On the local constancy of certain mod p Galois representations

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Introduction

Let p be an odd prime. Let $f = \sum_{n\geq 1} a_n q^n$ be a normalized cuspidal eigenform of weight $k \geq 1$, character ψ and level $\Gamma_1(N)$ such that $p \nmid N$.

We note that f being an eigenform implies that it is an **eigenfunction** for all the **Hecke operators** T_n with the eigenvalues given by a_n (for all $n \in \mathbb{N}$).

The work of Deligne, Deligne-Serre, and Eichler-Shimura associate to f a p-**adic Galois representation** $\rho_f : \mathrm{Gal}\left(\bar{\mathbb{Q}}/\mathbb{Q}\right) \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ such that ρ_f is **unramified** at all primes $l \nmid pN$.

Local structure of ρ_f at a prime $l \neq p$

Further, the **characteristic polynomial** of $\rho_f(\mathrm{Frob}_l)$ is given by $X^2 - a_l X + l^{k-1}\psi(l)$, where a_l is **the T_l-eigenvalue** of f.

By **local structure** of ρ_f at a prime l, we mean $\rho_f|_{G_l}$, where $G_l = \mathrm{Gal}\left(\bar{\mathbb{Q}}_l/\mathbb{Q}_l\right)$, identified as decomposition subgroup of Gal $\left(\bar{\mathbb{Q}}/\mathbb{Q}\right)$ at the prime l.

We note that **the local structure** of ρ_f at a prime l such that $l \nmid pN$ is determined by (a_l, k, ψ).

Local structure (LS) of ρ_f at a prime $l = p$: D_{cris}

In the ordinary case, a result of Deligne determines the **mod p reduction** $\bar{\rho}_f|_{G_p}^{ss}$ at the decomposition group G_p of p for weights $k \geq 2$. In the non-ordinary case, Fontaine and Edixhoven determine $\bar{\rho}_f|_{I_p}$ for $2 \leq k \leq p+1$.

Faltings et al proved that if $p \nmid N$ ($k \geq 2$), then $\rho_f|_{G_p}$ is a **crystalline** representation of **Hodge-Tate weights** $(0, k-1)$.

Colmez and Fontaine proved that the functor D_{cris}, defined as $D_{cris}(V) := \left(B_{cris} \otimes_{\mathbb{Q}_p} V\right)^{G_p}$, is an **equivalence** of categories:

$$\boxed{\text{Category of crystalline representations of } G_p \text{ over } \mathbb{Q}_p} \xrightarrow{D_{cris}} \boxed{\text{Category of weakly admissible filtered } \phi\text{-modules over } \mathbb{Q}_p.}$$

LS of ρ_f at a prime p: Definition of V_{k,a_p}

For an integer $k \geq 2$ and $0 \neq a_p \in \bar{\mathbb{Q}}_p$ with $\nu(a_p) > 0$, let D_{k,a_p} be the **weakly admissible filtered ϕ-module** of Scholl (of dimension 2). Then there **exists** a 2-dimensional **crystalline** representation V_{k,a_p} of G_p such that $D_{\text{cris}}(V^*_{k,a_p}) \cong D_{k,a_p}$ where V^*_{k,a_p} is the dual of V_{k,a_p}. The representation V_{k,a_p} is an **irreducible** crystalline representation with **Hodge-Tate weights** $(0, k-1)$, and the characteristic polynomial of Frobenius ϕ is given $X^2 - a_pX + p^{k-1}$.

We note that up to an unramified twist, $\rho_f|_{G_p}$ is isomorphic to V_{k,a_p}. We note that **the local structure** of ρ_f at prime p is determined by (a_p, k, ψ).

Results computing $\bar{V}_{k,a_p}$

Let $\overline{V}_{k,a_p}$ be the **mod** p **reduction** of a G_p-stable lattice of V_{k,a_p} upto semisimplification.

Note that $\overline{V}_{k,a_p}$ has been studied for various weight and slope ranges, and here, we are mentioning some of the results computing $\overline{V}_{k,a_p}$. It is known for the following ranges of weights and slopes:

- (Breuil) For all $\nu(a_p) > 0$ and $2 \leq k \leq 2p + 1$.
- (Berger-Li-Zhu) For all $\nu(a_p) > \lfloor \frac{k-2}{p-1} \rfloor$.
- (Bergdall-Levin) For all $\nu(a_p) > \lfloor \frac{k-1}{p} \rfloor$.
- (Buzzard-Gee) For all $\nu(a_p) \in (0, 1)$.
- (Ganguli-Ghate, Bhattacharya-Ghate,

Bhattacharya-Ghate-Rozensztajn, Ghate-Rai) For all $\nu(a_p) \in [1, 2)$.

- (Ghate-Vangala, Chitrao-Ghate-Yasuda) Related to local

constancy.

Local constancy (LC): Definition

We note that V_{k,a_p} is completely determined by a_p and the weight k, and so is $\overline{V}_{k,a_p}$. **Fixing** a_p, we define the map $k \to \overline{V}_{k,a_p}$ on **the weight space**. We study the question of **local constancy** of this map.

In general, local constancy may not exist for given values of k and a_p. The **zig-zag conjecture** of Ghate provides important **counterexamples** of local constancy when $k = 2\nu(a_p) + 2$.

$\mathrm{ind}(\omega_2^a)$: Definition

We fix a **fundamental character** ω_2 of the inertia subgroup I_p at prime p of level 2.

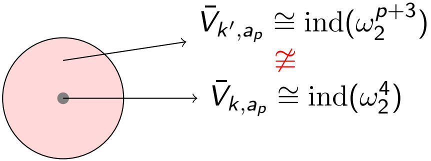
For $a \in \mathbb{Z}_{\geq 0}$ such that $(p+1) \nmid a$, let $\mathrm{ind}(\omega_2^a)$ denote the **unique two dimensional irreducible** representation of G_p such that $\mathrm{ind}(\omega_2^a)|_{I_p} \cong \omega_2^a \oplus \omega_2^{ap}$ and **the determinant character** is given by ω^a, where ω is the mod p reduction of the p-adic cyclotomic character.

LC dependency on a_p: A counterexample

Let $k = 5$, $p \geq 7$ and $a_p = p^{3/2} \in \bar{\mathbb{Q}}_p$.

- By a result of Breuil, we have that $\overline{V}_{k,a_p} \cong \mathrm{ind}(\omega_2^4)$.
- A result of Ghate-Rai

(Zig-zag conjecture for $\nu(a_p) = 3/2$) gives that $\overline{V}_{k',a_p} \cong \mathrm{ind}(\omega_2^{p+3})$ $\forall\ k' \in k + p^t(p-1)\mathbb{Z}_{>0}$ and for all $t \geq 1$.

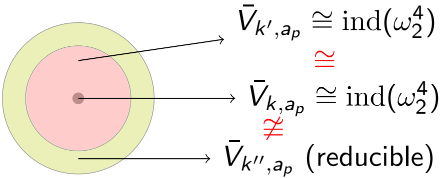


- Thus, local constancy does not exist around $k = 5$ for $a_p = p^{3/2}$ as $\overline{V}_{k',a_p} \ncong \overline{V}_{k,a_p}$ for all $k' \in k + p^t(p-1)\mathbb{Z}_{>0}$ and for all $t \geq 1$.

LC dependency on a_p: An example

Let $k = 5$, $p \geq 7$, and $a_p = p^{3/2}(1 + p^{1/2})^{1/2} \in \bar{\mathbb{Q}}_p$.

- By the result of Breuil, we have $\overline{V}_{k,a_p} \cong \mathrm{ind}(\omega_2^4)$.
- For all $k' \in k + p^t(p-1)\mathbb{Z}_{>0}$, the result of Ghate-Rai gives that $\overline{V}_{k',a_p} \cong \mathrm{ind}(\omega_2^4)$ for all $t \geq 2$ and $\overline{V}_{k',a_p}$ is reducible for $t = 1$.



- Thus, $k + p^2(p-1)\mathbb{Z}_{\geq 0}$ is the largest disk in the weight space on which $\overline{V}_{k',a_p}$ is constant. Hence, we have that p^{-2} is the radius of local constancy for k and a_p as given above.

Local constancy: Existing results

The first result proving the existence of local constancy is due to Berger. He proves the following result.

Theorem (Berger)

Suppose $a_p \neq 0$ *with* $\nu(a_p) > 0$ *and* $k > 3\nu(a_p) + \frac{(k-1)p}{(p-1)^2} + 1$, *then there exists* $m = m(k, a_p)$ *such that* $\overline{V}_{k',a_p} \cong \overline{V}_{k,a_p}$, *if* $k' \in k + p^{m-1}(p-1)\mathbb{Z}_{\geq 0}$.

The above theorem does not give **any explicit bounds** on $m(k, a_p)$ (equivalently on the radius of local constancy) and also does not determine **the explicit structure** of $\overline{V}_{k,a_p}$ within the disk of local constancy.

Continued...

Bhattacharya gives **the first explicit upper bound** on $m(k, a_p)$ for small weights by **computing** $\overline{V}_{k,a_p}$ **explicitly**, where $k > 2\nu(a_p) + 2$. The bound depends only on the slope $\nu(a_p)$. Bhattacharya proves the following result.

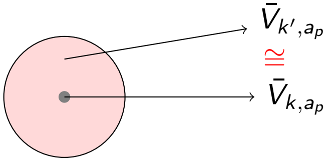
Theorem (Bhattacharya)

For $c \in \{0, 1, 2, 3\}$, let $b \geq 2c$ and suppose $k = b + c(p-1) + 2$, $2 \leq b \leq p-1$. In the range $c < \nu(a_p) < p/2 + c$ of slopes, if $k > 2\nu(a_p) + 2$ and $k \not\equiv 3 \mod (p+1)$, then Berger's constant $m(k, a_p)$ exists and is bounded above by $2\nu(a_p) + 1$. Moreover, $\overline{V}_{k,a_p} \cong \mathrm{ind}\left(\omega_2^{k-1}\right)$.

Our key ideas to prove LC

We prove local constancy by showing that $\overline{V}_{k',a_p}$ is **constant** for all $k' \in k + p^t(p-1)\mathbb{Z}_{>0}$, where $t \geq t_0$ by **explicitly computing** $\overline{V}_{k',a_p}$.

This gives local constancy in the punctured disk $\{k' |\ k' \in k + p^t(p-1)\mathbb{Z}_{>0}\ \&t \geq t_0\}$ around k in the weight space.



Next, we determine **the structure of** $\overline{V}_{k,a_p}$ by either applying Berger's local constancy theorem or the following result of Berger-Li-Zhu to establish local constancy in the whole disk.

Theorem (Berger-Li-Zhu)

If $\nu(a_p) > \lfloor \frac{k-2}{p-1} \rfloor$ *then*

$$\overline{V}_{k,a_p} \cong \begin{cases} \mathrm{ind}(\omega_2^{k-1}) & \textit{if } (p+1) \nmid (k-1) \\ \left(\mu_{\sqrt{-1}} \oplus \mu_{-\sqrt{-1}}\right) \otimes \omega^{\frac{k-1}{p+1}} & \textit{if } (p+1) \mid (k-1). \end{cases}$$

We now fix some notations. Let $k = b + c(p-1) + 2$ with $2 \le b \le p,\ 0 \le c \le p-2$ and $p \ge 7$.

Main result

We prove the following result.

Theorem (Ganguli,K)

Fix a_p *such that* $c < \nu(a_p) < min\{\frac{p}{2} + c - \epsilon,\ p - 1\}$ *and* $k > 2\nu(a_p) + 2$. *Assume that* $(b, c) \notin E$.

1. *Then* $\overline{V}_{k',a_p} \cong \overline{V}_{k,a_p}$ *for all* $k' \in k + p^t(p-1)\mathbb{Z}_{\geq 0}$, *where* $t \geq \lceil 2\nu(a_p) \rceil + \epsilon$.
2. *Moreover,* $\overline{V}_{k,a_p} \cong \mathrm{ind}\left(\omega_2^{k-1}\right)$.

In the above theorem, E is a finite "sporadic set" given by $k \equiv 1, 3 \bmod (p+1)$.

In the context of the above theorem, we make the following comments.

- The above theorem shows that **Berger's constant exists** such that $m(k, a_p) \leq \lceil 2\nu(a_p) \rceil + \epsilon + 1$.
- The set $E = \{(2c-1, c), (2c-2-p, c), (2c+1, c), (2c-p, c), (p, 0)\}$.
- The ordered pairs (b, c) in E are those points for which $\overline{V}_{k', a_p}$ may possibly be **reducible**.
- If local constancy **exists** for k, then using the result of Berger-Li-Zhu, we expect that $\overline{V}_{k', a_p}$ will always be **reducible** if $k \equiv 1 \bmod (p+1)$, and it will be **irreducible** in all other cases.

$\overline{V}_{k',a_p}$ via the mod p LLC

The mod p local Langlands correspondence is an injection between:

$$\left\{\begin{array}{c}\text{semisimple 2-dimensional } \bar{\mathbb{F}}_p \\ \text{representation of } \mathrm{Gal}\left(\bar{\mathbb{Q}}_p/\mathbb{Q}_p\right)\end{array}\right\} \xrightarrow{LL} \left\{\begin{array}{c}\text{semisimple finite length} \\ \text{smooth admissible } \bar{\mathbb{F}}_p \\ \text{representation of } \mathrm{GL}_2(\mathbb{Q}_p)\end{array}\right\}$$

- By a **smooth representation**, we mean that the stabilizer of each vector of the representation is an open subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$.
- A smooth representation is called an **admissible representation** if its invariant spaces under every compact open subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$ are finite-dimensional.

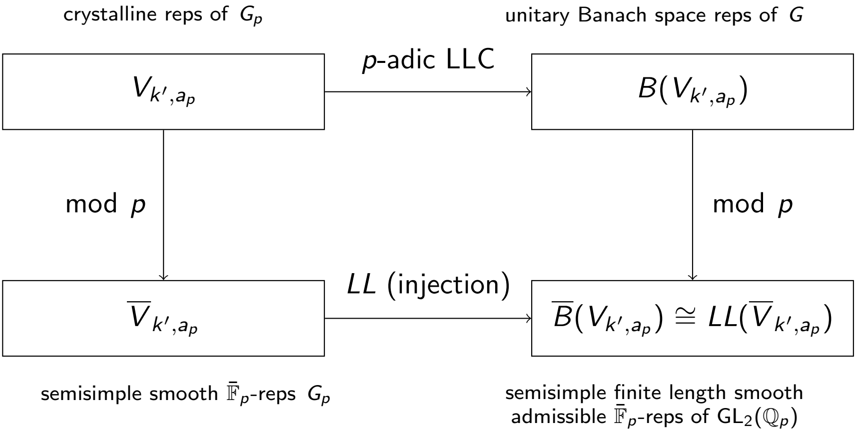
Compatibility of the p-adic and mod p LLC

The p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ associates to V_{k,a_p} a **unitary Banach space** representation $B(V_{k,a_p})$ of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Breuil gave a **locally algebraic** representation Π_{k,a_p} of $\mathrm{GL}_2(\mathbb{Q}_p)$, such that $B(V_{k,a_p})$ is a suitable **completion** of Π_{k,a_p} with respect to a G-**invariant norm**, and a $\mathrm{GL}_2(\mathbb{Q}_p)$-stable lattice Θ_{k,a_p} in Π_{k,a_p}.

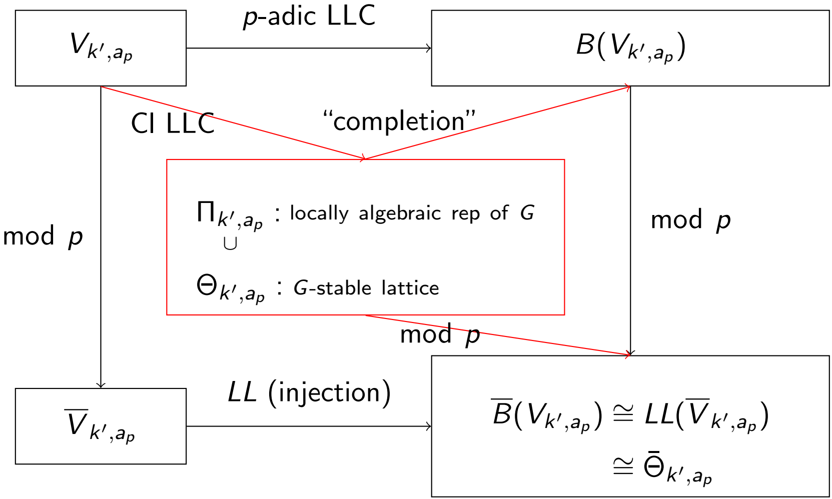
Berger proved **the compatibility** of the p-adic and the mod p local Langlands correspondences that gives $\bar{\Theta}^{ss}_{k,a_p} \cong LL(\overline{V}_{k,a_p})$, where $\bar{\Theta}_{k,a_p} := \Theta_{k,a_p} \otimes \bar{\mathbb{F}}_p$.

Compatibility of p-adic and mod p LLC: Pictorial View



"rep = representation".

Compatibility of p-adic and mod p LLC



Our key idea to determine $\bar{\Theta}_{k',a_p}$

For $r = k' - 2 \geq 0$, let $V_r = \mathrm{Sym}^r(\bar{\mathbb{F}}_p^2)$ be the symmetric power representation of $\mathrm{GL}_2(\mathbb{F}_p)$.

Let $\theta := \mathbf{x^p y} - \mathbf{x y^p} \in V_{p+1}$. We note that $\mathrm{GL}_2(\mathbb{F}_p)$ acts on θ by the **determinant character**. For $m \in \mathbb{N}$, let us denote

$$V_r^{(m)} = \{f \in V_r \mid \theta^m \text{ divides } f \text{ in } \bar{\mathbb{F}}_p[x, y]\}$$

which is a subrepresentation of V_r. From Buzzard-Gee [BG09], we get a surjective map

$$P : \mathrm{ind}_{KZ}^{G}\left(\frac{V_r}{V_r^{(\nu+1)}}\right) \twoheadrightarrow \Theta_{k',a_p} \otimes \bar{\mathbb{F}}_p.$$

An important filtration

We consider the following chain of submodules

$$0 \subseteq \mathrm{ind}_{KZ}^{G}\left(\frac{V_r^{(\nu)}}{V_r^{(\nu+1)}}\right) \subseteq \mathrm{ind}_{KZ}^{G}\left(\frac{V_r^{(\nu-1)}}{V_r^{(\nu+1)}}\right) \subseteq \cdots \subseteq \mathrm{ind}_{KZ}^{G}\left(\frac{V_r}{V_r^{(\nu+1)}}\right).$$

For $0 \leq m \leq \nu$, observe that $\mathrm{ind}_{KZ}^{G}\left(\frac{V_r^{(m)}}{V_r^{(m+1)}}\right)$ are the **successive quotients** in the above filtration.

In order to prove our main result, we show that the map P in fact **surjects** from $\mathrm{ind}_{KZ}^{G}\left(\frac{V_r^{(c-\epsilon)}}{V_r^{(c+1-\epsilon)}}\right)$, i.e.,

$$P : \mathrm{ind}_{KZ}^{G}\left(\frac{V_r^{(c-\epsilon)}}{V_r^{(c+1-\epsilon)}}\right) \twoheadrightarrow \Theta_{k',a_p} \otimes \bar{\mathbb{F}}_p.$$

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Thank You

Notations

- $G_p := \mathrm{Gal}\left(\bar{\mathbb{Q}}_p/\mathbb{Q}_p\right)$, $G := \mathrm{GL}_2(\mathbb{Q}_p)$, $K := \mathrm{GL}_2\left(\mathbb{Z}_p\right)$, and $Z := \mathbb{Q}_p^*$.
- We denote $q(i) := x^{r-(b-m+i(p-1))}y^{b-m+i(p-1)}$ for all $n_0 \leq i \leq c$, where $n_0 = 0$ if $b \geq m$ and 1 otherwise.
- We define ϵ as follows

$$\epsilon = \begin{cases} 0 & \text{if} \quad 2c-1 \leq b \leq p \\ 1 & \text{if} \quad 2(c-1)-p \leq b \leq 2(c-1) \\ 2 & \text{if} \quad 2 \leq b \leq 2(c-1)-(p+1). \end{cases}$$