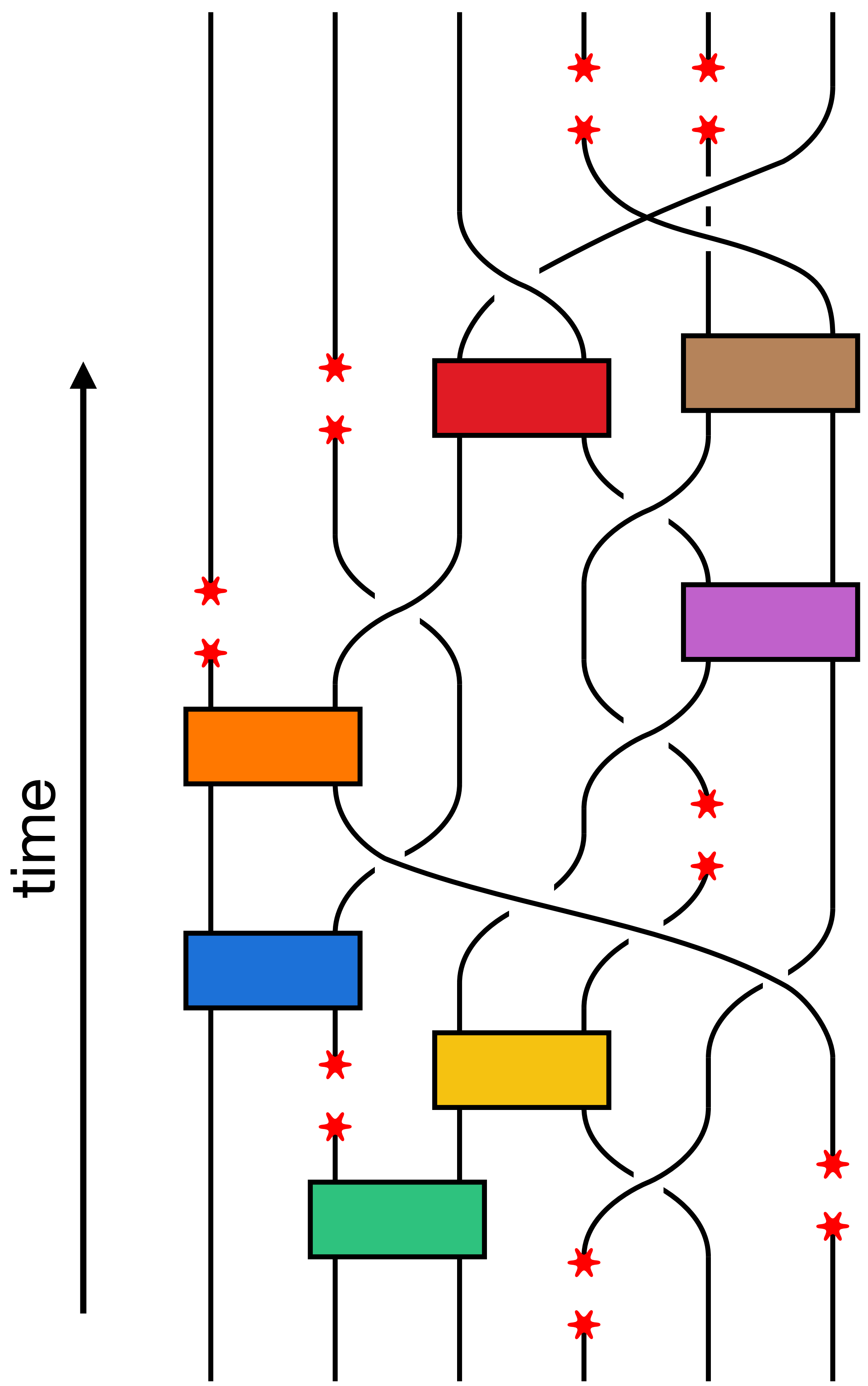
Measurement-induced entanglement transition

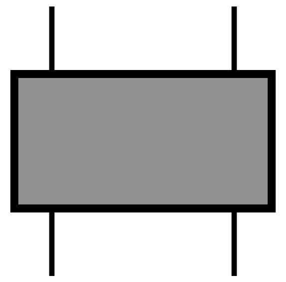


Competition between *entangling* unitary dynamics and *disentangling* (projective) measurements

Skinner et al. 18; Li et al. 18; Chan et al. 18; Choi et al. 20; Szyniszewski et al. 19; Gullans, Huse 19, 20;…..

All-to-all connected tensor network: an exactly solvable model?





$= 4 \times 4$ Haar random unitary

with rate $\propto 1 - r$

$=$

with rate $\propto r$

True measurements

- $|\psi\rangle \to \dfrac{P_{i,\uparrow}|\psi\rangle}{|P_{i,\uparrow}|\psi\rangle|}$ with Born rule probability $(1 + \langle\psi|\sigma_i^z|\psi\rangle)/2$
- Non-trivial, correlated outcome probabilities

Forced measurements

- $|\psi\rangle \to \dfrac{P_{i,\uparrow}|\psi\rangle}{|P_{i,\uparrow}|\psi\rangle|}$
- All outcomes spin-up, *postselection*

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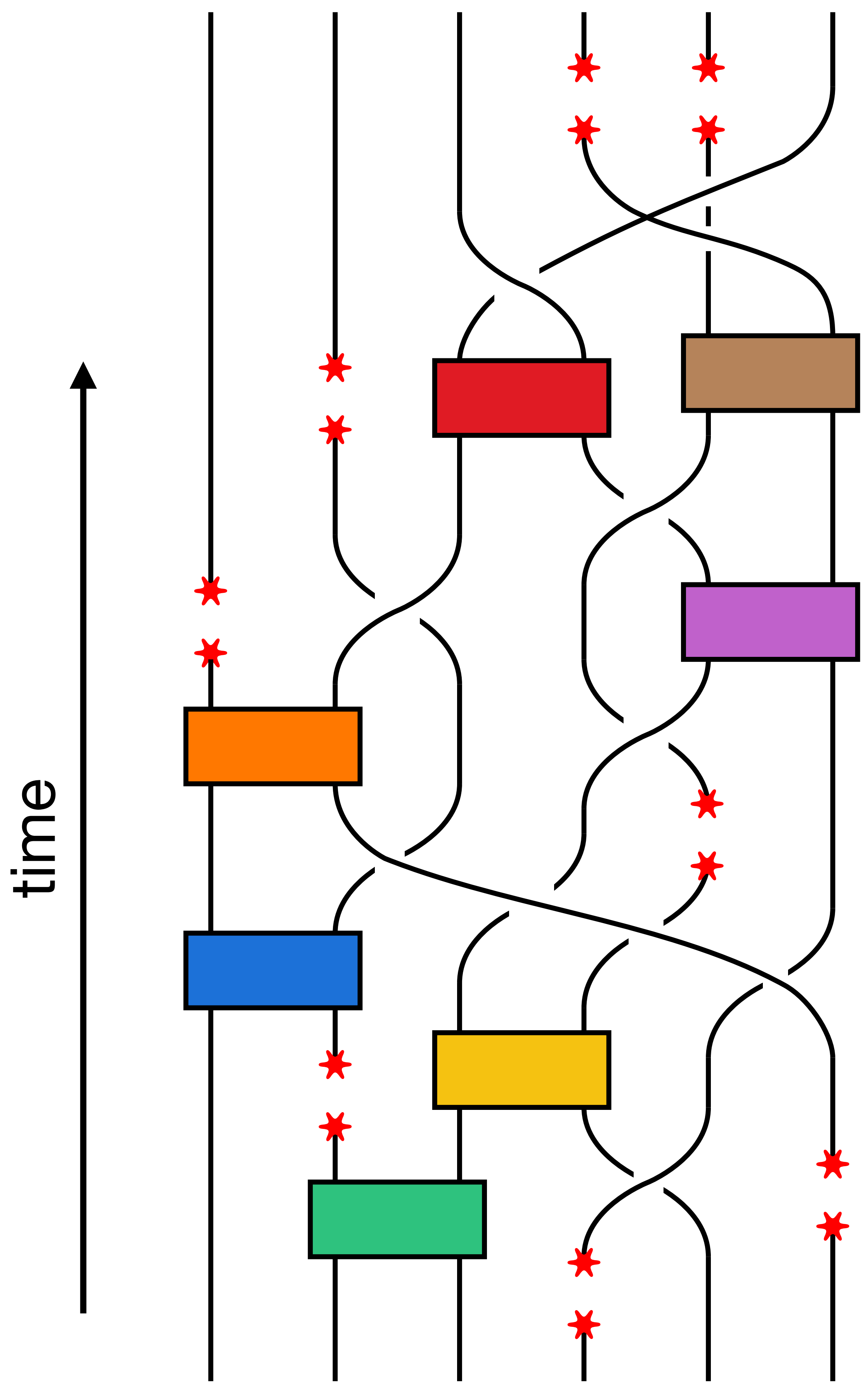

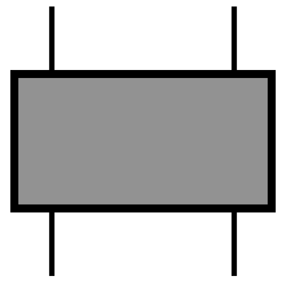

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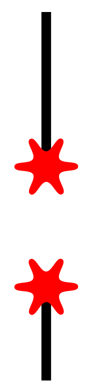

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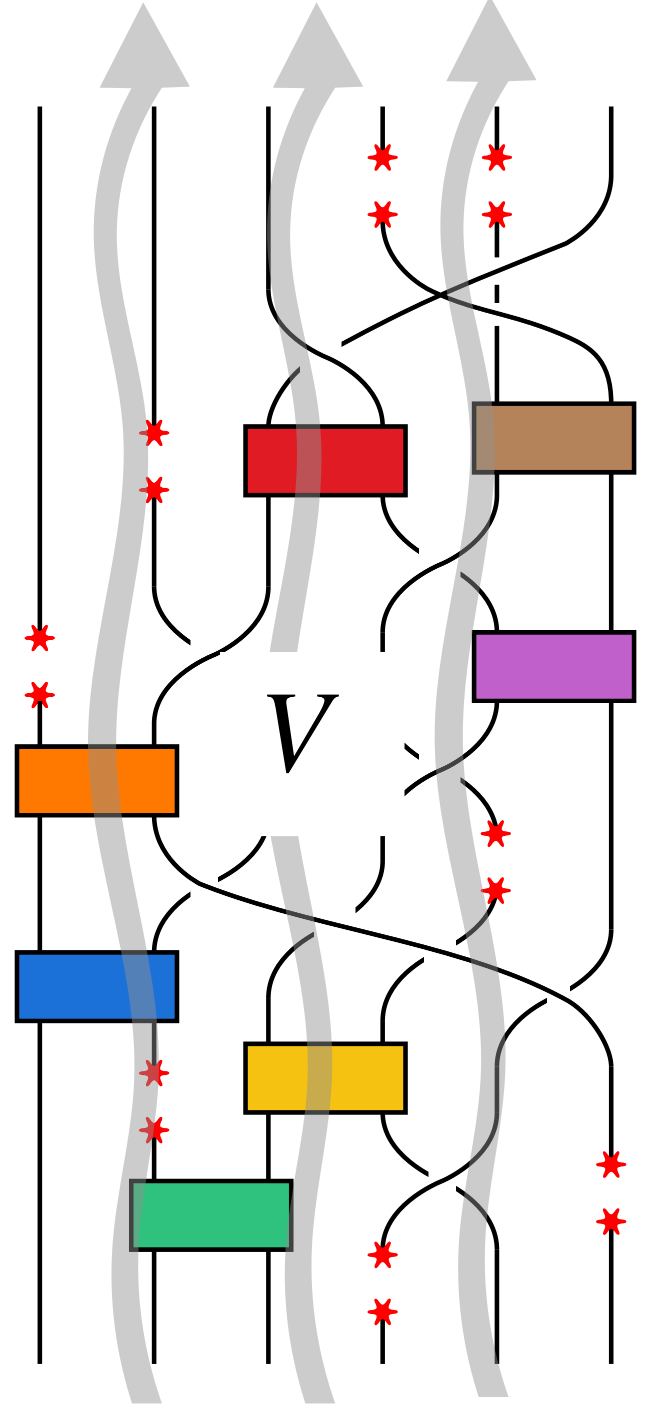
- No spatial structure $\Rightarrow$ area and volume laws have no real meaning
- A new diagnostic of the phase transition?

Operator Entanglement Entropy



How much quantum information flows through the circuit in time ?

— quantified by the Operator Entanglement Entropy of V

Operator Entanglement Entropy

Singular value decomposition $V(t) = \sum_{j=1}^{2^N} \lambda_j |j_t\rangle\langle j_0|$ with normalisation $\sum_{j=1}^{2^N} \lambda_j^2 = 1$

n^{th} opEE $\quad S_n(t) = \frac{1}{1-n} \ln\left(\sum_j \lambda_j^{2n} \right)$

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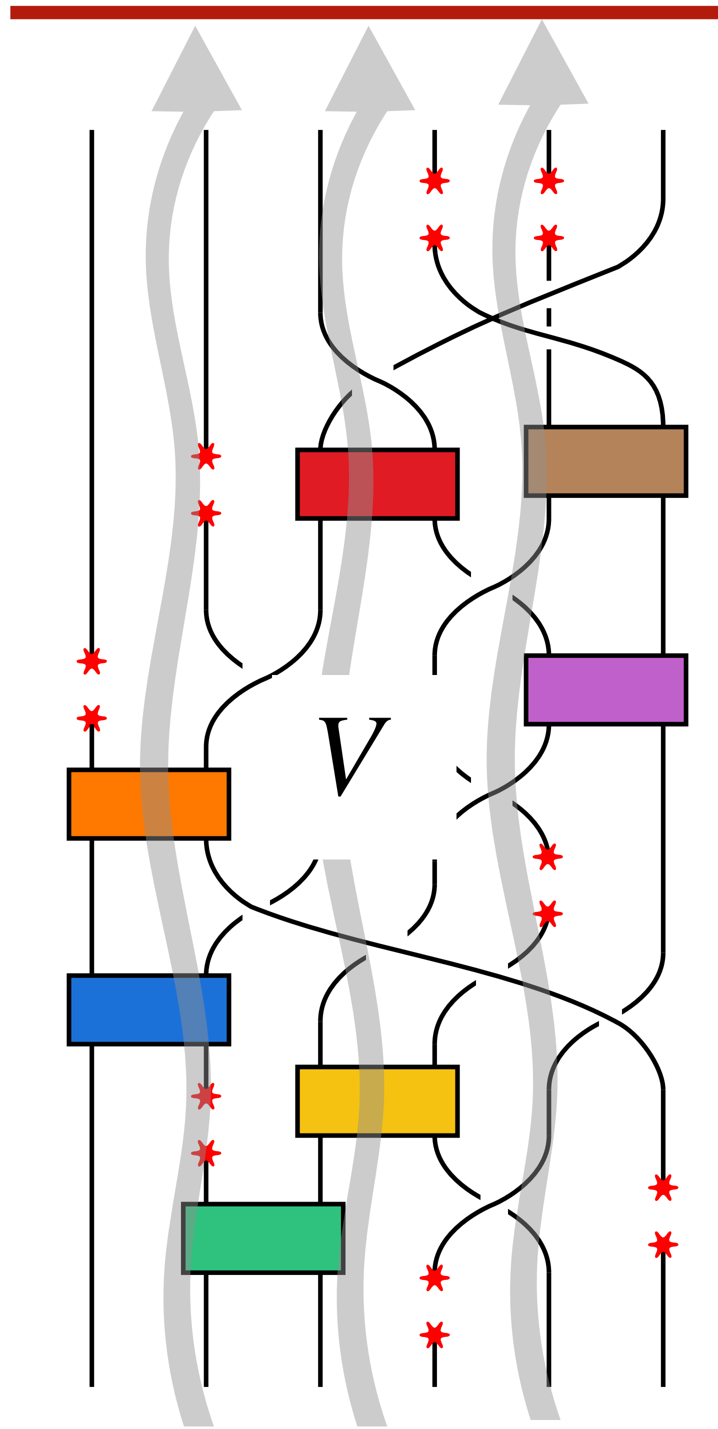

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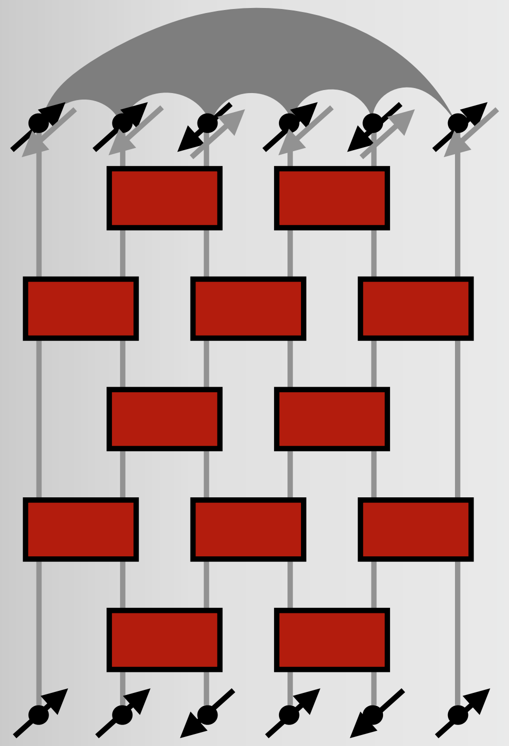
Two 'trivial' limits:

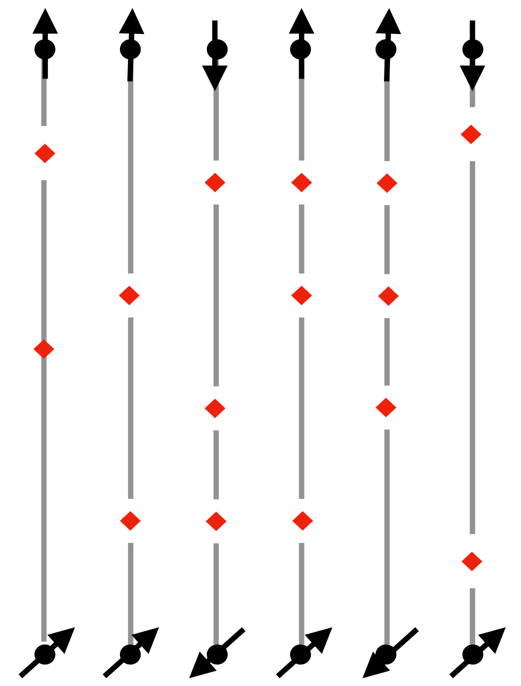


- No measurements
- $V \Rightarrow$ unitary
- $\lambda_j = 1/2^N$ for all j
- $S_n(t) = N \ln 2$ at all times
- Maximal and extensive opEE

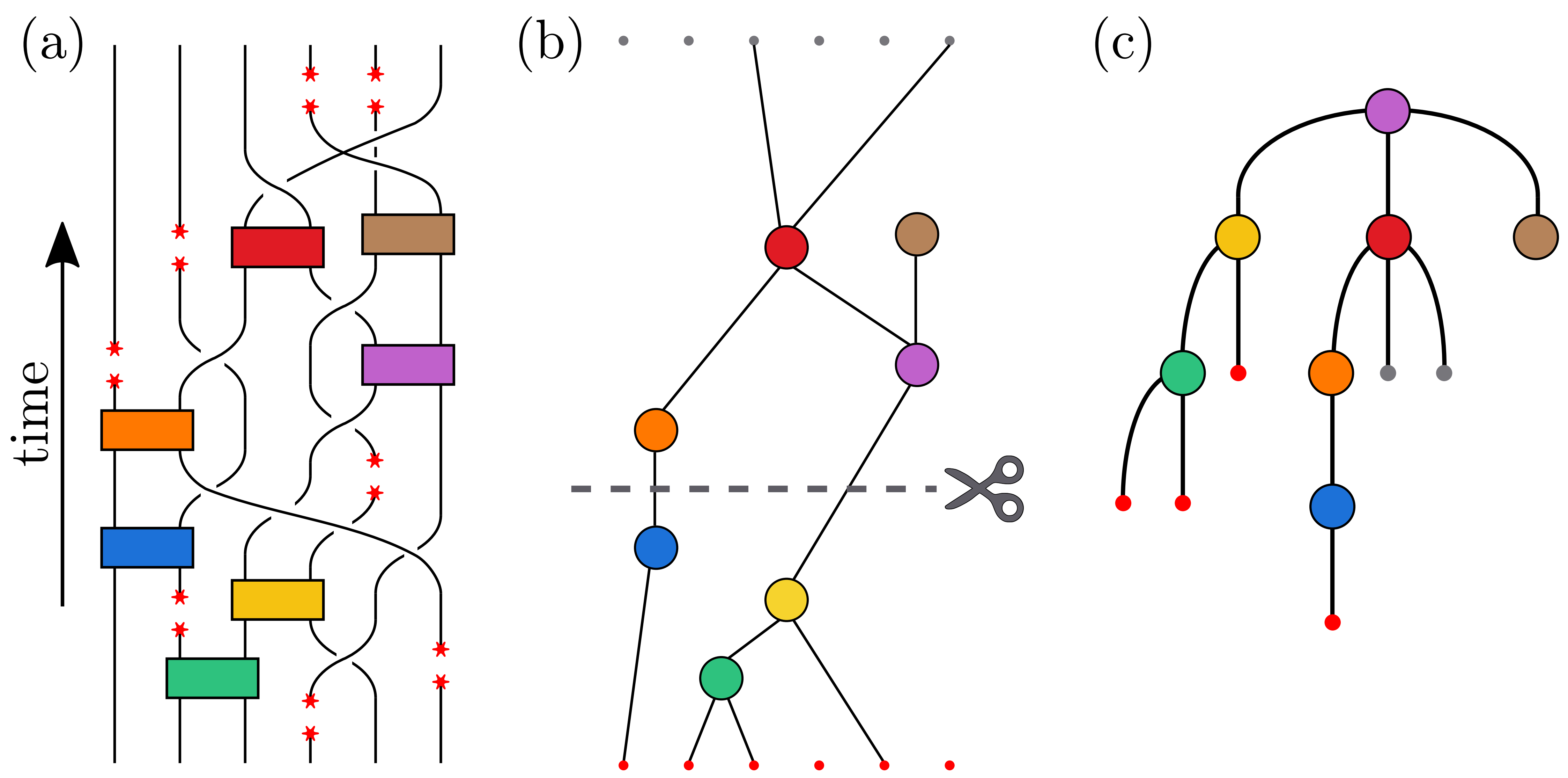
- Only measurements
- $\lambda_j = 1$ for all a particular j and 0 otherwise

  $|j_0\rangle = \uparrow \downarrow \downarrow \downarrow \uparrow \uparrow$

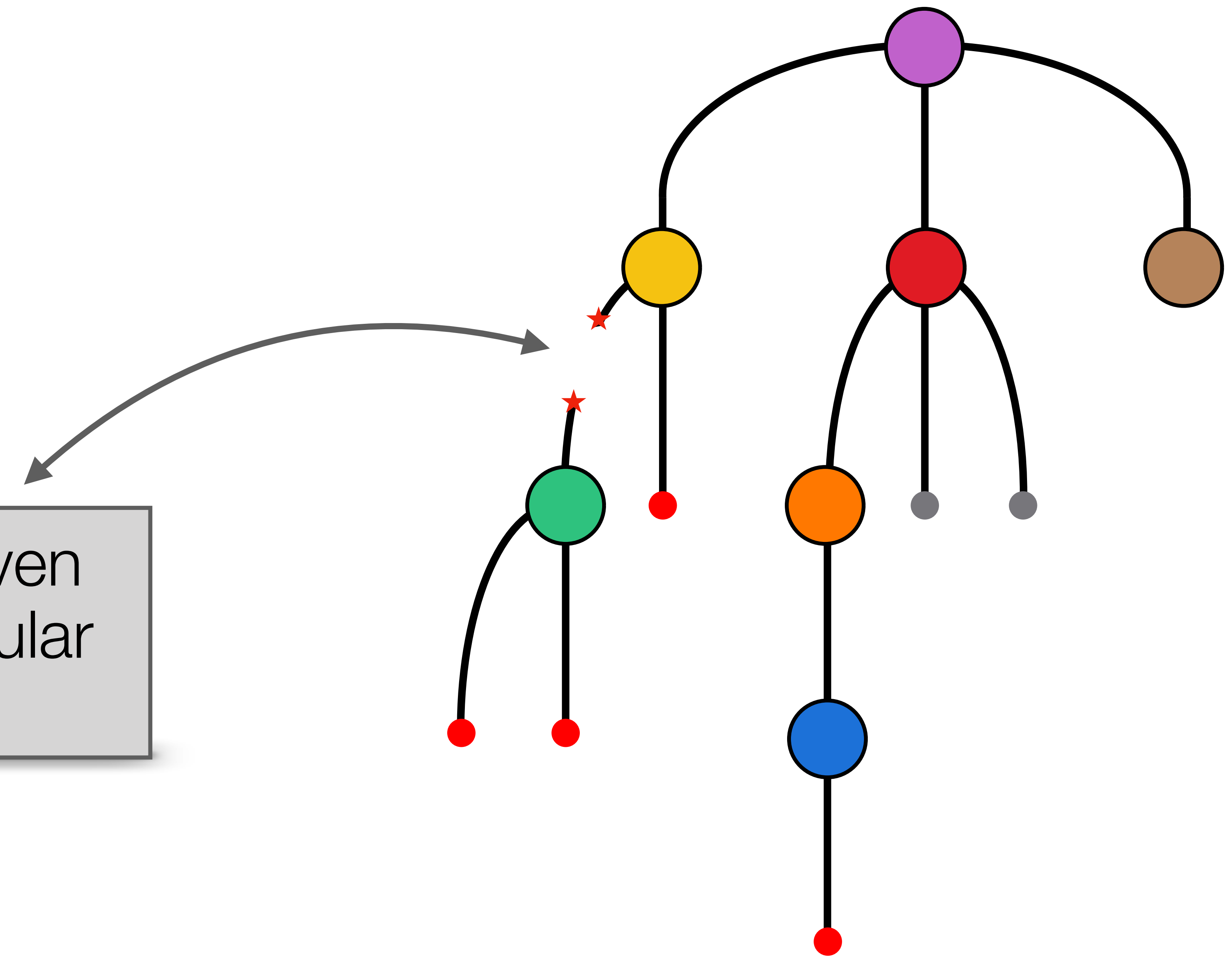
  $|j_t\rangle = \uparrow \uparrow \downarrow \uparrow \uparrow \downarrow$
- $S_n(t) = 0$



All-to-all connected tensor network as a tree tensor network



- Map the unitaries to nodes of the graph
- Map the worldlines of spins to edges of the graph
- Some of the edges absent due to measurements



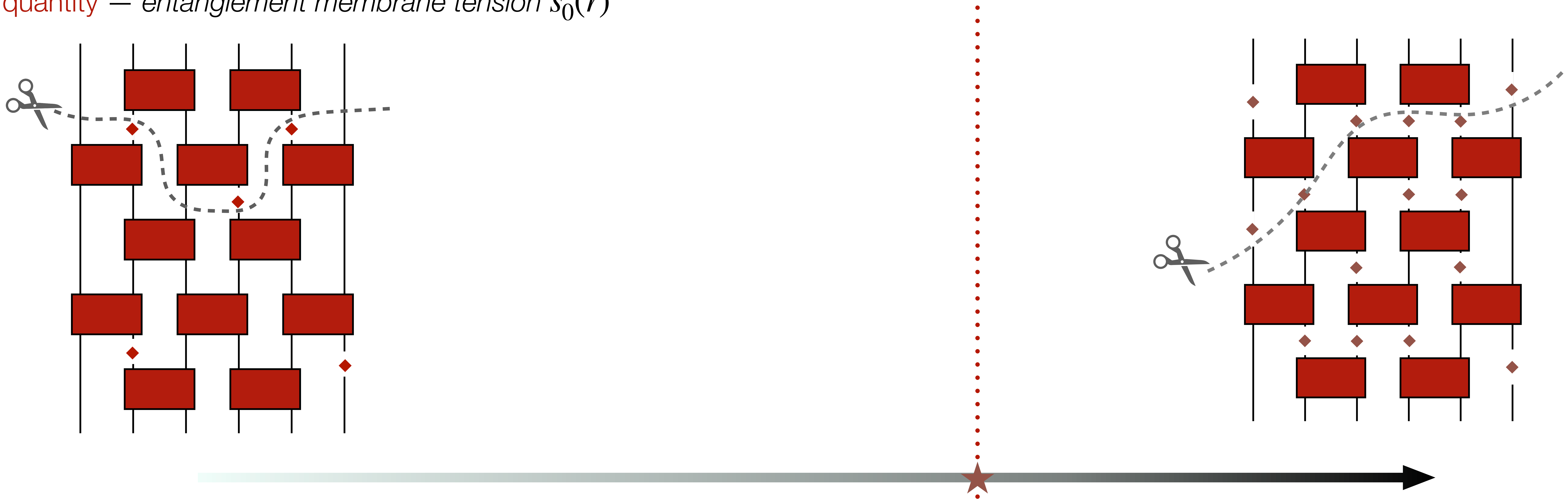
Probability that a site is measured before any unitary acts on it:

$$p = \frac{r}{2(1-r)+r} = \frac{r}{2-r}$$

Also the probability that a given branch is killed after a particular node

Classical phase transition — a bound for the quantum phase transition

- Idea of minimal cut — minimum number of links to cut to break the circuit apart
- Key quantity — *entanglement membrane tension* $s_0(r)$



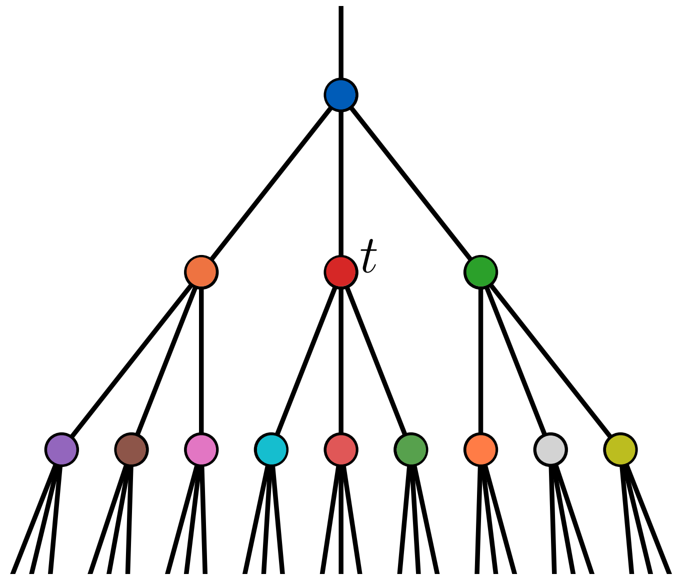
$r_c^{\text{classical}}$

- Circuit classically disconnected between initial and final times
- Cost of cutting the network is vanishing
- Classical percolation transition determined by circuit geometry

Classical phase transition — a bound for the quantum phase transition

- Idea of minimal cut — minimum number of links to cut to break the circuit apart
- Key quantity — *entanglement membrane tension* $s_0(r)$

Critical point:



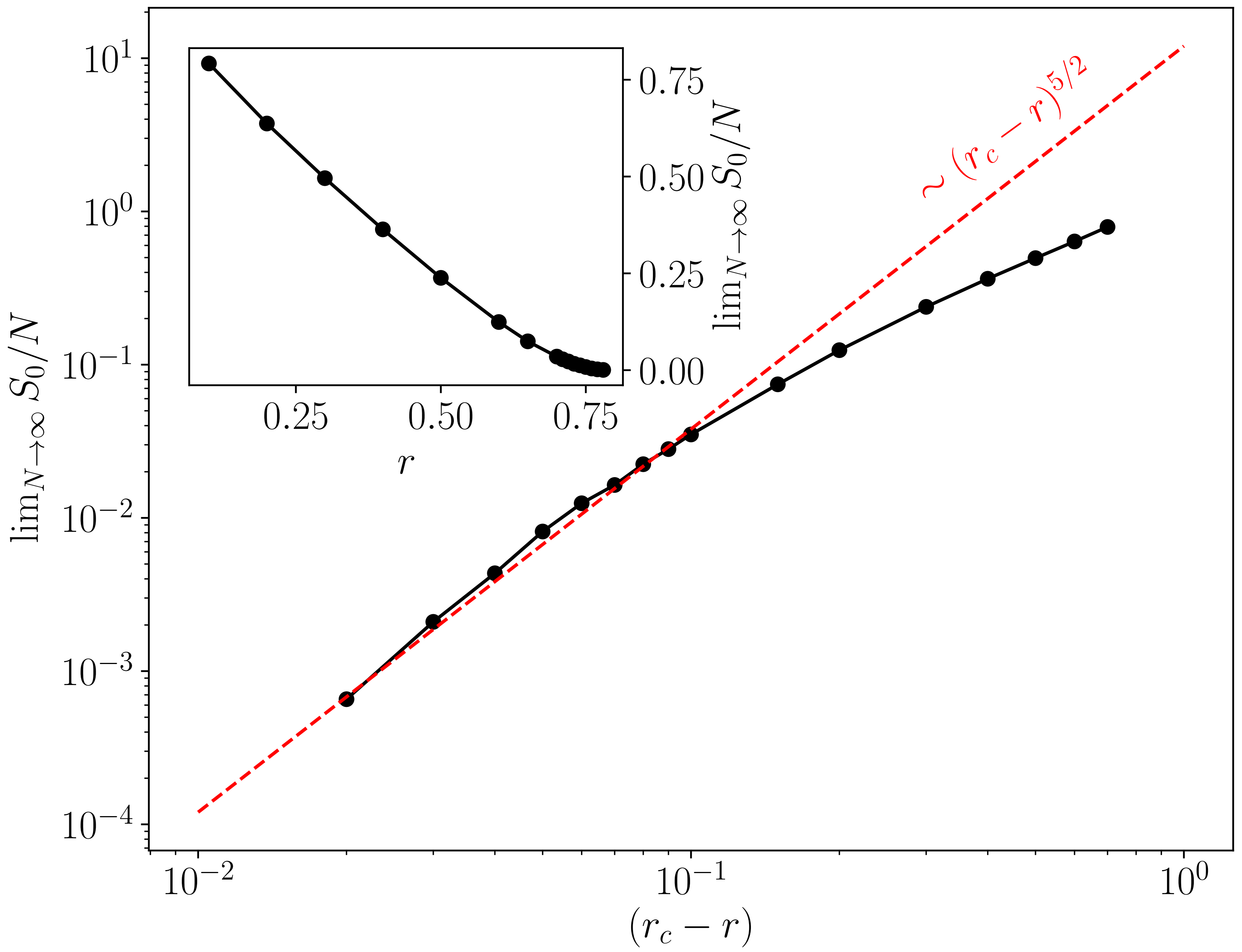
For the tree to survive forever

$$3(1-p) > 1$$

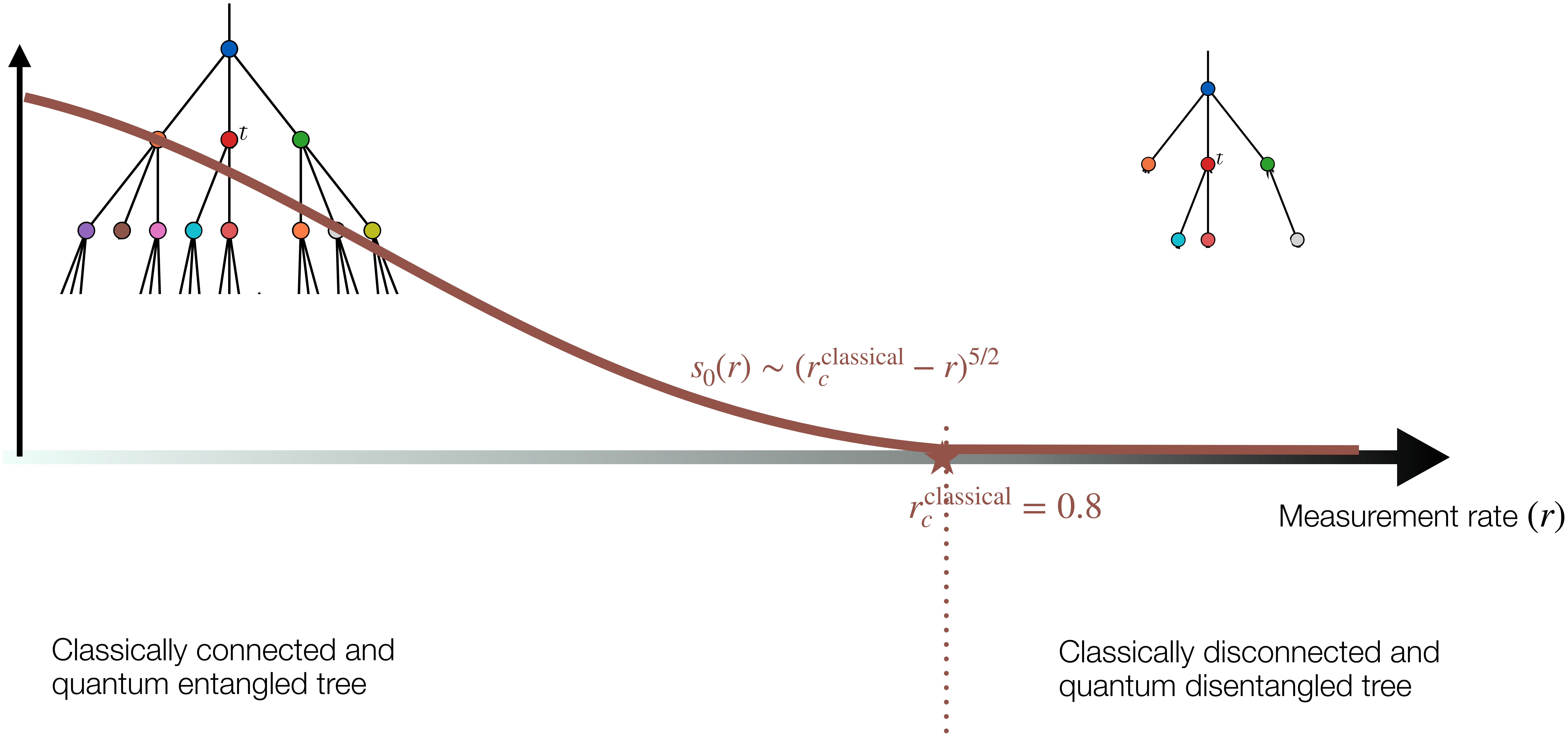
$$p_c = \frac{2}{3} \qquad r_c^{\text{classical}} = \frac{4}{5}$$

Critical scaling:

- $s_0(r) \sim (r_c^{\text{classical}} - r)^{5/2}$
- analytically from by mapping to layered Erdös-Rényi graphs



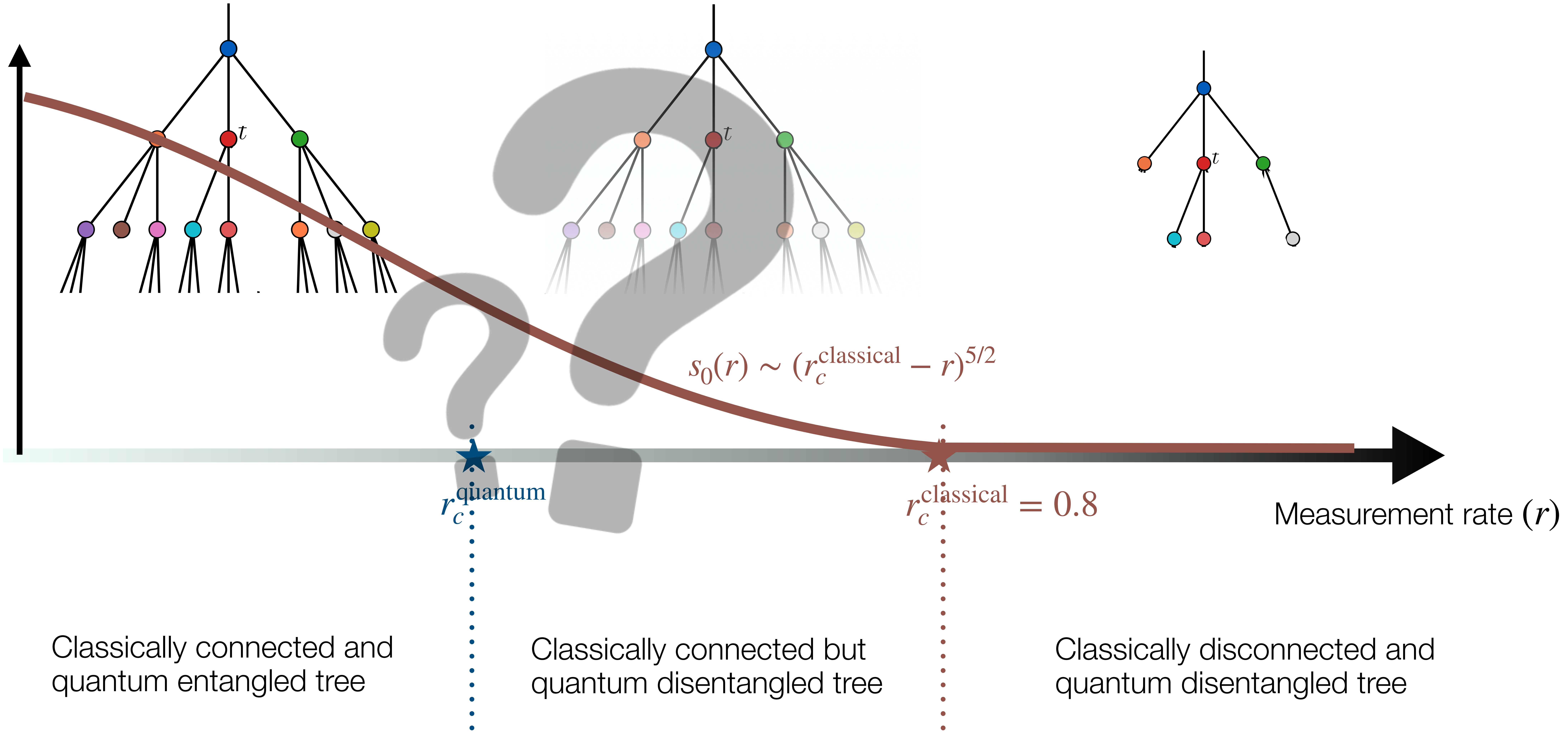
Phase diagram so far…



Classically connected and quantum entangled tree

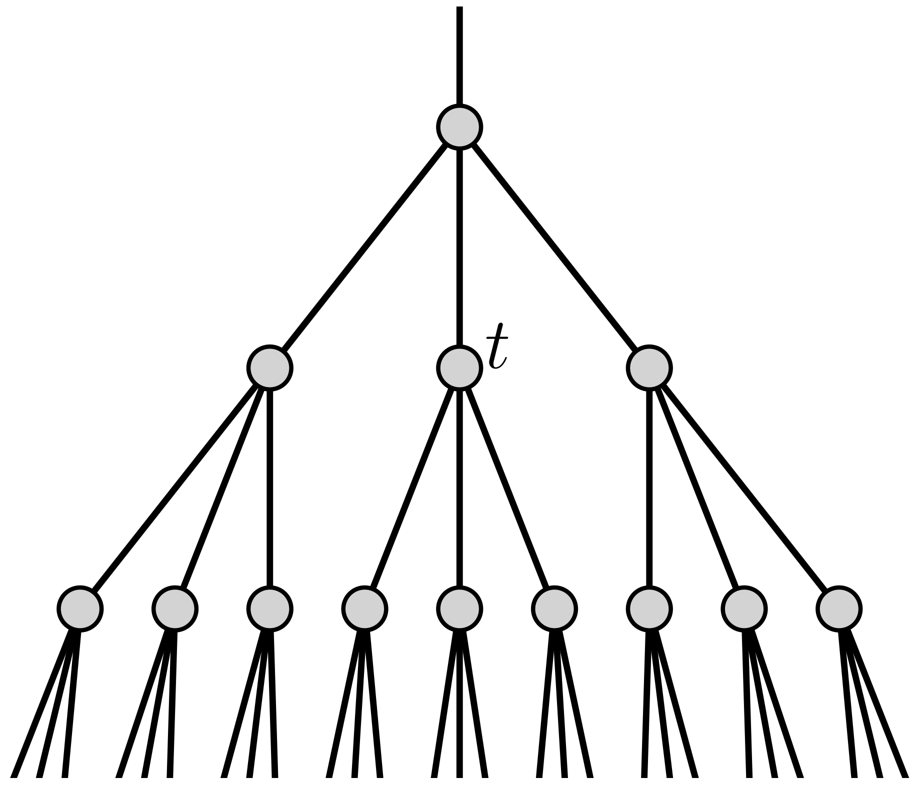
Classically disconnected and quantum disentangled tree

Phase diagram so far…

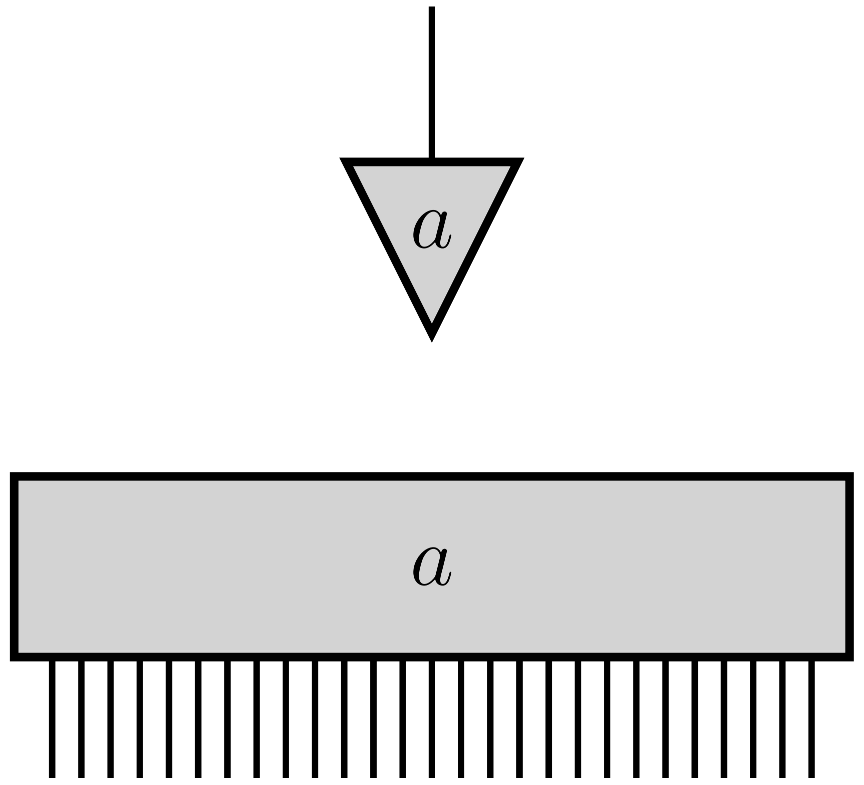


Entanglement in a tree tensor network

Quantum information flowing between the base and the apex of tree ?



$$= \sum_{a=1}^{2} \lambda_a$$



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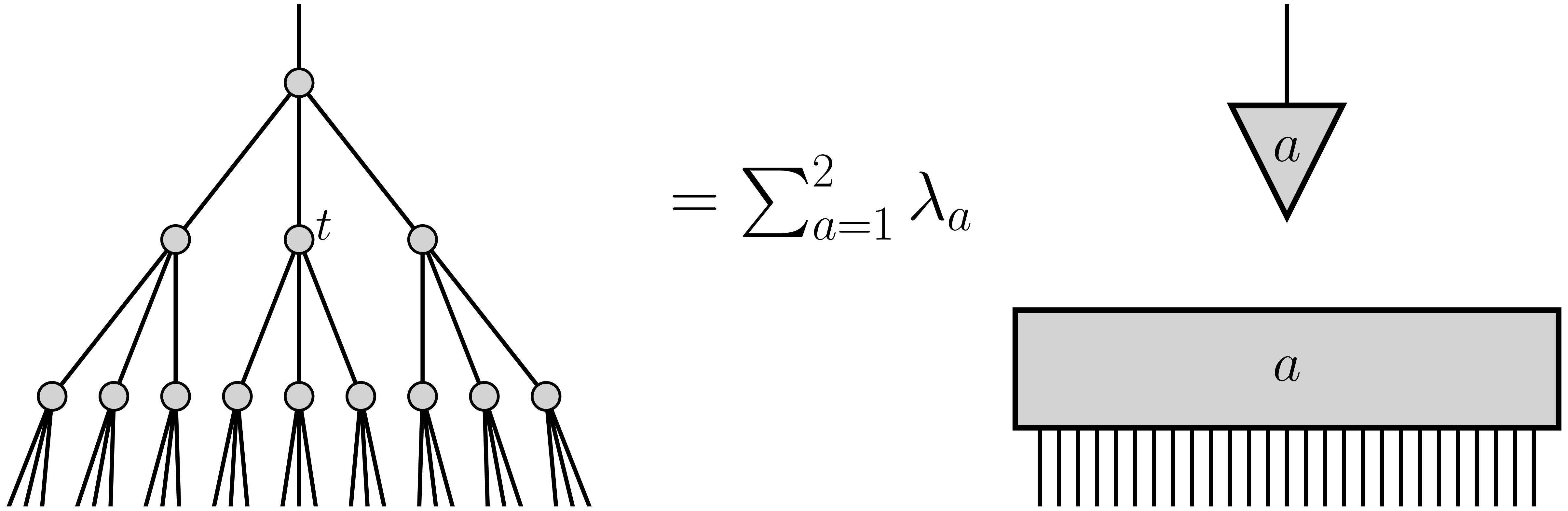


- quantified by the entanglement entropy between apex and base of tree
- Bond dimension of 2; two normalised singular values

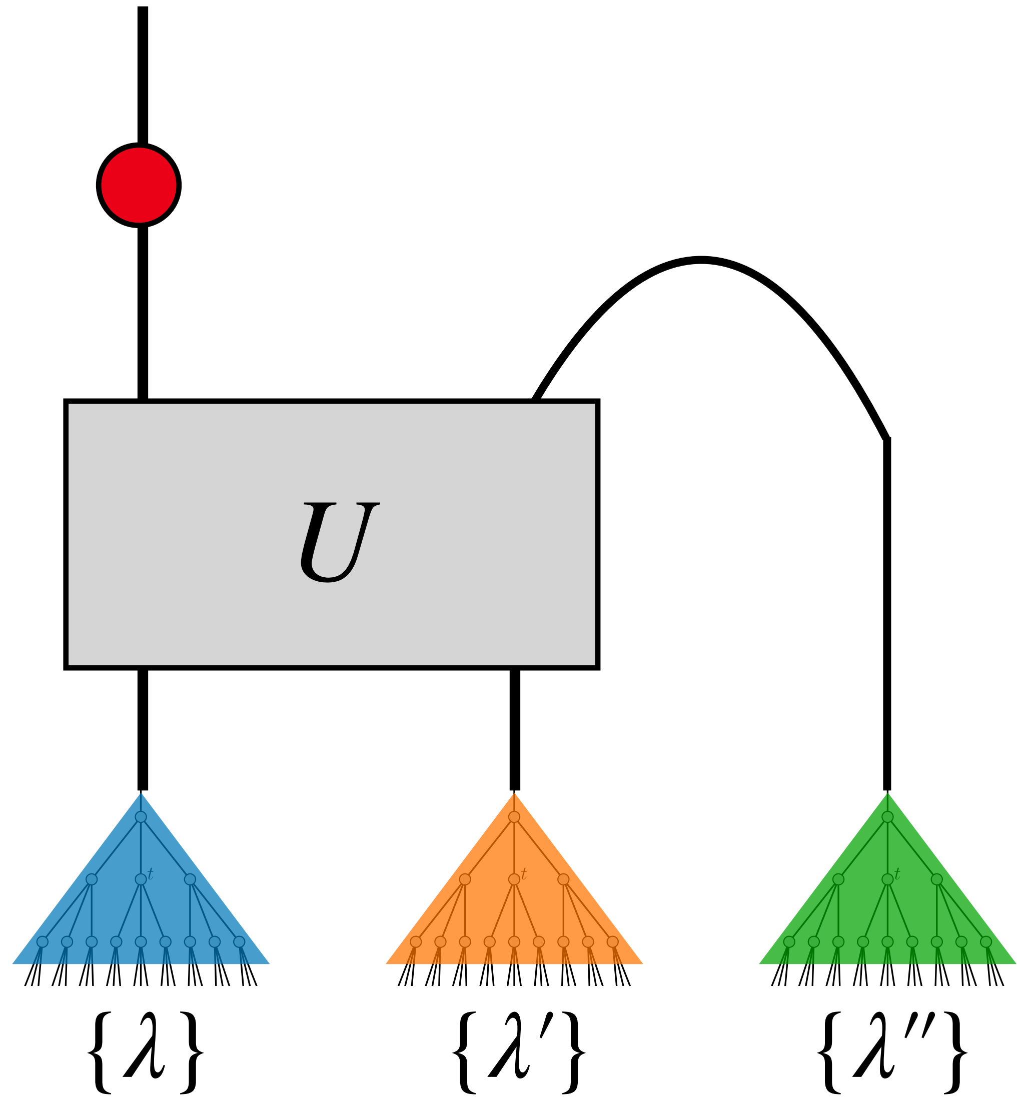
$$\lambda_{\max}^2 + \lambda_{\min}^2 = 1$$

- For a tree with k generations, denote $Z_k = \lambda_{\min}^2$
- Rényi entropy between apex and base

$$S_n = \frac{1}{1-n} \ln[Z_k^n + (1 - Z_k)^n]$$

Recursion relation for entanglement in a tree tensor network

A tree with $k+1$ generations can be generated from the singular values of three trees with k generations



$$T_{k+1} = \sum_{a,b,c,d=1}^{2} t^a_{bcd} \lambda_b \lambda'_c \lambda''_d |a\rangle_{\text{top}} \langle bcd|_{\text{bottom}}$$

Recursion relation for the singular values

- Non-linear recursion relation

$$Z_{k+1} = \begin{cases} F(Z_k, Z'_k, Z''_k) & \text{with probability } 1-p \\ 0 & \text{with probability } p \end{cases}$$

- Near a phase transition, $Z_{k+1} \ll 1$, so study a *linearised* recursion relation

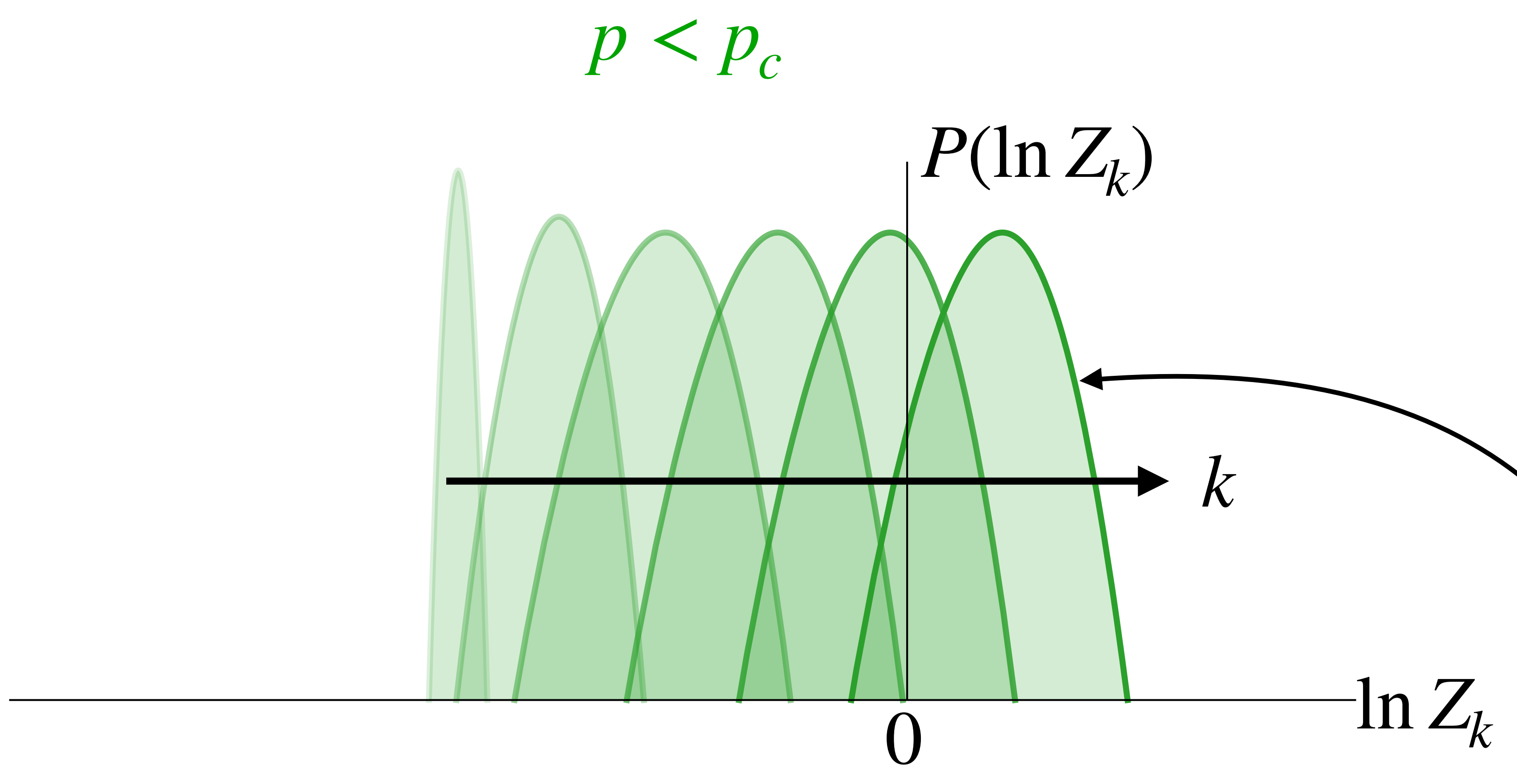
$$Z_{k+1} = \begin{cases} A_1 Z_k + A_2 Z'_k + A_3 Z''_k & \text{with probability } 1-p \\ 0 & \text{with probability } p \end{cases}$$

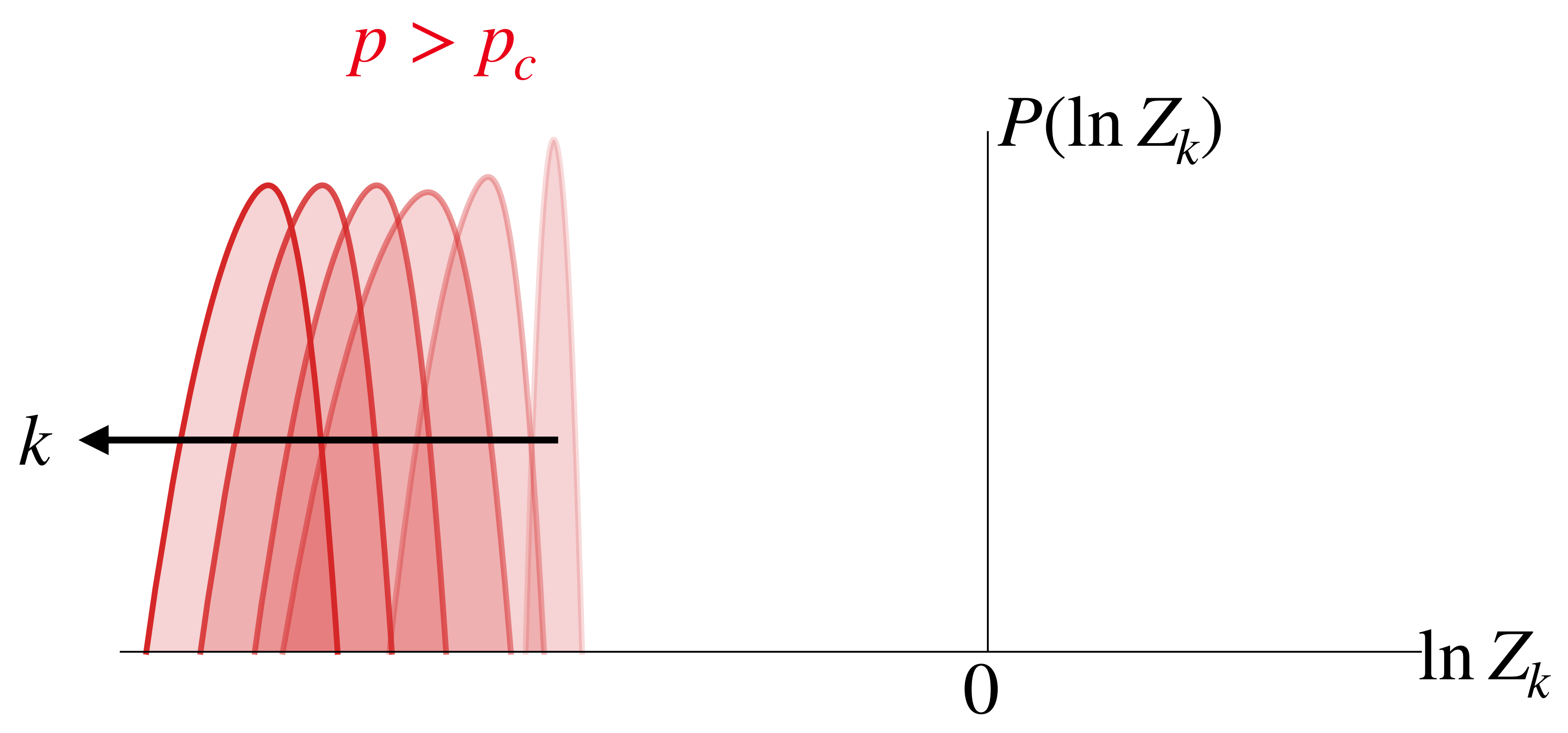
- F and A_i depend explicitly on the matrix elements of U (analytically tractable)

Recursion relation for entanglement in a tree tensor network

Linear recursion relation

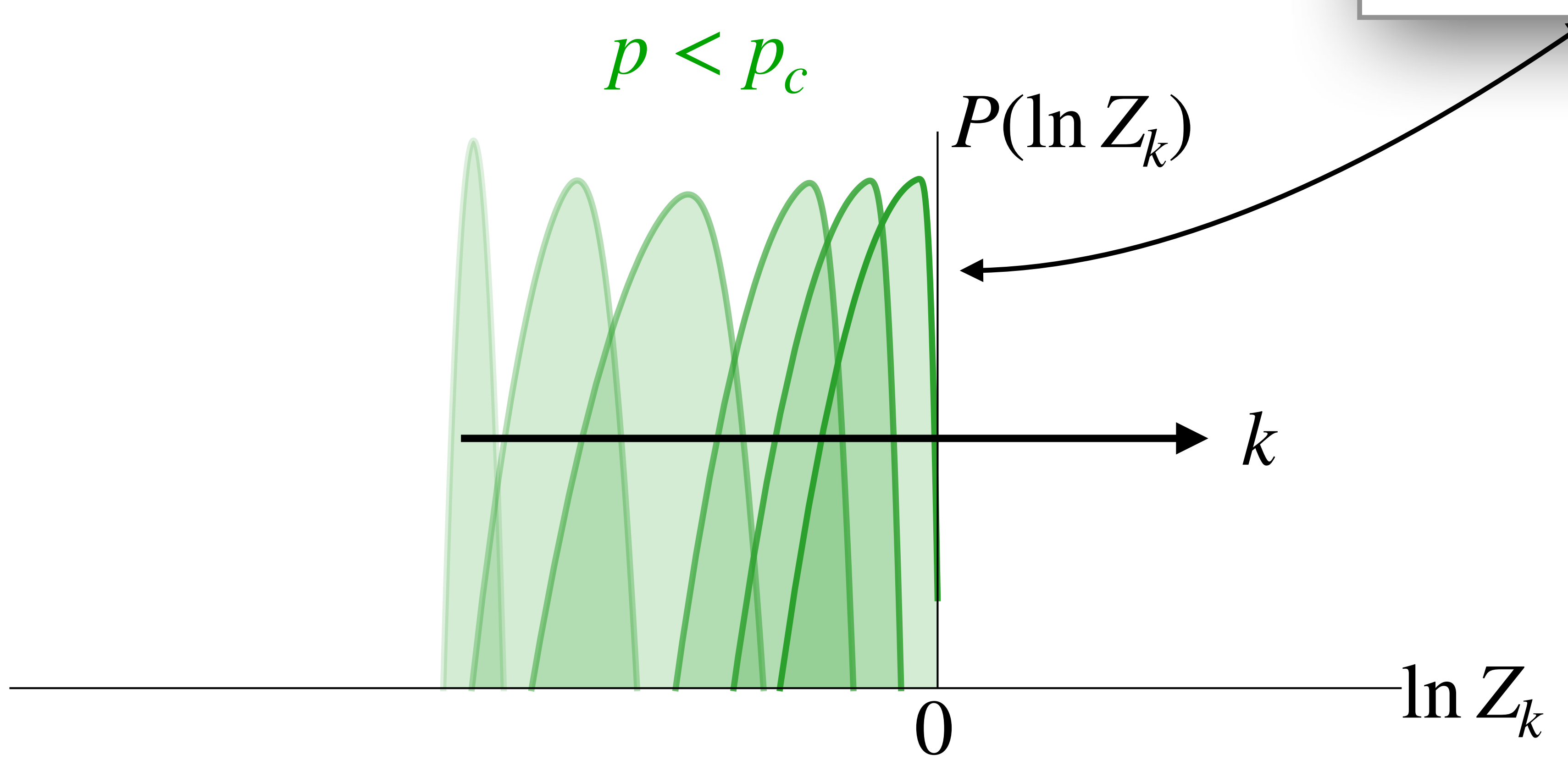
$$S_n = \frac{1}{1-n} \ln[Z_k^n + (1 - Z_k)^n]$$

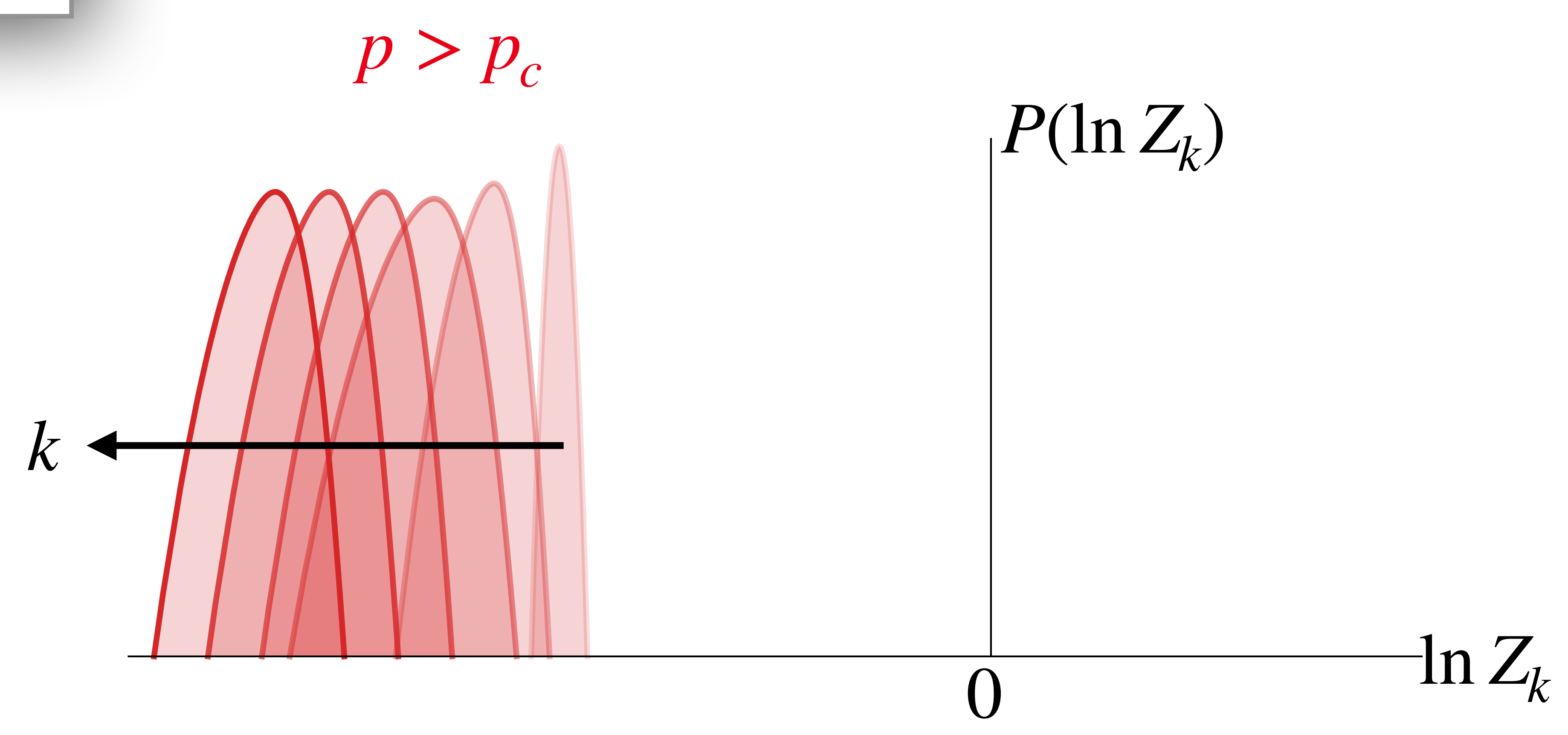




Non-linearity restricts $Z_{k\to\infty}$ to $\mathcal{O}(1)$ values

Non-linear recursion relation





Recursion relation for entanglement in a tree tensor network

- Linear recursion related to a travelling wave equation

  Derrida+Spohn'88

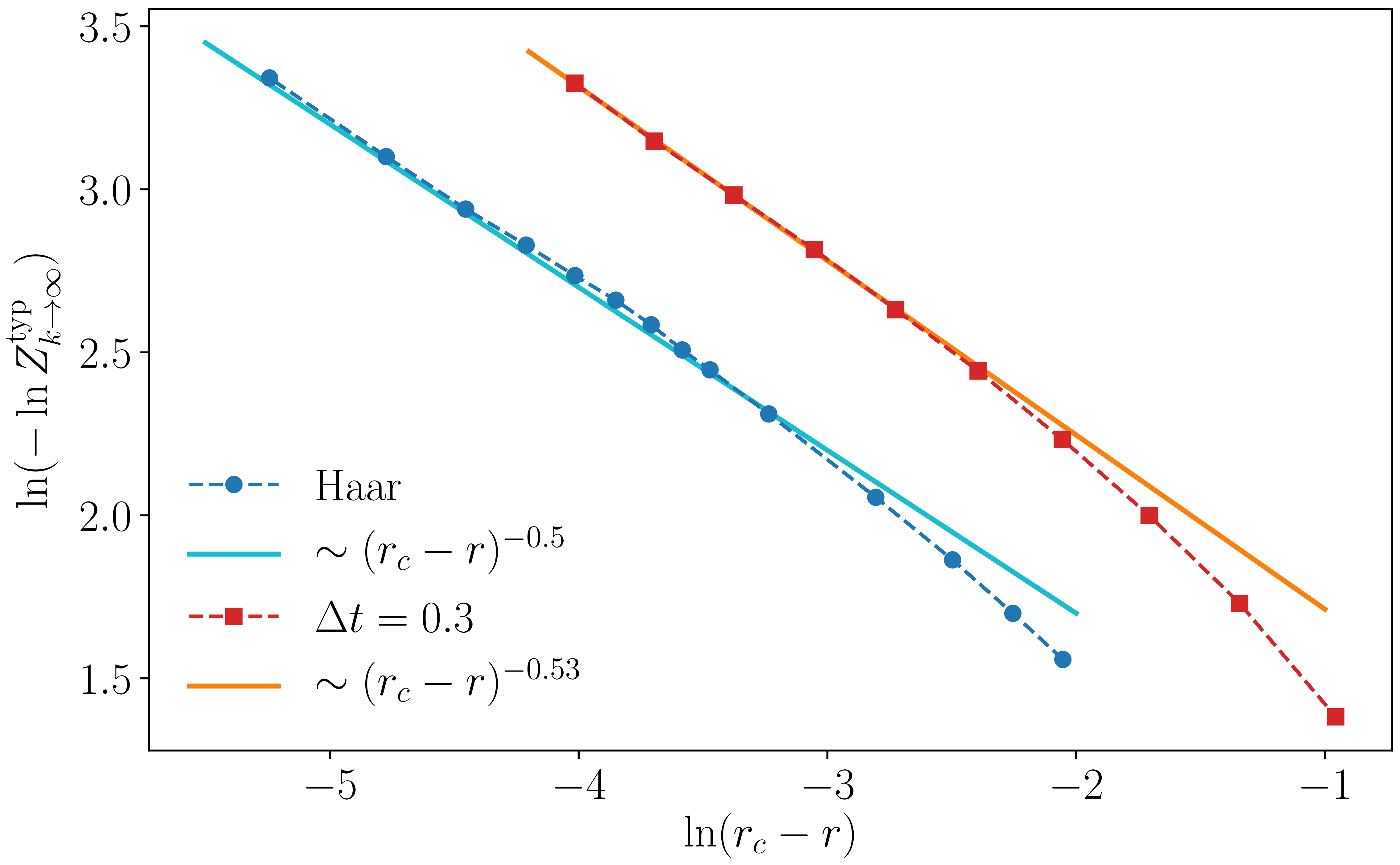
- $\ln Z_{\text{typ}}(k) \to$ front of travelling wave in fictitious time k
- Considerations of velocity selection give at criticality

$$c_{p_c} = 2\ln\left[(1-p_c)\sum_{i=1}^{3}\langle A_i^{1/2}\rangle\right]$$

- Critical point thus obtained from $c_{p_c} = 0$ as

$$p_c = 1 - 1/\sum_{i=1}^{3}\langle A_i^{1/2}\rangle$$

- The averages $\langle A_i^{1/2}\rangle$ can be computed analytically for the Haar ensemble yielding the exact critical point

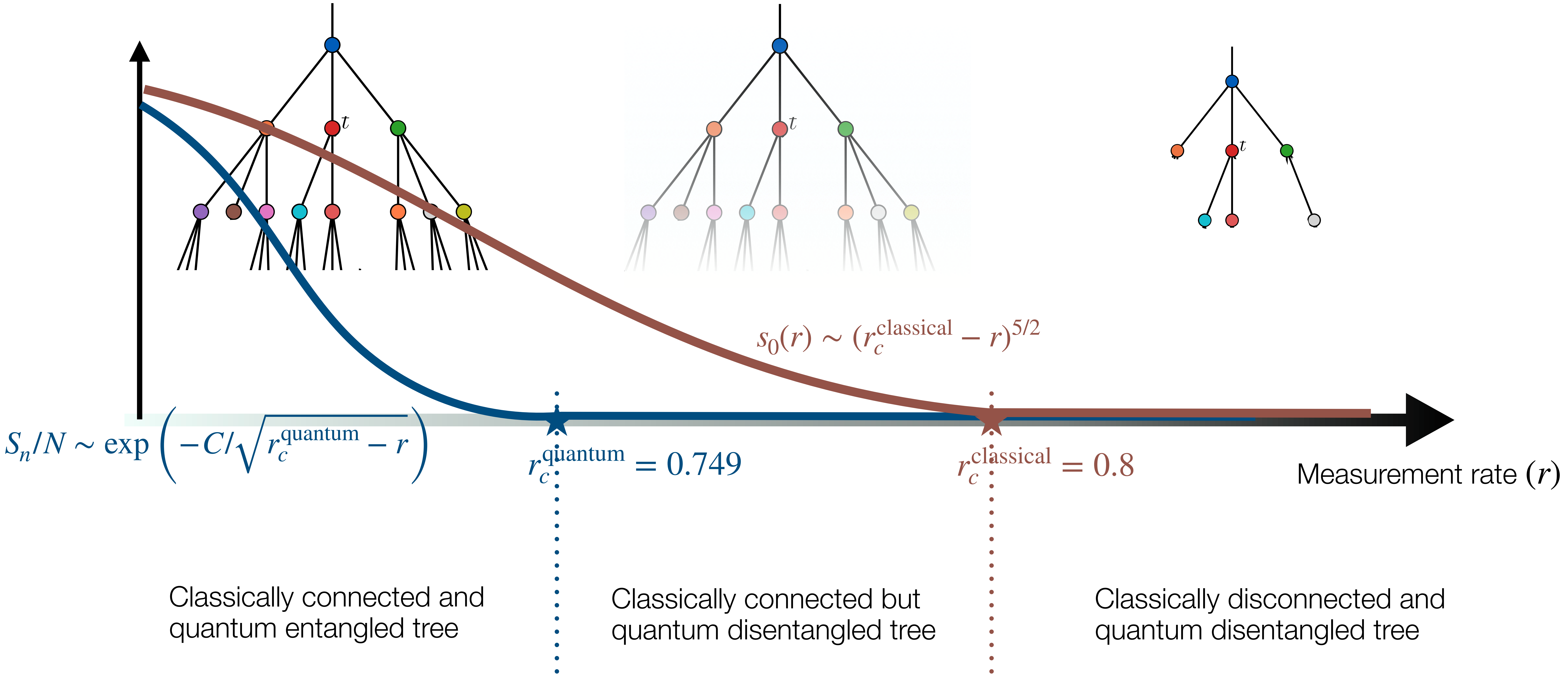


- Travelling-wave problem in the presence of the non-linearity to leading order maps onto a variant of Fisher-KPP equation
- Critical point stays the same as the linear recursion's travelling wave equation
- Critical scaling :

$$r_c = \frac{212 + 75\pi}{362 + 75\pi} \approx 0.749$$

$$Z_{\text{typ}}(k \to \infty) \sim \exp\left(-C/\sqrt{r_c - r}\right)$$

Completing the phase diagram



Summary

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