Notes on the Unruh effect and Hawking radiation

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Abstract

This is a lecture note written for a tutorial session at ICTS-KAWS winter school 2022.

1 Unruh effect

The Unruh effect [1] states that the Minkowski vacuum looks thermal from the point of view of an accelerating observer. There are different ways to understand this statement. In the following I will present several complementary perspectives.

1.1 Euclidean perspective

We start from the metric of D dimensional flat space:

$$ds^2 = -dt^2 + d\vec{x}_d^2, \quad d = D - 1. \tag{1}$$

To prepare the Minkowski vacuum $|vac\rangle$ at time $t = 0$, we can first do a Wick rotation, $\tau = it$, and do a Euclidean path integral from $\tau = -\infty$ to $\tau = 0$. To check the sign, we note that under this evolution, $e^{-iHt} = e^{-H(+\infty)}$ suppresses all the finite energy states.

Similarly, if we were to compute expectation values of operators in the Minkowski vacuum, we could first insert them in the Euclidean path integral (from $\tau = -\infty$ to $+\infty$), and then continue their locations into the Lorentzian signature.

The Euclidean perspective is particular interesting because there is an equivalent way of looking at the calculation. This stems from a simple coordinate transformation. Starting from

$$ds^2 = d\tau^2 + d\vec{x}_d^2 \tag{2}$$

We go to the polar coordinate of the Euclidean time and one of the spatial coordinates, say x_1

$$\tau = r\sin\theta, \quad x_1 = r\cos\theta, \tag{3}$$

and the metric becomes

$$ds^2 = r^2 d\theta^2 + dr^2 + d\vec{x}_{d-1}^2, \quad \theta \in [0, 2\pi). \tag{4}$$

The new perspective is to think of the variable θ as the Euclidean "time" (see fig. 1). Since the variable θ has a finite period 2π, the path integral has the form of computing a thermal trace:

$$\text{Tr}\left[e^{-2\pi H_\theta}\right], \tag{5}$$

where H_θ is the Hamiltonian conjugated to translation in θ.

A few words of caution here. First, strictly speaking, since the "time" circle shrinks to a point at $r = 0$, the calculation does not really have a trace interpretation. In this lecture

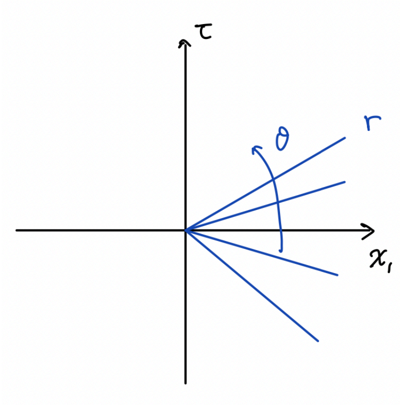

Figure 1: Another view point of the Euclidean path integral.

we always imagine cutting a tiny tube around $r = 0$ so we do not need to deal with this UV issue. Readers can consult [2] for a rigorous treatment.

Second, it is tempting to associate the 2π as being the inverse temperature β. However, this is wrong because the proper length of the circle is actually $2\pi r$. We could, however, say that the temperature as measured by a local observer at fixed r is

$$T(r) = \frac{2\pi}{r}. \tag{6}$$

To see the connection with an accelerating observer, we need to go to the Lorentzian signature. What is an observer that stays at constant radius r? It is an observer that accelerates with constant acceleration (see fig. 2). To see this, we note that an observer that travels along $r = r_0, \eta = i\theta$ is at

$$t = r_0 \sinh \eta, \quad x_1 = r_0 \cosh \eta. \tag{7}$$

The (proper) accelaration is given by $a = \frac{d}{dt}\frac{dx_1}{d\tau} = 1/r_0$. Again, an observer measures energy (and therefore temperature) in terms of their proper time, and the proper time τ is related to η via $\tau = r_0 \eta = \eta/a$.

We should emphasize that the above Euclidean derivation is very general, which applies to both free theories and interacting theories. But as a concrete example, we can consider a massless scalar field in four dimensions. Its correlator in Euclidean space is simply (omitting the prefactor)

$$\langle \phi(\tau_1, \vec{x}_{d,1}) \phi(\tau_2, \vec{x}_{d,2}) \rangle = \frac{1}{(\tau_1 - \tau_2)^2 + (\vec{x}_{d,1} - \vec{x}_{d,2})^2}. \tag{8}$$

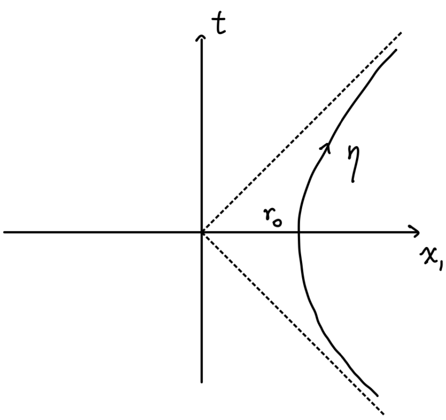

Figure 2: The trajectory of an observer with constant acceleration.

In terms of r, θ coordinates, we have (suppose the two operators have the same $\vec{x}_{d-1}$)

$$\langle \phi(r_1, \theta_1) \phi(r_2, \theta_2) \rangle = \frac{1}{(r_1 - r_2)^2 + r_1 r_2 \sin^2 \frac{\theta_1 - \theta_2}{2}}. \tag{9}$$

From this we also see that it is a periodic function in θ with period 2π.

It is easy to continue this to the Lorentzian signature, and we find

$$\langle \phi(r_1, \eta_1) \phi(r_2, \eta_2) \rangle = \frac{1}{(r_1 - r_2)^2 - r_1 r_2 \sinh^2 \frac{\eta_1 - \eta_2}{2}}. \tag{10}$$

Notice that it decays exponentially with the separation in η rather than as a power law (contrast it with the dependence on the Minkowski time). This is the typical behavior in a thermal state.

Before ending the discussion of the Euclidean perspective, we pass by noticing that such construction is quite general and is not restricted to flat space. For example, in de Sitter space, one can use similar arguments to understand that the Hartle-Hawking vacuum is thermal from the point of view of a static patch observer. Another example is the two-sided black hole, which will be relevant to out later discussion.

1.2 Lorentzian perspective

The same physics can also be derived from a Lorentzian perspective, without referring too much to the Euclidean path integral.[1]

Let's consider a free scalar field in two dimensions. We can expand the field operator as

$$\phi = \int \frac{dk}{\sqrt{2\pi\omega}} \left(a_k e^{-i\omega t + ikx} + a_k^\dagger e^{i\omega t - ikx} \right). \tag{11}$$

where $\omega = \sqrt{k^2 + m^2} > 0$ is the energy of the excitation. Notice that the annihlation operator multiplies the positive energy solution.

On the other hand, we can expand the same field operator in terms of the Rindler modes

$$\phi = \int dk' \left(b_{k',L} v_{k'} + b_{k',L}^\dagger v_{k'}^* + b_{k',R} u_{k'} + b_{k',R}^\dagger u_{k'}^* \right) \tag{12}$$

where $v_{k'}$ are functions defined on the left Rindler wedge only, and $u_{k'}$ are only defined on the right Rindler wedge. $u_{k'}$ are the positive frequency solutions associated with Rindler time η, while importantly, $v_{k'}$ are negative frequency solutions in η, since the η coordinate decreases towards the future in the left wedge. Here I'm using k' to distinguish it from the momentum k with respect to the Minkowski momentum k.

We note that from the Rindler perspective, we have two sets of modes because we now have two subsystems. The Minkowski vacuum state is an entangled state of these two subsystems.

Knowing the precise mode functions $v_{k'}, u_{k'}$, It is a standard exercise to work out the Bogoliubov transformation between the operators $\{a_k, a_k^\dagger\}$ and $\{b_{k',L}, b_{k',L}^\dagger, b_{k',R}, b_{k',R}^\dagger\}$. However, there is a quicker route due to Unruh [1] even without knowing the details of the functions.

The original mode functions in (12) are only defined in half spaces. Recall from the Euclidean discussion that we can go from the right wedge to the left wedge through $\eta \to \eta \pm i\pi$. Starting from the mode $v_{k'}$, after applying this analytic continuation, we would end up with a mode function in the left wedge. The idea is that we can construct a purely positive frequency Minkowski mode from the analytic continuation.[2]

[1] We still need someway to define the vacuum state. Usually people describe it using the $i\epsilon$ prescription, which is secretly the same as choosing the Euclidean vacuum.

[2] After the function is pinned down on the left and the right Rindler wedges, the function in the future and past wedges are determined as well.

The question is which analytic continuation we should use, such that the combination of the mode functions will correspond to a Minkowski mode with positive frequency.

What we know about the positive frequency Minkowski mode is that, it is regular in the lower half complex plane of t. This actually dictates which analytic continuation we should use. Recall the coordinate transformation in the right wedge:

$$t - x = -re^{-\eta}, \quad t + x = re^{\eta}, \tag{13}$$

holding x, r real, $\eta \to \eta - i\pi$ is the analytic continuation which leave $\mathrm{Im}(t) < 0$. Under this analytic continuation:

$$v_{k'} \sim e^{-i\omega'\eta} \to e^{-\pi\omega'} e^{-i\omega'\eta} \sim e^{-\pi\omega'} u^*_{k'} \tag{14}$$

Strictly speaking we need to also flip the sign of the momentum in $u^*_{k'}$, here I'm neglecting it since I never really defined k' in the first place. Therefore we concluded that the combination

$$v_{k'} + e^{-\pi\omega'} u^*_{k'} \tag{15}$$

involves only positive Minkowski components. Suppose v and u are already properly nomalized (according to the Klein-Gordon norm), it is easy to check that the following expression is also properly normalized:

$$f_{k'} = \sqrt{\frac{e^{\pi\omega'}}{2\sinh \pi\omega'}} \left(v_{k'} + e^{-\pi\omega'} u^*_{k'} \right). \tag{16}$$

In above, we started from the right wedge and did the analytic continuation. We could have also started from the left wedge and do the analytic continuation. That yields a different mode

$$g_{k'} = \sqrt{\frac{e^{\pi\omega'}}{2\sinh \pi\omega'}} \left(u_{k'} + e^{-\pi\omega'} v^*_{k'} \right). \tag{17}$$

The complex conjugate f^*, g^* will only involve negative Minkowski frequencies.

So we can expand the field operator in these operators (since we will not be talking about the plane wave modes anymore, let me drop the primes from now on)

$$\phi = \int dk \left(c_k f_k + c^\dagger_k f^*_k + d_k g_k + d^\dagger_k g^*_k \right). \tag{18}$$

We can now compare this with the expansion in Rindler modes

$$\phi = \int dk \left(b_{k,L} v_k + b^\dagger_{k,L} v^*_k + b_{k,R} u_k + b^\dagger_{k,R} u^*_k \right) \tag{19}$$

since we know how the mode functions are related, it is a simple algebra exercise to work out the Bogoliubov transformation. Remarkably, we've only used some analytic continuation arguments, but we never even write down the explicit form of the mode functions!

The operators c_k, d_k have the nice property that they annihlate the Minkowski vacuum:

$$c_k \left| vac \right\rangle = d_k \left| vac \right\rangle = 0. \tag{20}$$

From this one can work out how the Minkowski vacuum translates into an entangled state of the two Rindler wedges. The end result is

$$\left| vac \right\rangle = \prod_k \sqrt{2e^{-2\pi\omega_k} \sinh \pi\omega_k} \sum_{n_k=0}^{\infty} e^{-\pi\omega_k n_k} \left| n_k \right\rangle_L \left| n_k \right\rangle_R. \tag{21}$$

In other words, the Minkowski vacuum is a tensor product of thermofield double states for different modes on the two Rindler wedges, with inverse temperature 2π. The reduced density matrix in each wedge has the form of a thermal density matrix, as we derived from the Euclidean path integral.

1.3 Stress tensor

We could also verify that the Minkowski vacuum looks thermal from a Rindler observer point of view by computing the stress tensor that the observer sees.

One immediately puzzle is that, the stress tensor in the Minkowski vacuum, is zero by definition:

$$\langle T_{\mu\nu} \rangle = 0. \tag{22}$$

This is true in any coordinate systems, including the Rindler coordinates. So how come that the Rindler observer sees a nonzero stress tensor?

The resolution to the puzzle lies in that the value of the stress tensor depends on how we subtract the UV divergence. A Minkowski observer will subtract the UV divergence in the Minkowski vacuum, while for a Rindler observer, it is only natural to subtract the UV divergence in the Rindler vacuum. The difference in the two subtractions will lead to a finite answer.

Let's demonstrate this with a simple example: a massless boson in 2d. We want to understand what's the UV divergence that a Rindler observer will subtract. To see that, it is convenient to write the Rindler coordinate in the following way:

$$ds^2 = -r^2 d\eta^2 + dr^2 = e^{2\chi}(-d\eta^2 + d\chi^2), \quad -\infty < \chi < \infty. \tag{23}$$

Since the massless boson is a conformal field, if it is in the Rindler vacuum, its correlation function will take the form as if it is in the Minkowski vacuum of the $\{\eta, \chi\}$ coordinates. We can further introduce the null coordinates

$$u = \eta - \chi, \quad v = \eta + \chi, \quad ds^2 = e^{-(u-v)}(-du dv) \tag{24}$$

and we have

$$\langle \partial_u \phi(u_1, v_1) \partial_u \phi(u_1, v_1) \rangle_R = -\frac{1}{4\pi(u_1 - u_2)^2}, \tag{25}$$

and a similar formula for v. The subscript R emphasizes this is in the Rindler vacuum. The expression of the stress tensor is

$$T_{uu} = \partial_u \phi \partial_u \phi, \tag{26}$$

which diverges as we bring the two $\partial\phi$ together. A Rindler observer will always substract the vacuum contribution in (39) to get a finite result.

Now suppose we are in fact in the Minkowski vacuum. The two point function will no longer be given by (39). What would it be? We can write the metric as

$$ds^2 = -dU dV, \quad U = -e^{-u}, V = e^{v}, \tag{27}$$

and

$$\langle \partial_U \phi(U_1, V_1) \partial_U \phi(U_2, V_2) \rangle_M = -\frac{1}{4\pi(U_1 - U_2)^2} = -\frac{1}{4\pi(e^{-u_1} - e^{-u_2})^2} \tag{28}$$

$$\langle \partial_u \phi(u_1, v_1) \partial_u \phi(u_2, v_2) \rangle_M = -\frac{e^{-u_1 - u_2}}{4\pi(e^{-u_1} - e^{-u_2})^2}. \tag{29}$$

So the stress tensor of the Minkowski vacuum, as seen by a Rindler observer, is

$$T_{uu} = \lim_{u_1 \to u_2} \left(-\frac{e^{-u_1 - u_2}}{4\pi(e^{-u_1} - e^{-u_2})^2} + \frac{1}{4\pi(u_1 - u_2)^2} \right) = \frac{1}{48\pi}. \tag{30}$$

We can repeat the discussion for T_{vv} and get the same answer.

We could check that the stress tensor is consistent with the temperature (as seen by the conformal field) being $1/(2\pi)$.[3]

The same conclusion can be reached by computing the conformal anomaly. Above we are just using a more explicit calculation to illustrate the physical meaning.

[3] The Casimir energy on a cylinder with circumference 2π is equal to $E = -\frac{c}{12}$.

1.4 Detector

In Unruh's original paper [1], it is shown that an accelerating detector will indeed click as if it is in a thermal evironment. From the Rindler observer point of view, this is easy to describe, it is simply that the detector absorbs a particle from the thermal environment.

How do we describe this from the point of view of a Minkowski observer? It seems to be a puzzle because there is no particles in the Minkowski vacuum to begin with, so where does the particle that the detector absorbs come from?

The resolution to this puzzle was nicely explained in a paper by Unruh and Wald [3]. Consider a detector that is a two level system, with level spacing Ω. Suppose the detector contains a single spin, which starts in the down state $|\downarrow\rangle$, and accelerates with acceleration a. Further suppose that the detector couples to the scalar field we discussed in sec. 1.2 via a term like $\epsilon(\eta)\sigma_x\phi$, where $\epsilon(\eta)$ is a small time dependent coupling. We are not going into the details of the set up, but to see the resolution, we only need to know that the state of the system at late time looks like

$$|\psi(\eta\to\infty)\rangle = |vac,\downarrow\rangle - i\tilde{\epsilon}b_{\omega=\Omega/a,R}\,|vac,\uparrow\rangle \tag{31}$$

where $\tilde{\epsilon}$ can be computed from the details of the set-up and is proportional to ϵ. b_R is the annihlation operator in the right Rindler wedge, which we defined in sec. 1.2. To see what a Minkowski observer should see, we can express b_R in terms of the creation/annihlation operators with respect to the Minkowski time. We've discussed this in sec. 1.2, the answer is

$$\begin{aligned}|\psi(\eta\to\infty)\rangle &= |vac,\downarrow\rangle - i\tilde{\epsilon}\sqrt{\frac{e^{\frac{\pi\Omega}{a}}}{2\sinh\frac{\pi\Omega}{a}}}\left(d_{\omega=\frac{\Omega}{a}} + e^{-\frac{\pi\Omega}{a}}c^\dagger_{\omega=\frac{\Omega}{a}}\right)|vac,\uparrow\rangle \\ &= |vac,\downarrow\rangle - i\tilde{\epsilon}\sqrt{\frac{1}{2\sinh\frac{\pi\Omega}{a}}}c^\dagger_{\omega=\frac{\Omega}{a}}\,|vac,\uparrow\rangle\end{aligned} \tag{32}$$

So from the Minkowski observer point of view, a particle was emitted rather than being absorbed!

A higher order paradox is the following. From the Minkowski observer point of view, since an excitation is emitted by the detector, the energy of the fields in the right wedge should go up. How does one explain this from the Rindler observer point of view, since an excitation is absorbed? The answer to this can be found in [3].

2 Black hole

2.1 Near horizon geometry

For a black hole, there is a real horizon due to spacetime curvature, rather than a fictituous horizon as seen by an accelerating observer in flat space. However, they certainly bear a lot of similarities due to the equivalence principle.

In fact, if we zoom in into the near horizon region of a black hole, the geometry is approximately that of a Rindler space. Let's illustrate this with a Schwarzschild black hole:

$$ds^2 = -\left(1-\frac{r_s}{r}\right)dt^2 + \frac{dr^2}{1-\frac{r_s}{r}} + r^2 d\Omega_2^2. \tag{33}$$

Expand around the horizon to leading order in $(r-r_s)$,

$$ds^2 \approx -\frac{r-r_s}{r_s}dt^2 + \frac{r_s dr^2}{r-r_s} + r_s^2 d\Omega_2^2, \tag{34}$$

introduce

$$d\rho = \sqrt{\frac{r_s}{r-r_s}}dr, \quad \rho = 2r_s\sqrt{r-r_s}, \tag{35}$$

we find

$$ds^2 \approx -\frac{\rho^2}{4r_s^2}dt^2 + d\rho^2 + r_s^2 d\Omega_2^2. \tag{36}$$

We note that the radial and the time part of the metric takes the form of a Rindler space, with Rindler coordinate $\eta = t/(2r_s)$. So the inverse temperature in term of the coordinate t is

$$\beta = 2\pi(2r_s) = 4\pi r_s. \tag{37}$$

Again, this is not the temperature that an observer at fix r would measure, since t is not the proper time of the observer. We need to take into account the gravitational blue/red shift. However, t is the proper time of an observer sitting very far away from the black hole, so (37) is indeed the inverse temperature from their point of view. In fact, for a black hole that is in thermal equilibrium with the environment, the temperature as measured by an observer at radius r is given by

$$T = \frac{1}{4\pi r_s \sqrt{1 - \frac{r_s}{r}}}. \tag{38}$$

Note that in this discussion, I've secretly assumed that the region $\rho < 0$ exists, and is connected to the $\rho > 0$ region in the same way as the Minkowski space.

2.2 Various types of vacua

In the previous discussion of Minkowski vs Rindler, we've mainly focused on the Minkowski vacuum. In the context of black holes, such a vacuum is called the *Hartle-Hawking vacuum* (see fig. 3).

As we argued, in a Minkowski vacuum, the Rindler observer sees nonzero incoming and outgoing energy fluxes T_{uu}, T_{vv}. In the context of black hole, this means that a far away observer also sees incoming and outgoing radiations. The black hole manages to stay at thermal equilibrium due to both radiations. The black hole will have a future horizon and a past horizon and the stress tensor will be regular on both.

Similar to the Minkowski vacuum, we can prepare the Hartle-Hawking vacuum through a Euclidean path integral. This is presumably covered in Tom Hartman's lecture.

On the other hand, in the context of black holes, there are other types of vacuum that are important. The most physically relevant one is the so called *Unruh vacuum*. This is defined as a vacuum where the asymptotic observer only sees outgoing radiation but not incoming ones.

We can again understand the property of this vacuum by looking at our 2d toy model in sec. 1.3. Consider now the incoming modes are in the Rindler vacuum, in other words, the two point function of incoming modes $\partial_v \phi$ are given by

$$\langle \partial_v \phi(u_1, v_1) \partial_v \phi(u_1, v_1) \rangle_R = -\frac{1}{4\pi(v_1 - v_2)^2}. \tag{39}$$

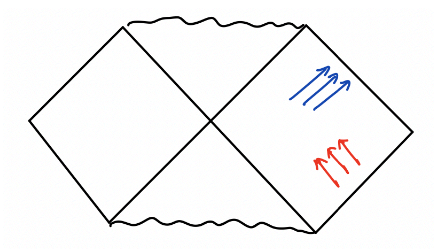

Figure 3

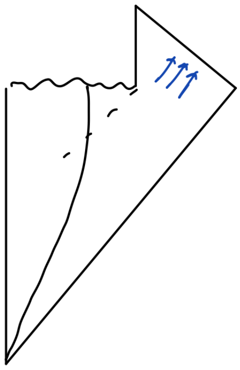

Figure 4

What would be the stress tensor that a Minkowski observer sees? Different from sec. 1.3, we would now want to substract the UV divergence corresponding to the Minkowski vacuum. Therefore

$$\begin{aligned} T_{VV} &= \lim_{V_1\to V_2}\left(-\frac{e^{-v_1-v_2}}{4\pi(v_1-v_2)^2}+\frac{1}{4\pi(V_1-V_2)^2}\right) \\ &= \lim_{V_1\to V_2}\left(-\frac{1}{4\pi V_1V_2(\log V_1-\log V_2)^2}+\frac{1}{4\pi(V_1-V_2)^2}\right) \\ &= -\frac{1}{48\pi V^2}. \end{aligned} \tag{40}$$

We see that the stress tensor is divergent in the past horizon!

A careful analysis on the black hole backround is much more complicated. Nonetheless, the main features we learned from the above discussion still holds for the black hole [4].

It might seem to be a problem that the stress tensor diverges on the past horizon. But we should remind ourselves that for an astrophysical black hole that forms from gravitational collapse, there isn't really a past horizon. What replaces it is the collapsing matter shell, in which the gravity solution is no longer given by the black hole metric (see fig. 4). The singular past horizon will however arise if we were to evolve the black hole and the radiations backwards while removing the matter shell.

There is also the *Boulware vacuum*, in which a far away observer sees no radiation. In this vacuum, the stress tensor is singular on both the future and the past horizon.

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