On-shell functions on the Coulomb branch of $\mathcal{N} = 4$ SYM

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Outline

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Motivation

- Most of the developments in on-shell methods have been for massless particles, with the amplitudes of $\mathcal{N} = 4$ SYM as the prototype/simplest amplitudes.
- Massive spinor-helicity formalism in a little group covariant manner was developed in 2017 by Arkani-Hamed, Huang and Huang.
- Extended to include on-shell supersymmetry by Herderschee, Koren and Trott in 2019. In a parallel paper, they also developed the formalism to compute $\mathcal{N} = 4$ SYM Coulomb branch tree amplitudes for a general gauge group breaking $U(\sum N_i) \to \prod_i U(N_i)$.
- They showed that under the massive super-BCFW shift, the tree amplitudes of this theory are BCFW constructible.

- Using old massive spinor helicity techniques, it was known that triangle master integralss don't contribute for one-loop amplitudes by Boels in 2010 and using dual conformal invariance, coefficients of box diagrams were computed by Alday, Henn, Plefka, Schuster 2009.
- Can one compute these loop amplitude coefficients by using modern on-shell methods? In this talk, we will discuss some developments in this direction done by us.

- Further for massless planar $\mathcal{N} = 4$ SYM case there exists an on-shell function formulation, which manifests the fact that loop integrands are dlog forms.
- On-shell functions are obtained by joining three point amplitudes and all the internal legs are also on-shell with integration over their Lorentz Invariant Phase Space (LIPS).
- In this approach, the BCFW bridge construction plays a key role, and is a first step towards manifesting the dlog representation of the loop integrand and lead to the Grassmannian description of the scattering amplitudes. We will perform this construction for the Coulomb branch.

Outline

Spinor-helicity variables

- Little group covariant massive spinor-helicity formalism:

$$p_{\alpha\dot{\beta}} = \sum_I |p_I]_\alpha \langle p^I|_{\dot{\beta}}, \tag{1}$$

  where $I = 1, 2$ are $SL(2, \mathbb{C})$ little group indices and we impose, $\langle p^I p^J \rangle = m\epsilon^{IJ}, \quad [p^I p^J] = -m\epsilon^{IJ}.$
- One can represent the $1/2-$BPS multiplet by using the long multiplet of $\mathcal{N} = 2$ supersymmetry.
- On-shell superfield:

$$\mathcal{W} = \phi + \eta_I^a \psi_a^I - \frac{1}{2}\eta_I^a \eta_J^b (\epsilon^{IJ}\phi_{(ab)} + \epsilon_{ab} W^{(IJ)}) + \frac{1}{3}\eta_I^b \eta_{Jb}\eta^{Ja}\tilde{\psi}_a^I + \eta_1^1 \eta_1^2 \eta_2^1 \eta_2^2 \tilde{\phi} \tag{2}$$

- Four point amplitude was constructed using super-BCFW by Herderschee, Koren and Trott:

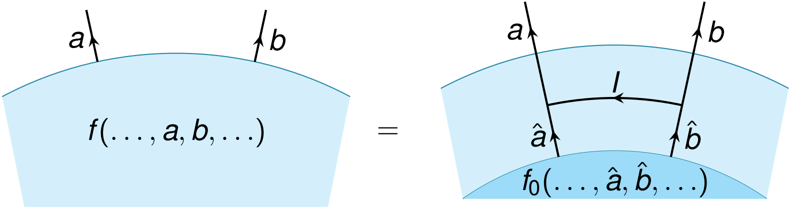
$$A^{(4)}(W, \overline{W}, W, \overline{W}) = \frac{\delta^{(4)}(Q)\delta^{(4)}(Q^{\dagger})}{s_{12}s_{14}}, \tag{3}$$

where $s_{ij} = -(p_i + p_j)^2 - (m_i \pm m_j)^2$.
- For non-zero amplitudes, the masses need to obey a condition due to central charge conservation/color neutrality: $\sum m_{\text{BPS}} = \sum m_{\text{anti-BPS}}$.
- In the non supersymmetric massless case, some BCFW shifts were constructible. On the other hand, tree level non supersymmetric massive amplitudes are not BCFW constructible. Shifts in the Grassmann variables renders the super-amplitude BCFW constructible.

Outline

BCFW bridge on the Coulomb branch

- There are both massive and massless (due to unbroken generators) spin-one fields on the Coulomb branch.
- There are various possible on-shell diagrams. One has to satisfy central charge conservation for each three point amplitude.
- The bridge can be added to two massless legs (BCFW), one massive and one massless legs (massive-massless shift by Ballav, Manna), two massive legs (massive super-BCFW shift by Herderschee, Koren, Trott).
- One can also have a more general situation where the bridge momentum p_I is massive. We are able to relate the bigger on-shell functions to smaller ones in all these cases.



- The relation is,

$$f(\ldots, a, b, \ldots) = \sqrt{\frac{s_{ab}}{s_{ab} - 4m_a m_b}} \int \frac{\mathrm{d}z}{z}\, f_0(\ldots, \hat{a}, \hat{b}, \ldots) \tag{4}$$

- The intermediate (bridge) momentum is,

$$p_I = -z\,|b^2\rangle[a^2| + \frac{2m_I}{\sqrt{s_{ab} - 4m_a m_b}}\Big(|b^2\rangle[a^1| + |b^1\rangle[a^2|\Big) - \frac{1}{z}\frac{m_I^2}{m_a m_b}\left(\frac{s_{ab}}{s_{ab} - 4m_a m_b}\right)|b^1\rangle[a^1| \tag{5}$$

- In terms of the three particle special kinematics u-variables the left and right three-point amplitudes dictate that,

$$\begin{aligned} p_I &\sim |u_L\rangle[\bullet| + |\bullet\rangle[u_L| \\ p_I &\sim |\bullet\rangle[u_R| + |u_R\rangle[\bullet|. \end{aligned} \tag{6}$$

- We indeed obtained the bridge momentum to be,

$$p_I = \frac{1}{\sqrt{s_{ab}}}\big(|u_L\rangle[u_R| - |u_R\rangle[u_L|\big) \tag{7}$$

- Whenever $m_I = 0$, we have two equal mass and one massless three particle special kinematics described by Arkani-Hamed, Huang, Huang.

$$2p_a \cdot p_I = [I|p_a|I\rangle = 0 \implies |I] \propto p_a|I\rangle. \tag{8}$$

Similarly for $p_{\hat{a}}$. Analogous relations also appear for the right hand three point amplitude. Therefore, we have,

$$x_L\, p_a|I] = -m_a|I\rangle\ , \qquad\qquad x_R\, m_b|I] = p_b|I\rangle. \tag{9}$$

- We find that $\frac{x_L}{x_R}$ is fixed in terms of external kinematics. The other independent combination gives the BCFW z deformation. Thus we have an interpretation of the z variable in terms of three particle special kinematics. For the most general case, we have an interpretation in terms of three particle special BPS kinematics u-spinors.

Quadruple cut at one-loop

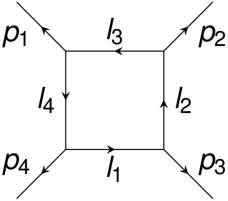
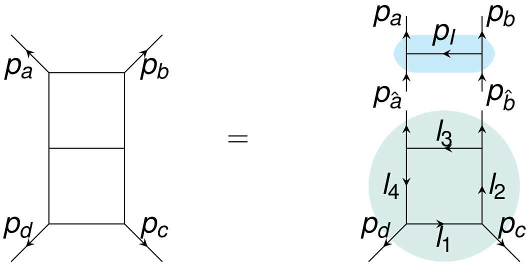


Figure: Generic one loop box diagram

The maximal cut for the diagram reads as,

$$\text{Maximal Cut}\left(\mathcal{A}_4^{1\text{ loop box}}\right) = \mathcal{A}_4^{\text{tree}}\sqrt{\frac{s_{12}s_{14}}{s_{12}s_{14} - 4m_{l_1}m_{l_3}s_{12} - 4m_{l_2}m_{l_4}s_{14}}} \tag{10}$$

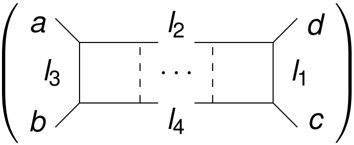
Maximal cuts at higher loops



$$\text{Max Cut}\left(\mathcal{A}_4^{2\text{ boxes}}[a,b,c,d]\right)$$

$$= \mathcal{A}_4^{\text{tree}} \frac{s_{ab}\, s_{ad}}{\sqrt{s_{ab} - 4m_a m_b}} \int \frac{\mathrm{d}z}{z} \frac{1}{\sqrt{s_{\hat{a}d}\left(s_{ab}s_{\hat{a}d} - 4m_{l_1}m_{l_3}s_{ab} - 4m_{l_2}m_{l_4}s_{\hat{a}d}\right)}} \tag{11}$$

- Max Cut



$$= \mathcal{A}_4^{\text{tree}} \frac{s_{ab}^{n/2}}{(s_{ab} - 4m_{l_2}m_{l_4})^{(n-1)/2}} \sqrt{\frac{s_{ad}}{s_{ab}s_{ad} - 4m_{l_2}m_{l_4}s_{ad} - 4m_{l_1}m_{l_3}s_{ab}}} \tag{12}$$

Note that n is the number of ‘boxes’ in the graph.

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Future directions

- Understanding dual conformal invariance of the Coulomb branch amplitudes and building higher point tree and loop amplitudes with this guidance.
- With Veronica Calvo Cortes, Yassine El Maazouz, Amit Suthar, we have a symplectic Grassmannian description of the three point Coulomb branch amplitude and the general structure of the higher point amplitudes cf. Amit Suthar's talk.
- The BCFW bridge construction we discussed will provide guidance on how to amalgamate these three point symplectic Grassmannians and perhaps the appropriate choice of minors for the higher point symplectic Grassmannian integral.

- Thank you!