# Crossed modular categories and the Verlinde formula for twisted conformal blocks

Tanmay Deshpande

(joint work with Swarnava Mukhopadhyay)

Tata Institute of Fundamental Research
Mumbai, India

# Twisted affine Lie algebras

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- The twisted affine Lie algebra is defined as

$$\widehat{L}(\mathfrak{g}, \gamma) := (\mathfrak{g} \otimes \mathbb{C}((t)))^{\gamma} \oplus \mathbb{C}c$$

with $c \in \widehat{L}(\mathfrak{g}, \gamma)$ central and Lie bracket given by

$$[X \otimes f, Y \otimes g] := [X, Y] \otimes fg + \frac{(X, Y)_{\mathfrak{g}}}{|\gamma|} \cdot \mathrm{Res}_{t=0}(g \cdot df) \cdot c,$$

where $(\cdot, \cdot)_{\mathfrak{g}}$ is the normalized Killing form such that $(\theta^{\vee}, \theta^{\vee}) = 2$ for any long root $\theta$ of $\mathfrak{g}$.

# Integrable level $\ell$ representations

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- For each $\gamma \in \Gamma$, $\mathcal{C}^\ell(\mathfrak{g},\gamma)$ is a finite semisimple abelian category with simple objects $\{\mathcal{H}_\lambda\}_{\lambda \in P^\ell(\mathfrak{g},\gamma)}$ parametrized by a certain finite subset $P^\ell(\mathfrak{g},\gamma) \subseteq P_+(\mathfrak{g}^\gamma)$.

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- For $\lambda \in P^\ell(\mathfrak{g},\gamma) \subseteq P_+(\mathfrak{g}^\gamma)$, let $V_\lambda$ be the finite dimensional irreducible $\mathfrak{g}^\gamma$-rep of highest weight $\lambda$. Via the evaluation, we can define an action of $\widehat{L}^+(\mathfrak{g},\gamma) = (\mathfrak{g} \otimes \mathbb{C}[[t]])^\gamma \oplus \mathbb{C}c$ on $V_\lambda$, where we let $c$ act by $\ell$.

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- Consider the parabolic Verma module $\mathrm{Ind}_{\widehat{L}^+(\mathfrak{g},\gamma)}^{\widehat{L}(\mathfrak{g},\gamma)} V_\lambda$. It has a unique maximal submodule, and the quotient is $\mathcal{H}_\lambda$.

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- For $\mathcal{H} \in \mathcal{C}^\ell(\mathfrak{g}, \gamma)$, its restricted dual $\mathcal{H}^\vee$ is naturally an object of $\mathcal{C}^\ell(\mathfrak{g}, \gamma^{-1})$ and we have a natural isomorphism $\mathcal{H}^{\vee\vee} = \mathcal{H}$. In other words we have a duality $\mathcal{C}^\ell(\mathfrak{g}, \gamma) \leftrightarrow \mathcal{C}^\ell(\mathfrak{g}, \gamma^{-1})^{op}$, which respects the Γ-action defined above.

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- The vacuum object $\mathbb{1} := \mathcal{H}_0 \in \mathcal{C}^\ell(\mathfrak{g}, 1)$ is a self-dual and Γ-invariant simple object.
- This says that $\mathcal{C}^\ell(\mathfrak{g}, \Gamma) := \bigoplus_{\gamma \in \Gamma} \mathcal{C}^\ell(\mathfrak{g}, \gamma)$ is a finite semisimple Γ-crossed abelian category. (In particular, we have a Γ-grading and a Γ-action on $\mathcal{C}^\ell(\mathfrak{g}, \Gamma)$.)

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- Crossed braid isomorphisms: $\beta_{X,Y} : X \otimes Y \xrightarrow{\cong} \gamma(Y) \otimes X$ for $X \in \mathcal{C}_\gamma, Y \in \mathcal{C}$ satisfying certain compatibilities like the hexagon axiom etc and such that $\mathcal{C}_1$ is a modular fusion category.

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### Theorem (D., Mukhopadhyay)

*Suppose the Γ-action on $\mathfrak{g}$ preserves a Borel subalgebra. Then the category $\mathcal{C}^\ell(\mathfrak{g}, \Gamma)$ has a natural structure of a Γ-crossed modular fusion category.*

*Ongoing work: The first restriction can be removed.*

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- This is equivalent to defining the multiplicity spaces $\mathrm{Hom}(A, A_1 \otimes \cdots \otimes A_n) = \mathrm{Hom}(\mathbb{1}, A_1 \otimes \cdots \otimes A_n \otimes A^\vee)$ for all $A \in \mathcal{C}_{\gamma_1 \cdots \gamma_n}$, and these spaces must satisfy suitable conditions.

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- In other words, given any finite collection of objects $A_i \in \mathcal{C}_{\gamma_i}$ with $\gamma_1 \cdots \gamma_n = 1$, we must assign a finite dimensional vector space to play the role of $\mathrm{Hom}(\mathbb{1}, A_1 \otimes \cdots \otimes A_n)$.

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### Theorem (D., Mukhopadhyay)

*Let $\mathcal{C}$ be a Γ-crossed abelian category. Then upgrading $\mathcal{C}$ to a Γ-crossed modular fusion category is equivalent to defining a $\mathcal{C}$-extended Γ-crossed modular functor.*

# Some history

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- The mathematical foundations and proofs were worked on by Tsuchiya-Ueno-Yamada, Beauville, Laszlo, Sorger, Faltings, Huang, Teleman and others.
- Twisted conformal blocks and their analogous properties in various settings have been studied by many authors like Birke-Fuchs-Schweigert, Shen-Wang, Frenkel-Szczesny, Damiolini, Hong-Kumar and others.

# Admissible Γ-covers of curves and their moduli stacks

- Let $\tilde{C} \to C$ be an admissible Γ-cover of (possibly nodal) complex projective curve of genus $g$ with smooth marked points $\vec{p} = (p_1, \cdots, p_n)$ on $C$ along with choice of lifts $\vec{\tilde{p}}$ on $\tilde{C}$ such that outside $\vec{p}$ and the nodes of $C$, we have an unramified Γ-covering.

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- The choice of the lifts $\tilde{p}_i$ of $p_i$ (and the root of unity $e^{\frac{2\pi\sqrt{-1}}{|\Gamma_{\tilde{p}_i}|}}$) determines a generator $\gamma_i$ of $\Gamma_{\tilde{p}_i} \leq \Gamma$. Hence a Γ-cover as above determines $\vec{\gamma} \in \Gamma^n$ associated with the marked points.

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- Also choose special formal local parameters $\tilde{t}_i$ at $\tilde{p}_i$ such that $\tilde{t}_i^{|\gamma_i|}$ is a formal local parameter $t_i$ at $p_i$. This choice gives us an identification $\mathcal{K}_{\tilde{p}_i} \cong \mathbb{C}((\tilde{t}_i))$ which respects the $\gamma_i$ action on both sides.

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- We have the moduli stacks $\widehat{\overline{\mathcal{M}}}^{\Gamma}_{g,n}$ of such $n$-marked admissible Γ-covers $(\tilde{C} \to C, \vec{\tilde{p}}, \vec{\tilde{t}})$.

# Twisted conformal blocks

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- Hence we may consider the vector space of coinvariants:
$$\mathcal{V}_{\vec{\mathcal{H}},\Gamma}(\tilde{C} \to C, \vec{\tilde{p}}, \vec{\tilde{t}}) := [\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n]_{\mathfrak{g}(\tilde{C} \setminus \Gamma \cdot \vec{\tilde{p}})^{\Gamma}}.$$

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- Working with families of admissible covers, we define the sheaf $\mathcal{V}_{\vec{\mathcal{H}},\Gamma}$ of twisted conformal blocks on $\widehat{\overline{\mathcal{M}}}^\Gamma_{g,n}(\vec{\gamma})$.

# Twisted conformal blocks and the Γ-crossed modular functor

Theorem (D., Mukhopadhyay)

*(Suppose that the action of* $\Gamma$ *on* $\mathfrak{g}$ *preserves a Borel.) The collection of the sheaves* $\mathcal{V}_{\vec{\mathcal{H}},\Gamma}$ *of twisted conformal blocks on the various moduli stacks* $\widehat{\overline{\mathcal{M}}}^{\Gamma}_{g,n}(\vec{\gamma})$ *satisfy the axioms of a* $\mathcal{C}^{\ell}(\mathfrak{g},\Gamma)$*-extended* $\Gamma$*-crossed modular functor. In particular:*

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- *The sheaves* $\mathcal{V}_{\vec{\mathcal{H}},\Gamma}$ *are in fact vector bundles and they furthermore admit a natural flat projective log-connection.*
- *The sheaves (equipped with the above connections) satisfy the propagation of vacua and factorization axioms.*

# Twisted conformal blocks and the $\Gamma$-crossed modular functor

### Theorem (D., Mukhopadhyay)

*(Suppose that the action of $\Gamma$ on $\mathfrak{g}$ preserves a Borel.) The collection of the sheaves $\mathcal{V}_{\vec{\mathcal{H}},\Gamma}$ of twisted conformal blocks on the various moduli stacks $\widehat{\overline{\mathcal{M}}}^{\Gamma}_{g,n}(\vec{\gamma})$ satisfy the axioms of a $\mathcal{C}^{\ell}(\mathfrak{g},\Gamma)$-extended $\Gamma$-crossed modular functor. In particular:*

- *The sheaves $\mathcal{V}_{\vec{\mathcal{H}},\Gamma}$ are in fact vector bundles and they furthermore admit a natural flat projective log-connection.*
- *The sheaves (equipped with the above connections) satisfy the propagation of vacua and factorization axioms.*

Hence we see that $\mathcal{C}^{\ell}(\mathfrak{g},\Gamma)$ has the structure of a $\Gamma$-crossed modular fusion category. Let us denote the monoidal fusion product on this category by $\odot$.

# Ranks of vector bundles of twisted conformal blocks

- Let $\mathcal{H}_i \in \mathcal{C}^\ell(\mathfrak{g}, \gamma_i)$, with $\gamma_1 \cdots \gamma_n = 1$. Consider the corresponding $\Gamma$-cover $(\widetilde{C} \to \mathbb{P}^1, \vec{\tilde{p}})$ with ramification data $\vec{\gamma}$. Then the multiplicity space $\mathrm{Hom}(\mathbb{1}, \mathcal{H}_1 \odot \cdots \odot \mathcal{H}_n)$ is defined to be the space of coinvariants $[\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n]_{\mathfrak{g}(\tilde{C} \setminus \Gamma \cdot \vec{\tilde{p}})^\Gamma}$.

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- More generally, the ranks of all the vector bundles $\mathcal{V}_{\vec{\mathcal{H}},\Gamma}$ on various $\widehat{\overline{\mathcal{M}}}^{\Gamma}_{g,n}(\vec{\gamma})$ equal the dimensions of suitable defined multiplicity spaces.
- In any $\Gamma$-crossed modular fusion category, we compute the dimensions of all the relevant multiplicity spaces in terms of certain "crossed S-matrices". This is the categorical twisted Verlinde formula.

# Crossed S-matrices

Let $\mathcal{C} = \bigoplus_{\gamma \in \Gamma} \mathcal{C}_\gamma$ be a $\Gamma$-crossed modular fusion category.

- Let $\gamma \in \Gamma$. Then the number of simple objects of $\mathcal{C}_\gamma$ equals the number of $\gamma$-stable simple objects of $\mathcal{C}_1$.
  Cor: $|P^\ell(\mathfrak{g}, \gamma)| = |P^\ell(\mathfrak{g})^\gamma|$.

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- For each $\gamma$-stable simple object $C$ in $\mathcal{C}_1$, choose an isomorphism $\gamma(C) \xrightarrow{\psi_C} C$. Let $M$ be a simple object of $\mathcal{C}_\gamma$.
- Define $\widetilde{S}^\gamma_{C,M}$ to be the categorical trace of the composition

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- The traces above define the unitary (after normalization) $\gamma$-crossed S-matrix $S^\gamma$ .
- For $\mathcal{C}^\ell(\mathfrak{g}, \Gamma)$, the crossed S-matrices can be explicitly computed. Hence the ranks of the vector bundles of conformal blocks can be calculated using the twisted categorical Verlinde formula.

# Twisted Verlinde formula for twisted conformal blocks

Consider a homomorphism $\pi_1(C \setminus \vec{p}, \star) \to \Gamma$ with image $\Gamma^\circ$ and the associated admissible cover $(\tilde{C} \to C, \vec{\tilde{p}})$ with monodromies $\vec{\gamma} \in (\Gamma^\circ)^n \subseteq \Gamma^n$.

## Theorem (D., Mukhopadhyay)

*Suppose that $\Gamma$ preserves a Borel subalgebra in $\mathfrak{g}$. Let $\lambda_i \in P^\ell(\mathfrak{g}, \gamma_i)$. Then*

$$\dim \mathcal{V}_{\vec{\lambda},\Gamma}(\tilde{C} \to C, \vec{\tilde{p}}) = \sum_{\lambda \in P^\ell(\mathfrak{g})^{\Gamma^\circ}} \frac{S^{\gamma_1}_{\lambda_1,\lambda} \cdots S^{\gamma_n}_{\lambda_n,\lambda}}{\left(S_{0,\lambda}\right)^{n+2g-2}}.$$

# Example

- Consider $\mathbb{Z}/3\mathbb{Z}$ acting on $\mathfrak{g}$ of type $D_4$ by diagram automorphisms rotating the Dynkin diagram. In this case $\mathfrak{g}^{\mathbb{Z}/3\mathbb{Z}}$ is of type $G_2$. The weight 0 of $G_2$ always lies in $P^{\ell}(D_4, \overline{1}) \subseteq P_+(G_2)$ for all $\ell$.

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- At level $\ell = 2$ we have: $\dim \mathbb{V}_{0,0,0,\mathbb{Z}/3\mathbb{Z}}((E \to \mathbb{P}^1, \vec{\tilde{p}})) = 3$

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- In ongoing joint work with S. Mukhopadhyay, we describe the group cohomological data that relate the $\Gamma$-crossed modular fusion categories corresponding to the actions $\phi$ and $\psi$.

*Thank You!*