A Conformal Field Theoretical Study of Quantum Ising model with Longer Range Interaction

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Motivation

Motivation of the study is the following: It is well known in the literature of quantum Ising model that the minimal model of cft is sufficient to describe the central charge behaviour, now the question is it also sufficient to describe the central charge behaviour for the quantum Ising model with longer range interaction.

This model Hamiltonian has also several gapless quantum critical lines along with multicritical points, depending on the presence or absence of transverse field. We also raise the question whether there is any relation between the central charge and quantum criticality and the transverse field.

In this study we raise the queation whether there is any possibility to find the emergence of quantum Lifshitz transition. There are several examples of cft criticality, we also raise the question whether there is any possibility non-cft criticality?

The Basic Hamiltonian: Starting Point

$$H = -\sum_i (\mu\sigma_i{}^x + \lambda_1\sigma_i{}^z\sigma_{i+1}{}^z + \lambda_2\sigma_i{}^x\sigma_{i-1}{}^z\sigma_{i+1}{}^z). \quad (1)$$

Here μ is the transverse field, λ_1 is the two spin interaction of nearest-neighbour (NN) sites and λ_2 is the three spin interactions. Thus we term this model Hamiltonian as lqIm with three spin interaction.

A few more variants of Model Hamiltonian: (1). Anderson Pseudo-Spin Hamiltonian

$$H = -\mu \sum_{i=1}^{N} (1 - 2c_i^\dagger c_i) - \lambda_1 \sum_{i=1}^{N-1} (c_i^\dagger c_{i+1} + c_i^\dagger c_{i+1}^\dagger + h.c) - \lambda_2 \sum_{i=2}^{N-1} (c_{i-1}^\dagger c_{i+1} + c_{i+1} c_{i-1} + h.c), \quad (2)$$

Performing Jordan-Wigner transformation $\sigma_i^x = 1 - 2c_i^\dagger c_i$ and $\sigma_i^z = -\prod_{j<i}(1 - 2c_j^\dagger c_j)(c_i + c_i^\dagger)$

The model Hamiltonian can be expressed in terms of pseudo spin-vector. The transition can be verified by investigating behavior of pseudo spin-vector in the parameter space[26,28]. This model Hamiltonian can be expressed as,

$$\mathscr{H}(k) = \chi(k).\sigma \quad (3)$$

The Bloch Hamiltonian of Eq.2, which is a 2×2 matrix, can be written as

$$\mathscr{H}(k) = \chi_z(k)\sigma_z - \chi_y(k)\sigma_y, \quad (4)$$

where $\chi_z(k) = -2\lambda_1 \cos k - 2\lambda_2 \cos 2k + 2\mu$, and $\chi_y(k) = 2\lambda_1 \sin k + 2\lambda_2 \sin 2k$. The excitation spectra can be obtained as

$$E_k = \pm\sqrt{\chi_z^2(k) + \chi_y^2(k)}. \quad (5)$$

This model supports topological distinct gapped phases (i.e $W = 0, 1, 2$) separated by the three quantum critical lines as shown in Fig.5. The energy gap closes at these quantum critical lines, $\lambda_2 = \mu + \lambda_1$, $\lambda_2 = \mu - \lambda_1$ and $\lambda_2 = -\mu$, obtained for momentum $k_0 = \pm\pi$, $k_0 = 0$ and $k_0 = \cos^{-1}(-\lambda_1/2\lambda_2)$ respectively. The topological angle can be written as $\phi_k = \tan^{-1}(\chi_y(k)/\chi_z(k))$.

(2). Majorana fermion presentation

The model Hamiltonian in Eq. (1) can be written in the Majorana basis, using $c_j^\dagger = \frac{a_j + ib_j}{2}$, $c_j = \frac{a_j - ib_j}{2}$, as

$$H = -i \sum_{\alpha=0,1,2} \left(\sum_{j=1}^{N-\alpha} \gamma_\alpha b_j a_{j+\alpha} \right), \tag{6}$$

where a_j and b_j are Majorana operators satisfying anti-commutation relation, $\gamma_0 = -\mu$, $\gamma_1 = \lambda_1$, and $\gamma_2 = \lambda_2$. The Hamiltonian can be translated into the Fourier space as $f(k) = \sum_{\alpha=0,1,2} \gamma_\alpha e^{ik\alpha}$. Considering $z = e^{ik}$ and interpreting $f(k)$ on the unit circle in a complex plane, the complex function associated with the Hamiltonian can be written as

$$H = -i \left[-\sum_{i=1}^{N} \mu b_j a_j + \lambda_1 \sum_{i=1}^{N-1} b_j a_{j+1} + \lambda_2 \sum_{i=1}^{N-1} b_i a_{i+2} \right],$$

The BDI Hamiltonian is equivalent to a polynomial $f(z)$

$\sum_\alpha t_\alpha e^{ik\alpha}$. Let us now interpret this as a complex function evaluated on the unit circle, $z = e^{ik}$. This motivates us to associate a complex function $f(z)$ (essentially a Laurent series around the origin) to every translation-invariant Hamiltonian in the BDI class:

$$\boxed{H = \sum_{\alpha \in \mathbb{Z}} t_\alpha H_\alpha \qquad \Rightarrow \qquad f(z) = \sum_{\alpha \in \mathbb{Z}} t_\alpha z^\alpha} . \tag{2.4}$$

Note that $f_k = f(e^{ik})$, i.e., the single-particle spectrum and Bogoliubov angle are determined by the function on the unit circle. However, it is much more powerful to work with

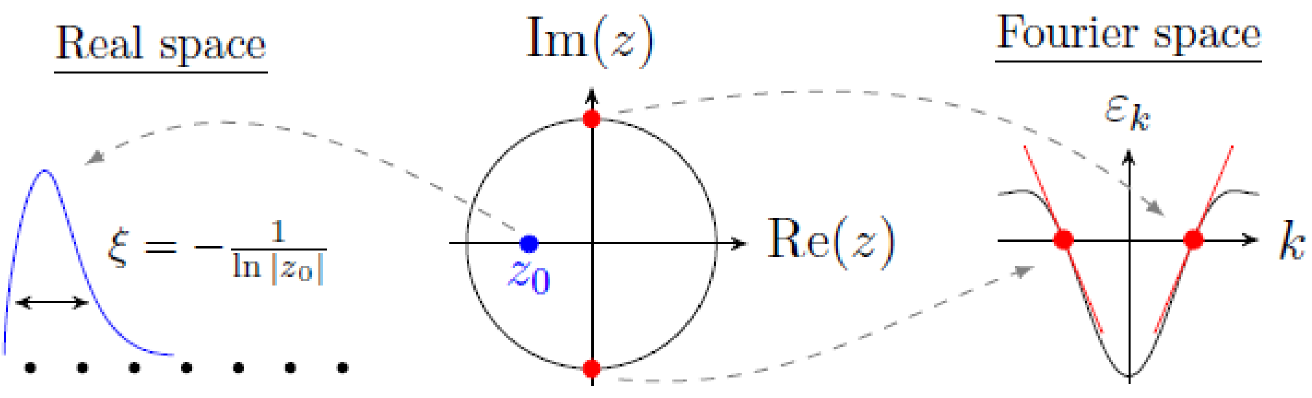
using Fourier transformation, $f(k) = \sum_{\alpha=0,1,2} \gamma_\alpha e^{ik\alpha}$, with $z = e^{ik}$. The complex function associated with the model Hamiltonian can be written as,

$$f(z) = \sum_{\alpha=0,1,2} \gamma_\alpha z^\alpha = -\mu + \lambda_1 z + \lambda_2 z^2, \tag{B·3}$$

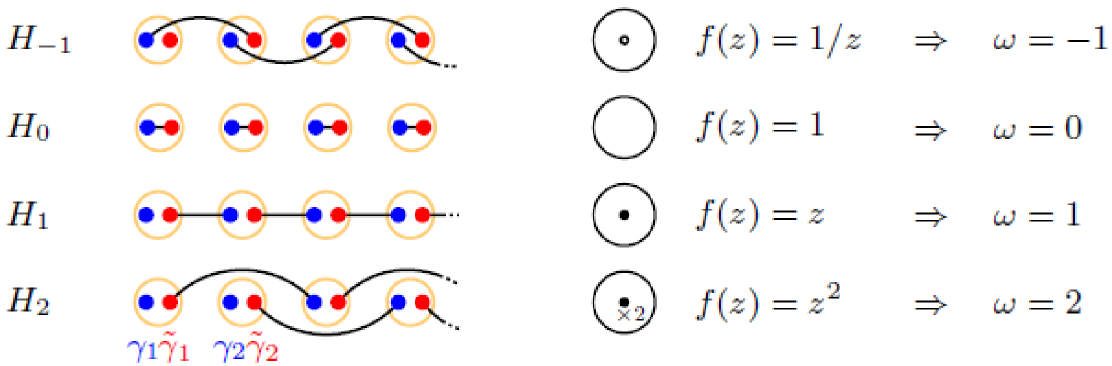
where for $\gamma_{0,1,2}$ are respectively $-\mu$, λ_1, and λ_2. Here $f(z)$ has two solutions z_1 and z_2 which can be written as,

$$z_{1,2} = \frac{-\lambda_1 \pm \sqrt{\lambda_1^2 + 4\mu\lambda_2}}{2\lambda_2}. \tag{B·4}$$

Some Interrelation of Polynomial function



Edge modes and criticality from the zeros of $f(z)$. The middle figure shows the zeros of $f(z)$. The zero z_0 within the disk (blue) corresponds to an edge mode (for each edge) with localization length $\xi = \frac{1}{|\ln|z_0||}$. Each zero on the unit circle (red) implies a massless Majorana field in the low-energy limit (contributing a central charge $c = \frac{1}{2}$).



$f(z)$ instead of f_k. Firstly, note that if the Hamiltonian has finite range, then $f(z)$ is a polynomial *after* we separate out the possible pole at the origin, $f(z) = \frac{1}{z^{N_p}} f_{\text{poly}}(z)$. We can now invoke the fundamental theorem of algebra to conclude that $f_{\text{poly}}(z)$ is completely determined by its set of zeros $\{z_i\}_i$ (up to a global prefactor), i.e.,

$$\boxed{f(z) = \frac{a}{z^{N_p}} \prod_i (z - z_i)} \,. \tag{2.5}$$

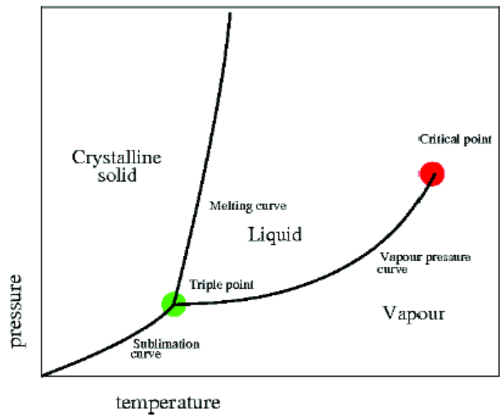
Since $t_\alpha \in \mathbb{R}$, we have that $f(z)^* = f(z^*)$, or, equivalently, that a is real and that the zeros are either real or come in complex-conjugate pairs. Conversely, any set of zeros satisfying these properties defines a unique model according to Eq. (2.4) (up to a global prefactor).

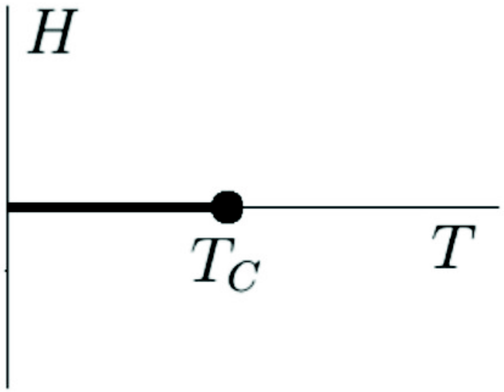
What is CFT

- A conformal field theory is a quantum field theory that is invariant under the conformal group.
- The conformal group is the set of transformations of spacetime that preserve angles (but not necessarily distances).
- Generally a conformal transformation is a coordinate transformation that is a local rescaling of the metric.
- A field theory with rotation, translation, and scale invariance is said to have conformal invariance. Fixed point theories, such as the free boson or the free fermion theories (and many interacting theories) are thus conformal field theories.

CFT and its relation to quantum phase transition and condensed matter physics

- A critical point is the point at the end of a phase equilibrium curve where a continuous phase transition occurs [1].

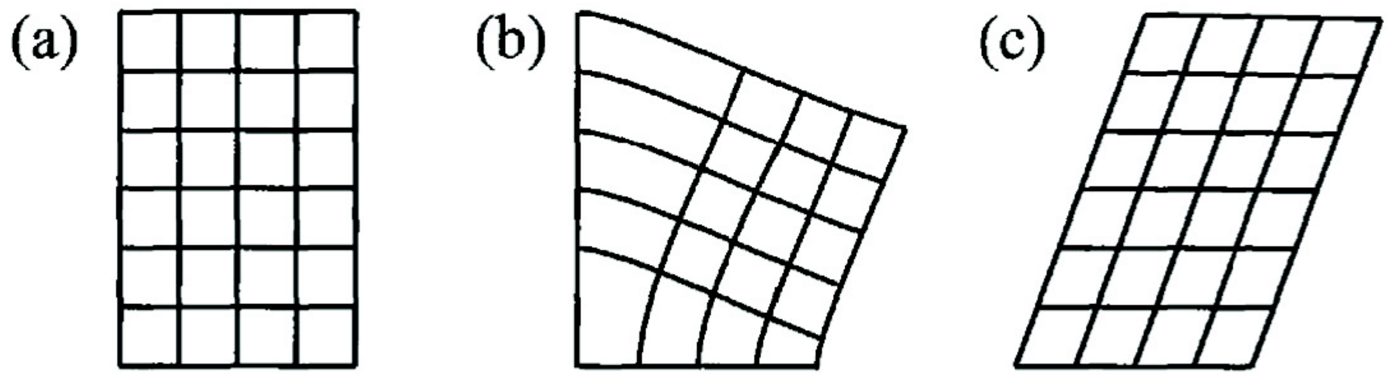




- **Examples :** The liquid-gas transition of water, or at the Curie temperature of a ferromagnet.
- Consider the case of a ferromagnet placed in an external magnetic field H. In this case T (temparature) is the tunable parameter and M (magnetization) is the order parameter.

- As T approaches some critical temperature T_c , thermal fluctuations become large and the material becomes paramagnetic.
- Right at the critical point, the correlation length $\xi \to \infty$ i.e, the corresponding field theory is therefore massless and becomes invariant under a dilation of the length-scales (Scale invariance) [2].
- In this regime we must be able to replace the lattice system with an effective-field theory without a lattice, i.e. we have effectively a continuum system, a system with these properties is said to be a conformal field theory [3].
- The system with scale invariance will have other spatial transformations which forms a class of conformal transformation.

- Conformal transformation $r \to r'$ is the one which locally corresponds to a combination of a translation, rotation and dilatation by preserving the angles between the lattice vectors [4].



- Operator Product Expansion (OPE) : For $|x_1 - x_2| << \xi$ i.e for local fields $\phi_p(x_1)$ and $\phi_q(x_2)$ one can write the operator algebra as,

$$\phi_p(x_1)\phi_q(x_2) = \sum_{r=0}^{\infty} c_{pq}^{r} \frac{1}{|x_1 - x_2|^{d_p+d_q-d_r}} \phi_r(x_2), \qquad (1)$$

where c_{pq}^r is structure constant of operator algebra and d_p, d_q, d_r are the dimensions of scaling fields ϕ_p,ϕ_q, ϕ_r respectively [5].

Central Charge

- The central charge (c) labels each universality class of the critical systems.
- For $0 < c < 1$ the number of primary fields is finite and their conformal weights are rational numbers which are determined by the Kac formula.
- In $c = 1$ theory any non-negative coformal weight is allowed and there exist infinite number of primary fields.
- The results for $0 < c < 1$ have been obtaind by making use of the representation theory of the Virasoro algebra. In $c = 1$ theory, the representation theory is not powerful enough to specify the theory.
- The $c = 1$ theory is the most relevant CFT for the description of 1D quantum liquids [7].

A Few More Words on CFT

Two dimesional CFT has been developed with the seminal paper of Belavian, Polyakov and Zamolochikov. It has been proven to be an extremely richness in mathematical physics with three main application in string theory, two-dimensional critical system and application to mathematics in general and group theory. In the present study, we do the CFT for 2 (= 1+1) dimensional critical phenomena. We will see that different kinds of criticality shows in the present study by means of CFT. The physics of quantum criticality in low dimensional (1+1) quantum many body system can be studied by using conformal field theory. At the quantum critical point the most important parameter is the central charge. which is related with the quantum fluctuations at zero temperature. The central charge can be calculated in many ways. At zero temperature it can be calculated from the zeros of the polynomial of the model Hamiltonian system.

Physical significance of central charge: The central charge counts the number of massless degrees of freedom. For the gapped system, there are no low energy degrees of freedom and central charge is zero. There are different physical significance between the integer and fractional central charge. Theories with integer central charge have free field representation. Theories with half-integer central charge, example of quantum Ising model (2D Ising model) have a free Majorana representation at the transition point. But in the present study we will see that for the longer range interaction it is not possible to express/derive the Dirac equation for Majorana fermions. Theories with fractional central charge other than quantum Ising model is gapped.
A conformal field theory is characterized by a number c called the conformal charge. This number is roughly a measure of the number of degrees of freedom of the model considered. By convention, the free boson theory has conformal charge $c = 1$. So does the free complex fermion. Conformal field theories with conformal charge $c < 1$ correspond to known critical statistical models, like the Ising model, solution to be equivalent to a free Majorana fermion). The central charge labels each universality class of the critical of the theory.

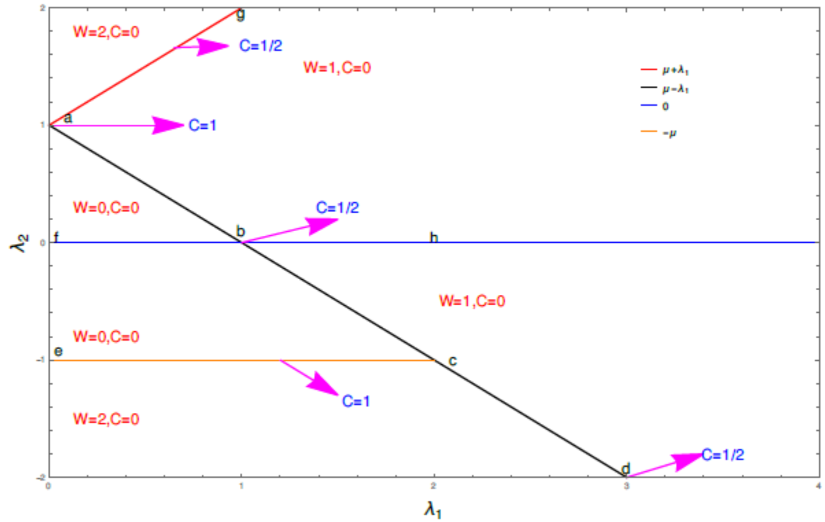


Figure 1. Topological phase diagram of model Hamiltonian for $\mu = 1$. Line 'ac' represents the critical line $\lambda_2 = \mu - \lambda_1$ (blue line), line 'be' represents the critical line $\lambda_2 = -\mu$ (magenta line) and line 'ad' represents the critical line $\lambda_2 = \mu + \lambda_1$ (red line). Points 'a' and 'b' are multi-critical points (green and black dots respectively) which differentiate between three distinct gapped phases with $W = 0, 1, 2$ and C is the central charge.

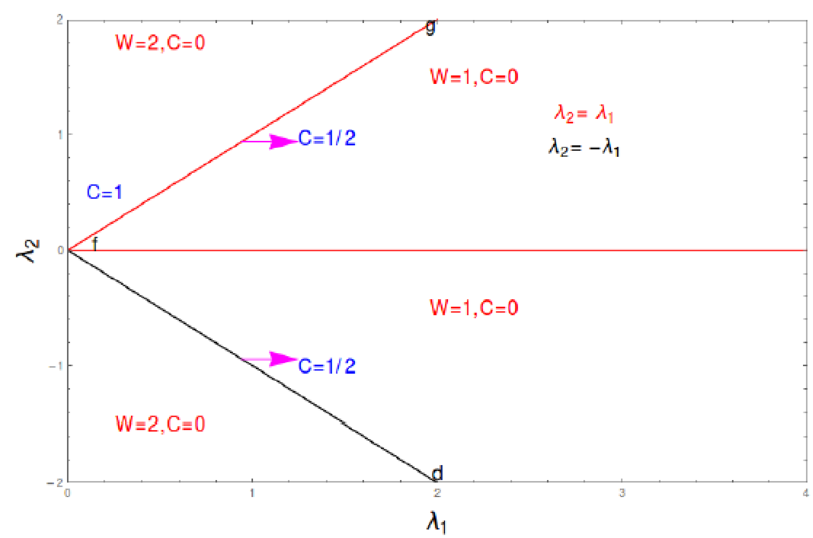


Figure 2. Topological phase diagram of model Hamiltonian for $\mu = 0$. line gf represents the critical line $\lambda_2 = \mu - \lambda_1$ line 'fd' represents the critical line $\lambda_2 = \mu + \lambda_1$ (red line). Point 'f' is the multicritical which differentiate between three distinct gapped phases with $W = 0, 1, 2$ and C is the central charge.

Unitary representions of Virasoro Algebra: Central Charge Analysis of Quantum Ising Model

This discrete set of points, where unitary representations of the Virasoro algebra are not excluded, occur at values of the central charge

$$c = 1 - \frac{6}{m(m+1)} \qquad m = 3, 4, \ldots \tag{4.6a}$$

($m = 2$ is the trivial theory $c = 0$). To each such value of c there are $m(m-1)/2$ allowed values of h given by

$$h_{p,q}(m) = \frac{\left[(m+1)p - mq\right]^2 - 1}{4m(m+1)} \tag{4.6b}$$

where p, q are integers satisfying $1 \leq p \leq m-1,\ \ 1 \leq q \leq p$.

Thus we see that the *necessary* conditions for unitary highest weight representations of the Virasoro algebra are ($c \geq 1$, $h \geq 0$) or ($4.6a, b$). That the latter of these two conditions is also sufficient, i.e. that there indeed exist unitary representations of the Virasoro algebras for these discrete values of c, h, was shown

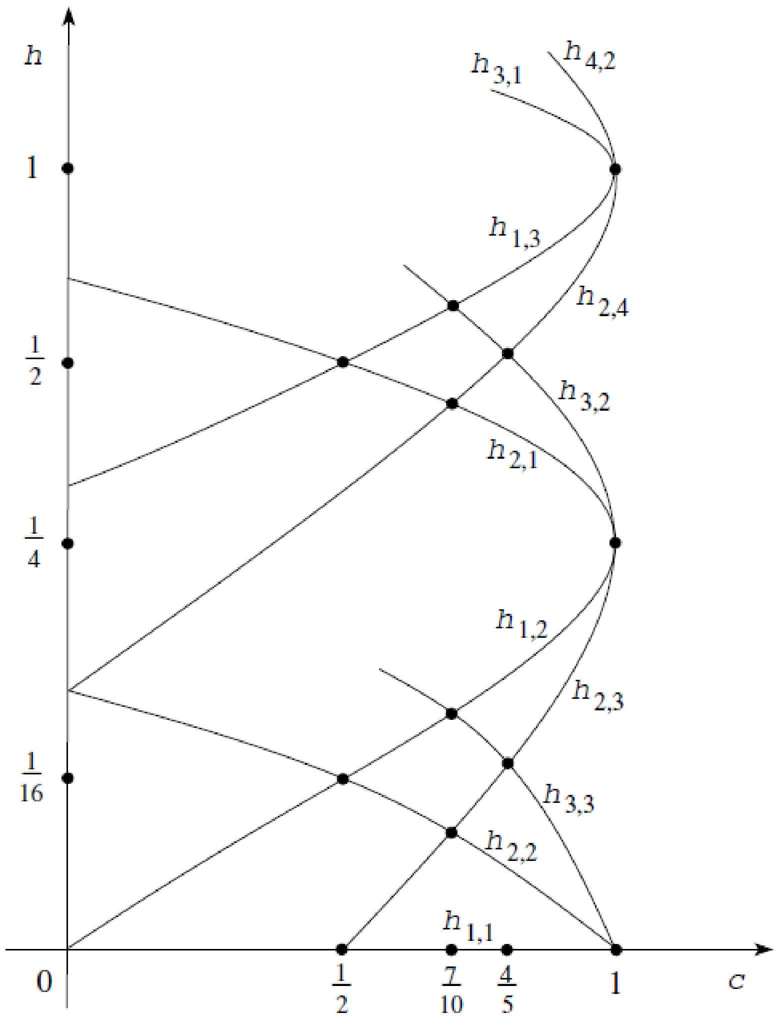


Fig. 6. First few vanishing curves $h = h_{p,q}(c)$ in the h, c plane.

While the $c < 1$ discrete series distinguishes a set of representations of the Virasoro algebra, it is not obvious that these should be realized by readily constructed statistical mechanical model at their critical points. The first few members of the series (4.6a) with $m = 3, 4, 5, 6$, i.e. central charge $c = \frac{1}{2}, \frac{7}{10}, \frac{4}{5}, \frac{6}{7}$, were associated in [18] respectively with the critical points of the Ising model, tricritical Ising model, 3-state Potts model, and tricritical 3-state Potts model,

How to Calculate Central Charge for longer range Ising model

The topological property of the system can be captured by the winding number (W) which counts the number of edge modes in the corresponding gapped or critical phases. The critical properties of the system can be captured from the C which measures the CFT of the corresponding criticality. Due to the one-to-one correspondence between the Hamiltonian and the associated complex function $f(\zeta)$, the central charge can be obtained using the zeros $(\zeta = 1,2,3)$ in the complex plane if they are non-degenerate. Among the zeros of the complex function $f(\zeta)$, that lie on the unit circle carries the information that the system is at criticality. Therefore, the number of zeros on the unit circle determines the value of C as

$$C = \frac{1}{2}(\text{number of zeros on the unit circle}). \tag{10}$$

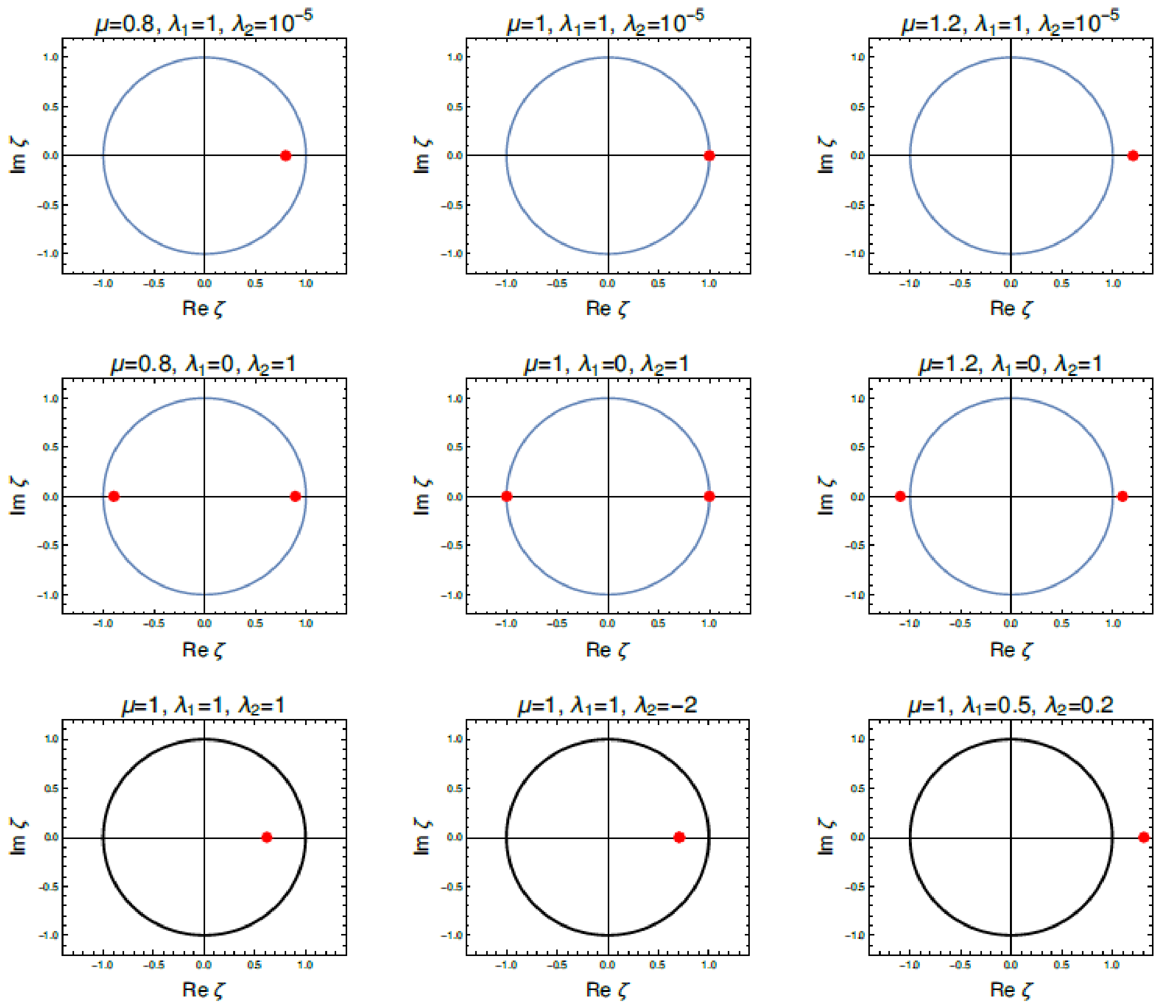
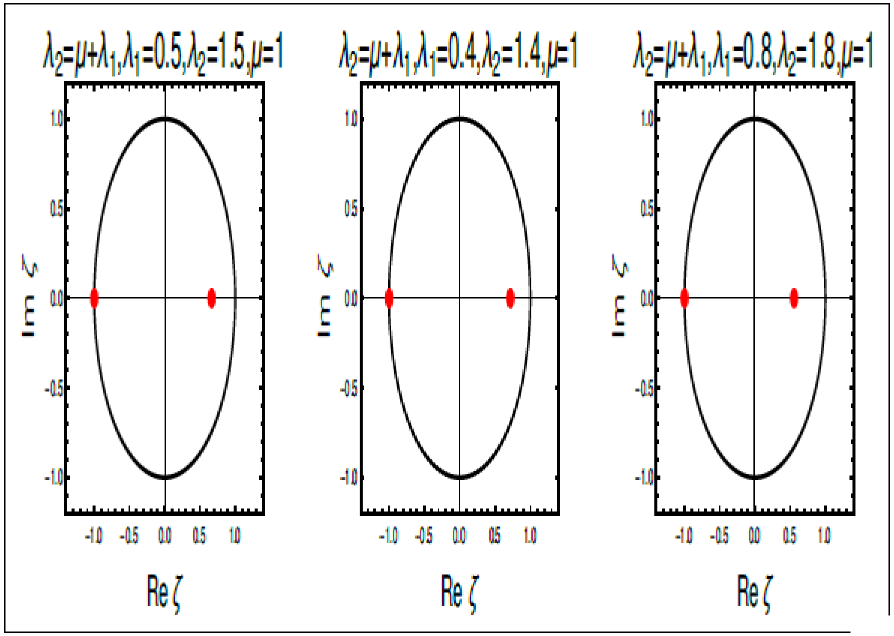
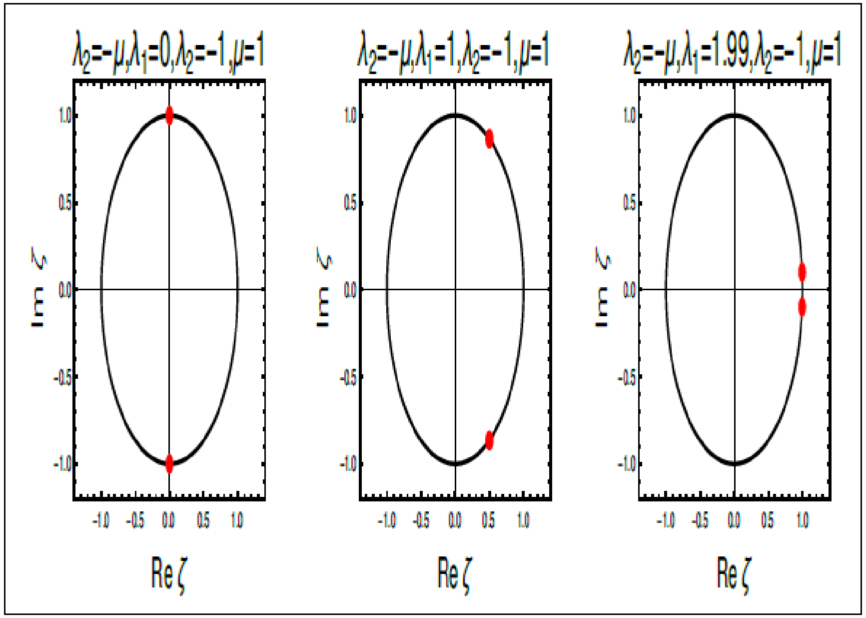
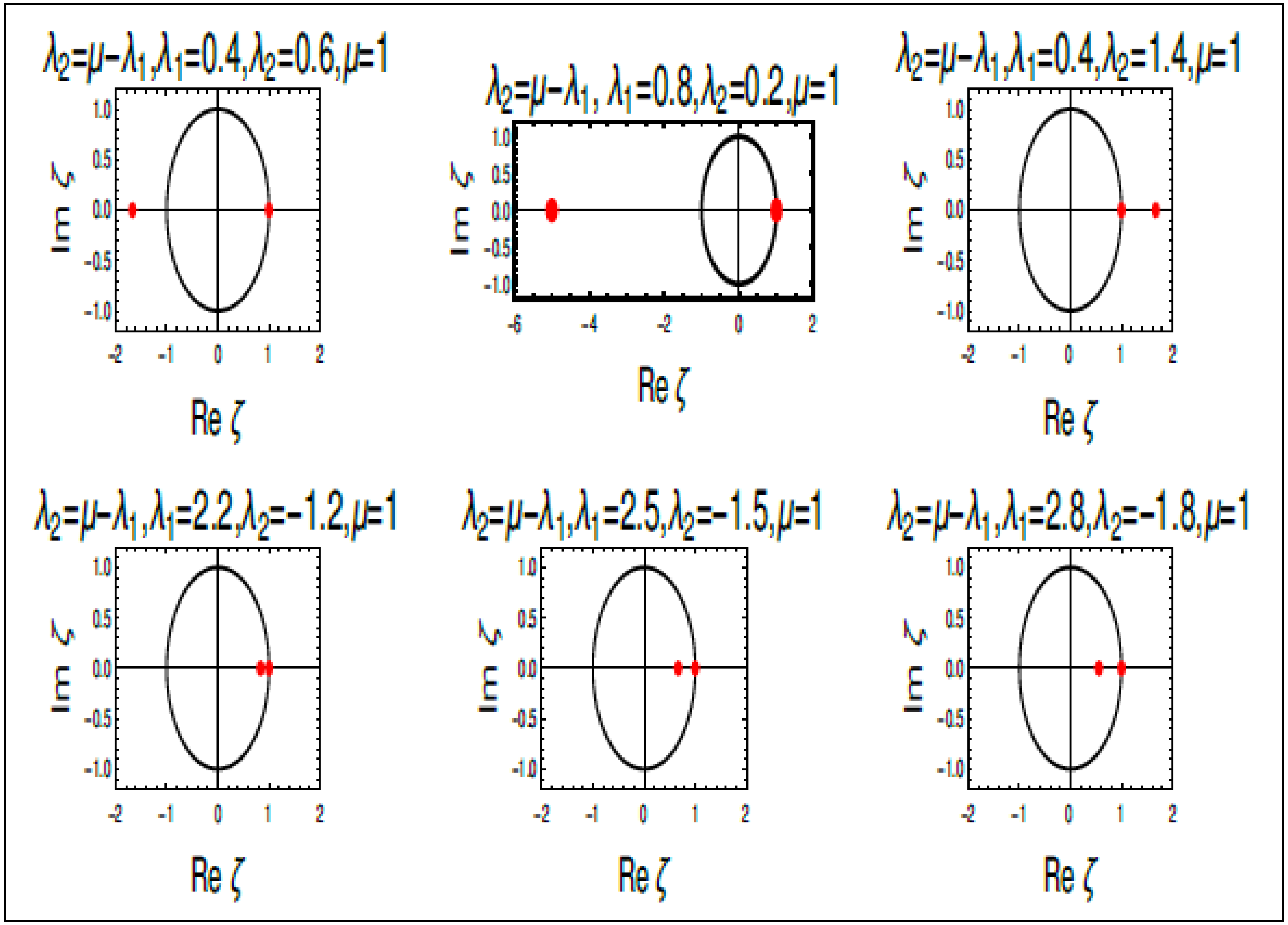


Figure 3. This figure present the zeros of the complex function (Eq. 9). Parameter space for this figure is $\lambda_1 = 1$, $\lambda_2 = 10^{-5}$. This figure panel consist of three figure for three different values of μ . The left, middle and right are for $\mu = 0.8, 1$ and 1.2 respecively. The upper and lower panel are respectively for $\lambda_2 = 0$ and $\lambda_1 = 0$ line solutions. The bottom panel shows the zero central charge for the gapped region.

Analysis of Central Charge for Quantum Critical lines






λ2=μ−λ1,λ1=0.4,λ2=0.6,μ=1
λ2=μ−λ1, λ1=0.8,λ2=0.2,μ=1
λ2=μ−λ1,λ1=0.4,λ2=1.4,μ=1
λ2=μ−λ1,λ1=2.2,λ2=−1.2,μ=1
λ2=μ−λ1,λ1=2.5,λ2=−1.5,μ=1
λ2=μ−λ1,λ1=2.8,λ2=−1.8,μ=1
Im ζ
Re ζ

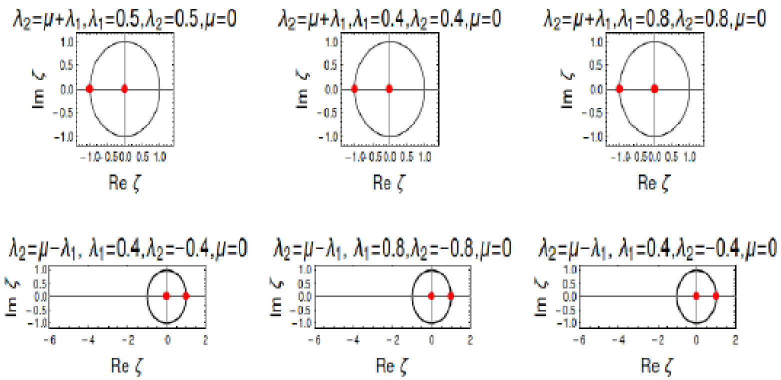


Figure 5. This figure present the zeros of the complex function (Eq.9). This figure consists of two panels. Upper, middle and lower panels are respectively for the quantum critical line $\lambda_2 = \mu + \lambda_1$, $\lambda_2 = \mu - \lambda_1$ and $_2 = -\mu$. For all panels we consider $\mu = 0$.

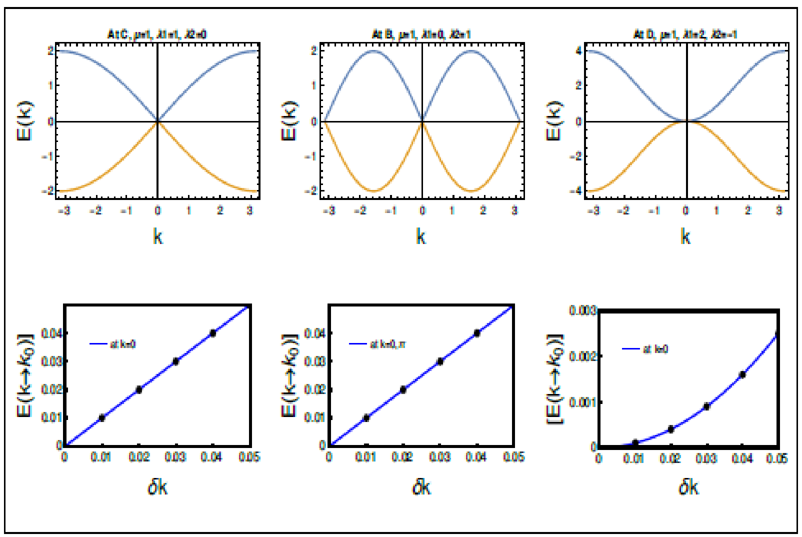


Figure 6. This figure panel consists of two rows and each panel consists three figures. The upper and lower rows are respectively for the energy dispersion and the corresponding scaling analysis for that region of parameter space. The left, middle and right figures are respectively for enegy dispersion of quantum Ising model and quantum Ising model with longer range interaction. All three figures are at the quantum critical points.

Conformal and non-Conformal Quantum Criticality

We have already discussed that the $f(z)$ is proportional to the energy dispersion. If $f(k)$ is nondegenerate, the energy dispersion $(\varepsilon(k)(k-k_0))$ is a relavistic dispersion. In order to treat the model Hamiltonian system using CFT, Lorentz symmetry has to be manifest, i.e., it must have a relativistic dispersion and a dynamical critical exponent $z=1$. For the case of m degenerate zeros, it implies a non-linear dispersion $\varepsilon(k) \sim (k-k_0)^z$, here is the breakdown of Lorentz invariance, for this situation CFT can not be used.

Results for Energy Dispersion and Scaling Relation

Fig.6 present the energy dispersion and the scaling relation for our model Hamiltonian. This figure panel consists of two rows and each row consists of three figures. The left, right and middle figures are respectively for the three quantum critical points. This dispersion base on eq. 5. In the upper row, the left panel is for the qIm at the topological quantum phase transition with $W = 1$ and $C = 1/2$ (point b), where both the dispersion branches meet at the origin.

The middle figure is for the dispersion is for the point point a ($W = 0$ and $C = 1$), It is one of the multicritical point, where there the dispersion branches meet phase at the origin and also at the Brillouin zone boundary. Thus it has no topological phase but with integer value of C. The right figure is for the another multicritical point, which is non conformal quantum critical point, where the physics of CFT is not applicable. One of the most interesting feature we obtain a flat feature of dispersion close to the origin.

In the lower panel we present the scaling relation of energy dispersion near to the energy dispersion and from that dispersion we extract the dynamical critical exponent ($z =$). It reveals from our study that $z = 1$ is for the left and middle but for the rigt figure it is 2.

This entails the fact that there is a topological quantum critical point where the Lifshitz universality class with $z = 2$ and $\nu = 1/2$, between two distinct gapless phases through muticritical point. The Lifshitz transition that emerges in our study only for the presence of transverse field. At this multicritical point system is in non-CFT criticality. To the best of our knowledge this is the first study of the existence conformal criticality and non-conformal criticality for this model Hamiltonian system.

Conclusions

(1). We have presented the results of conformal field theory study for quantum Ising model with longer range interaction in presence and absence of transverse field.

(2). We have shown explicitly that the central charge is zero for the gapped phase of the system but finite for gapless quantum critical lines.

(3). The topological properties is the same for the all quantum critical lines with out transverse field but it is different for a quantum critical line in presence of transverse field.

(4). We have predicted two different kind of muticritical points, one is conformal quantum critical point and the other is non-conformal quantum critical points.

(5). We have predicted the existence of quantum Lifshitz transition point.

Thank You