Quadratic twist of epsilon factor of symmetric cube transfers of modular forms.

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Introduction

- Let F be a number field.
- Consider the ring of adèles $\mathbb{A}_F = \{(x_v)_v \in \prod_{v \leq \infty} F_v | x_v \in \mathcal{O}_v$ for almost all finite $v\}$. Here $\mathcal{O}_v$ is the ring of integers of F_v. Let $\mathfrak{p}_v$ be the maximal ideal, $q_v = |\mathcal{O}_v/\mathfrak{p}_v|$.
- Let χ be a character of $\mathbb{A}_F^\times / F^\times$. Then $\chi = \otimes_v \chi_v$.
- Let $\phi_v \in \widehat{F_v}$ be non-trivial. Tate [1950] associated the local ε-factor with the local L-function by:

$$\varepsilon(s, \chi_v, \phi_v) = \frac{\gamma(s, \chi_v, \phi_v) L(s, \chi_v)}{L(1-s, \chi_v^{-1})},$$

where $\gamma(s, \chi_v, \phi_v,) \in \mathbb{C}(q_v^{-s})$.

- $\varepsilon(s, \chi_v, \phi_v)\ \varepsilon(1-s, \chi_v^{-1}, \phi_v) = \chi_v(-1)$.
- $\varepsilon(s, \chi_v, \phi_v) = q_v^{(1/2-s)n(\chi_v, \psi_v)} \varepsilon(1/2, \chi_v, \phi_v), n(\chi_v, \psi_v) \in \mathbb{Z}$.
- $L(s, \chi) = \epsilon(s, \chi) L(1-s, \chi^{-1})$.

- Let $a(\chi_v)$ be the smallest positive integer such that $\chi_v|_{1+\mathfrak{p}_v^{a(\chi_v)}} = 1$.
- Let $n(\phi_v) \in \mathbb{Z}$ such that $\phi_v|_{\mathfrak{p}_v^{-n(\phi_v)}} = 1$ but $\phi_v|_{\mathfrak{p}_v^{-n(\phi_v)-1}} \neq 1$.
- We have,

$$\varepsilon(\chi_v, \phi_v) = q_v^{-\frac{a(\chi_v)}{2}} \chi_v(c) \sum_{x \in \frac{\mathcal{O}_v^\times}{1+\mathfrak{p}_v^{a(\chi_v)}}} \chi_v^{-1}(x)\phi_v(\frac{x}{c})$$

where $c \in F_v^\times$ has valuation $a(\chi_v) + n(\phi_v)$.

Property of ε-factors:

1. $\varepsilon(\chi_v\theta_v, \phi_v) = \theta(\pi_v)^{a(\chi_v)+n(\phi_v)}\varepsilon(\chi_v, \phi_v)$, where θ_v is an unramified character of $F_v^\times$. The element π_v is a uniformizer of F_v.
2. $\varepsilon\left(\mathrm{Ind}_{W_K}^{W_{\mathbb{Q}_p}} \rho, \phi\right) = \varepsilon\left(\rho, \phi \circ \mathrm{Tr}_{K/\mathbb{Q}_p}\right)$, where ρ denotes a multiplicative character of a finite extension $K/\mathbb{Q}_p$.

- For an odd prime p, set $p^* := \left(\frac{-1}{p}\right) \cdot p$.
- $\mathbb{Q}(\sqrt{p^*})/\mathbb{Q}$ is ramified only at p.
- Let χ denote the quadratic character attached to $\mathbb{Q}(\sqrt{p^*})$.
- χ can be identified with a character of the idèle group, i.e., characters $\{\chi_q\}_q$ with $\chi_q : \mathbb{Q}_q^\times \to \mathbb{C}^\times$ satisfying the following conditions:

  1. For $q \neq p$, the character χ_q is unramified and $\chi_q(q) = \left(\frac{q}{p}\right)$.
  2. For $q = p$, $a(\chi_p) = 1$ and $\chi|_{\mathbb{Z}_p^\times}$ factors through the unique quadratic character of $\mathbb{F}_p^\times$ with $\chi_p(p) = 1$.

- Let $f \in S_k(N, \epsilon)$ be a cusp form of weight k, level N and nebentypus ϵ. Write $N = p^{N_p} N'$ with $p \nmid N'$.
- Let π_f be the automorphic representation of the adèle group $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ attached to $f \in S_k(N, \epsilon)$. We have

$$\pi_f = \bigotimes_p \pi_{f,p}.$$

- $\pi_{f,p}$ can be principal series, special or supercuspidal representation.
- Consider the variance number:

$$\varepsilon_p := \frac{\varepsilon\left(\pi_{f,p} \otimes \chi_p\right)}{\varepsilon\left(\pi_{f,p}\right)}.$$

Theorem (A. Pacetti, [1], 2013)

Let $f \in S_k(N)$ and $p|N$ be an odd prime with $\pi_{f,p}$ be the ramified supercuspidal representation i.e. $\pi_{f,p} = \mathrm{Ind}_{W_K}^{W_{\mathbb{Q}_p}}(\varkappa)$ with $[K/\mathbb{Q}_p] = 2$. Then,

1. $K = \mathbb{Q}_p[\sqrt{p^\star}]$ *if* $\varepsilon(\pi_f \otimes \chi_p) = \chi_p(N')\varepsilon(\pi_f)$.
2. $K = \mathbb{Q}_p[\delta\sqrt{p^\star}]$ *if* $\varepsilon(\pi_f \otimes \chi_p) = -\chi_p(N')\varepsilon(\pi_f)$ *where* δ *is any non-square.*

- Banerjee & Mandal [2020] generalized the results of [1] for arbitrary nebentypus ϵ.

sym^3 transfer

- Let $\pi = \bigotimes_p \pi_p$ be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$.
- For each prime p, let ϕ_p be the two dimensional representation of the Weil-Deligne group attached to π_p.
- consider the third symmetric power $\mathrm{sym}^3 : \mathrm{GL}_2 \to \mathrm{GL}_4$ of the standard representation.
- Then $\mathrm{sym}^3 \circ \phi_p$ is a four dimensional representation of the Weil-Deligne group. Using Local Langlands correspondence, this gives an irreducible representation of $\mathrm{GL}_4(\mathbb{Q}_p)$, denoted by $\mathrm{sym}^3(\pi_p)$.
- Define

$$\mathrm{sym}^3(\pi) := \bigotimes_p \mathrm{sym}^3(\pi_p).$$

Due to Kim & Shahidi (2002), $\mathrm{sym}^3(\pi)$ is an automorphic representation of $\mathrm{GL}_4(\mathbb{A}_\mathbb{Q})$.

Supercuspidal types

Let π_p be a supercuspidal representation of $\mathrm{Gl}_2(\mathbb{Q}_p)$ with p odd. Then $\pi_p = \mathrm{Ind}_{W_K}^{W_{\mathbb{Q}_p}}(\varkappa)$ with $[K : \mathbb{Q}_p] = 2$ and $\varkappa \in \widehat{K^\times}$ which is not trivial on $\ker(N_{K/\mathbb{Q}_p}(K^\times))$. Then,

$$\mathrm{sym}^3(\pi_p) = \mathrm{Ind}_{W_K}^{W_{\mathbb{Q}_p}}(\varkappa^3) \oplus \mathrm{Ind}_{W_K}^{W_{\mathbb{Q}_p}}(\varkappa^2\varkappa^\sigma), \quad \sigma \in W_{\mathbb{Q}_p} \backslash W_K.$$

We have the following types:

- **(Type I):** $\mathrm{Ind}_{W_K}^{W_{\mathbb{Q}_p}}(\varkappa^3)$ is irreducible and it is isomorphic to $\mathrm{Ind}_{W_K}^{W_{\mathbb{Q}_p}}(\varkappa^2\varkappa^\sigma)$.
- **(Type II):** $\mathrm{Ind}_{W_K}^{W_{\mathbb{Q}_p}}(\varkappa^3)$ is irreducible and it is not isomorphic to $\mathrm{Ind}_{W_K}^{W_{\mathbb{Q}_p}}(\varkappa^2\varkappa^\sigma)$.
- **(Type III):** $\mathrm{Ind}_{W_K}^{W_{\mathbb{Q}_p}}(\varkappa^3)$ is reducible.

Main Object

- Let $f \in S_k(N, \epsilon)$ be a newform with $N = p^{N_p} N', p \nmid N'$, $\epsilon = \epsilon_p \cdot \epsilon'$.
- Let $\pi_f = \otimes \pi_{f,p}$ be the attached automorphic representation.
- We aim to study:

$$\varepsilon_p := \frac{\varepsilon\left(\mathrm{sym}^3(\pi_{f,p}) \otimes \chi_p\right)}{\varepsilon\left(\mathrm{sym}^3(\pi_{f,p})\right)}.$$

- f is called p-minimal if the p-part of its level is the smallest among all its twists by Dirichlet character.

Proposition (Banerjee, Mandal, —, 2023)

Let f be a p-minimal newform with a_p being the p-th Fourier coefficient. Let $\pi_{f,p}$ be a ramified principal series representation.

- *Let $p \geq 5$. If $N_p > 1$, then the number*

$$\varepsilon_p = \begin{cases} \chi_p(c) p^{\frac{3-3k}{2}} a_p^3, & \text{if } p \equiv 1 \pmod 4, \\ i\chi_p(c) p^{\frac{3-3k}{2}} a_p^3, & \text{if } p \equiv 3 \pmod 4, \end{cases}$$

  where c has valuation $-3(N_p - 1)$ satisfying a "certain" property.

If $\pi_{f,p}$ is a special representation with $p \geq 3$, then ε_p is given by

$$\varepsilon_p = -p^{\frac{8-3k}{2}} a_p^3.$$

Proposition (Banerjee, Mandal, —, 2023)

Let p be an odd prime such that $\pi_{f,p} = \mathrm{Ind}_{W(K)}^{W(\mathbb{Q}_p)}(\varkappa)$, where $[K : \mathbb{Q}_p] = 2$. Then, ε_p is given as follows:

- *Assume that $K/\mathbb{Q}_p$ is ramified. For Type I and II representations, we have $\varepsilon_p = 1$ when $p \geq 5$ or $p = 3$ with $a(\varkappa) \equiv a(\varkappa^3) \pmod 2$. If $p = 3$ with $a(\varkappa) \not\equiv a(\varkappa^3) \pmod 2$, then we have*

$$\varepsilon_3 = \begin{cases} 1, & \text{if } (3, K/\mathbb{Q}_3) = 1, \\ -1, & \text{if } (3, K/\mathbb{Q}_3) = -1. \end{cases}$$

- *Let $\mathrm{sym}^3(\pi_{f,p})$ be of Type III. This is possible only when $p = 3$. If $a(\varkappa^3) > 1$ then we have $\varepsilon_3 = \chi_3(d)$ or $-\chi_3(d)$ depending upon $(3, K/\mathbb{Q}_3) = 1$ or $(3, K/\mathbb{Q}_3) = -1$ respectively. Here, d has valuation $-a(\varkappa^3) + 1$ satisfying a "certain" property. Otherwise, $\varepsilon_3 = 1$.*

Theorem (Banerjee, Mandal, —, 2023)

Let $\pi_{f,p} = \mathrm{Ind}_{W_K}^{W_{\mathbb{Q}_p}} \varkappa$ *with* $K/\mathbb{Q}_p$ *quadratic. If* N_p *is even, then* $K = \mathbb{Q}_p(\zeta_{p^2-1})$ *is the unique unramified quadratic extension of* $\mathbb{Q}_p$*. For* $a(\epsilon_p) \neq \frac{N_p}{2}$*, if* $p \geq 5$ *then we have the following:*

1. $\mathrm{sym}^3(\pi_{f,p})$ *is of Type I or II if* $\varepsilon(\mathrm{sym}^3(\pi_f) \otimes \chi_p) = \chi_p(M')\chi'_p(s_1)\varepsilon(\mathrm{sym}^3(\pi_f))$.
2. $\mathrm{sym}^3(\pi_{f,p})$ *is of Type III if* $\varepsilon(\mathrm{sym}^3(\pi_f) \otimes \chi_p) = \chi_p(M's_2)\chi'_p(s_3)\varepsilon(\mathrm{sym}^3(\pi_f))$.

where M' *denote the prime-to-p part of* $a(\mathrm{sym}^3(\pi_f))$ *and* $s_1, s_2, s_3 \in K^\times$ *have valuations* $-a(\varkappa)+1, -a(\varkappa)-a(\varkappa^3)+2$ *and* $-2a(\varkappa^3)+2$ *respectively satisfying "certain" properties.*

Theorem (Banerjee, Mandal, —, 2023)

Let $\pi_{f,p}$ be as above. If N_p odd, then $K/\mathbb{Q}_p$ ramified. Suppose $3|N$ with $a(\varkappa^3) \geq 3$ odd, and $\mathrm{sym}^3(\pi_{f,3})$ is of Type I or II, then the corresponding ramified extensions are determined as follows:

1. $K = \mathbb{Q}_3[\sqrt{-3}]$ *if* $\varepsilon(\mathrm{sym}^3(\pi_f) \otimes \chi_3) = \chi_3(M')\varepsilon(\mathrm{sym}^3(\pi_f))$.
2. $K = \mathbb{Q}_3[\delta\sqrt{-3}]$ *if* $\varepsilon(\mathrm{sym}^3(\pi_f) \otimes \chi_3) = -\chi_3(M')\varepsilon(\mathrm{sym}^3(\pi_f))$.

where δ is a non-square and M' denotes the prime-to-p part of the conductor of $\mathrm{sym}^3(\pi_f)$.

- *We have similar classification of the ramified extensions when* $\mathrm{sym}^3(\pi_{f,3})$ *is of Type III.*

References

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[3] D. Banerjee, T. Mandal and S. Mondal, *Two properties of symmetric cube transfer of a modular form.* arXiv preprint arXiv:2304.14555 (2023).

Thank You