Revisiting multiple thermal reservoir stochastic thermodynamics

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Based on:
arXiv:2303.14949

Markovian Evolution

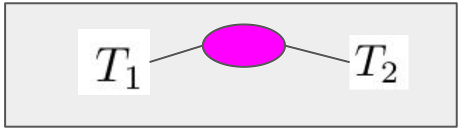
$$\frac{dP(\epsilon_i)}{dt} = \sum_{\epsilon_j} W_{\epsilon_i,\epsilon_j} P(\epsilon_j) - \sum_{j} W_{\epsilon_j,\epsilon_i} P(\epsilon_i)$$

Detailed Balance

$$W_{\epsilon_i,\epsilon_j} e^{-\beta \epsilon_j} = W_{\epsilon_j,\epsilon_i} e^{-\beta \epsilon_i}$$

System connected to a single thermal reservoir at fixed temperature

System connected to '2' thermal reservoirs



$$W_{\epsilon_i,\epsilon_j} = W^1_{\epsilon_i,\epsilon_j} + W^2_{\epsilon_i,\epsilon_j}.$$

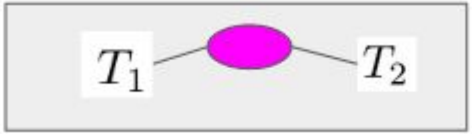
(In Stochastic Thermodynamics literature)

$$\frac{W^{1,2}_{\epsilon_i,\epsilon_j}}{W^{1,2}_{\epsilon_j,\epsilon_i}} = e^{-\beta^{1,2}(\epsilon_i-\epsilon_j)}$$

We show

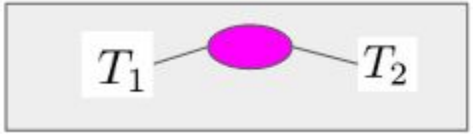
Inconsistent

Generic evolution is non-Markovian



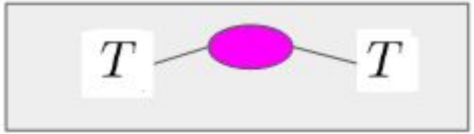
$$W_{\epsilon_i,\epsilon_j} = W^1_{\epsilon_i,\epsilon_j} + W^2_{\epsilon_i,\epsilon_j}.$$

True for all T_1, T_2.



limit

$T_1 = T_2 = T.$



$$W_{\epsilon_i,\epsilon_j} = W^1_{\epsilon_i,\epsilon_j} = W^2_{\epsilon_i,\epsilon_j}.$$

Contradiction

A more mathematical argument for the case

$$\mathbf{T_1} \neq \mathbf{T_2}$$

found in

arXiv:2303.14949

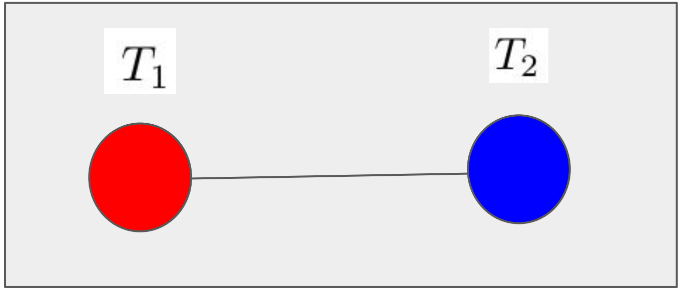
Two thermal reservoirs

would attach at

atleast

TWO points in space.

Simple System: Two points at different Temperatures

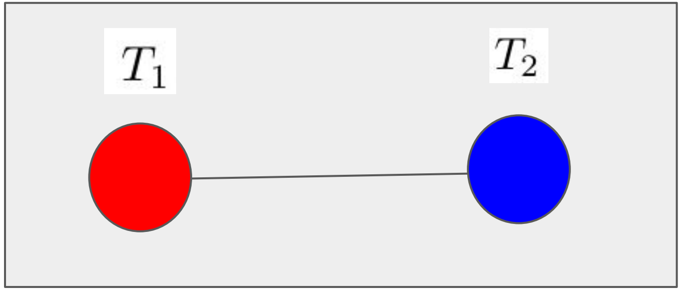


Each point can take energies

$$0, \epsilon, 2\epsilon, 3\epsilon, 4\epsilon$$

Show:
Generically, the composite evolution is non-Markovian

Simple System: Two points at different Temperatures



In time interval
dt
only one point
can exchange energy
with surroundings.

Assuming Markovian Evolution

$$\begin{aligned} P(3\epsilon \to 5\epsilon)dt \quad &= [w^2_{3\epsilon,\epsilon} + w^1_{4\epsilon,2\epsilon}]dtp^1(2\epsilon)p^2(\epsilon) \\ &+[w^1_{3\epsilon,\epsilon} + w^2_{4\epsilon,2\epsilon}]dtp^1(\epsilon)p^2(2\epsilon) \\ &+[w^2_{2\epsilon,0} + w^1_{5\epsilon,3\epsilon}]dtp^1(3\epsilon)p^2(0) \\ &+[w^1_{2\epsilon,0} + w^2_{5\epsilon,3\epsilon}]dtp^1(0)p^2(3\epsilon) \\ &= WW^{1,2}_{5\epsilon,3\epsilon}P(3\epsilon)dt \end{aligned}$$

$$= WW^{1,2}_{5\epsilon,3\epsilon}dt[p^1(2\epsilon)p^2(\epsilon) + p^1(\epsilon)p^2(2\epsilon) + p^1(3\epsilon)p^2(0) + p^1(0)p^2(3\epsilon)]$$

Assuming Markovian Evolution

$$\begin{aligned} P(5\epsilon \to 3\epsilon)dt \quad &= [w^1_{\epsilon,3\epsilon} + w^2_{0,2\epsilon}]dtp^1(3\epsilon)p^2(2\epsilon) \\ &+[w^2_{\epsilon,3\epsilon} + w^1_{0,2\epsilon}]p^1(2\epsilon)p^2(3\epsilon) \\ &+w^1_{2\epsilon,4\epsilon}dtp^1(4\epsilon)p^2(\epsilon) + w^2_{2\epsilon,4\epsilon}dtp^1(\epsilon)p^2(4\epsilon) \\ &+w^1_{3\epsilon,5\epsilon}dtp^1(5\epsilon)p^2(0) + w^2_{3\epsilon,5\epsilon}dtp^1(0)p^2(5\epsilon) \\ &= WW^{1,2}_{3\epsilon,5\epsilon}P(5\epsilon)dt \end{aligned}$$

$$= WW^{1,2}_{3\epsilon,5\epsilon}dt[p^1(2\epsilon)p^2(3\epsilon) + p^1(3\epsilon)p^2(2\epsilon) + p^1(5\epsilon)p^2(0) + p^1(0)p^2(5\epsilon) + p^1(4\epsilon)p^2(\epsilon) + p^1(\epsilon)p^2(4\epsilon)]$$

$$\mathbf{T_1} = \mathbf{T_2} = \mathbf{T}$$

$$w^1_{\epsilon,\epsilon'} = w^2_{\epsilon,\epsilon'} = w_{\epsilon,\epsilon'}$$

$$p^1(\epsilon) = p^2(\epsilon) = p(\epsilon).$$

$P(3\epsilon) = 2[p(3\epsilon)p(0) + p(2\epsilon)p(\epsilon)]$ and $P(5\epsilon) = 2[p(5\epsilon)p(0) + p(4\epsilon)p(\epsilon) + p(3\epsilon)p(2\epsilon)]$

$$
\begin{aligned}
P(5\epsilon \rightarrow 3\epsilon)dt \quad &= [w_{\epsilon,3\epsilon} + w_{0,2\epsilon}]dtp(3\epsilon)p(2\epsilon) \\
&+[w_{\epsilon,3\epsilon} + w_{0,2\epsilon}]p(2\epsilon)p(3\epsilon) \\
&+w_{2\epsilon,4\epsilon}dtp(4\epsilon)p(\epsilon) + w_{2\epsilon,4\epsilon}dtp(\epsilon)p(4\epsilon) \\
&+w_{3\epsilon,5\epsilon}dtp(5\epsilon)p^{-}(0) + w_{3\epsilon,5\epsilon}dtp(0)p(5\epsilon) \\
&= WW_{3\epsilon,5\epsilon}P(5\epsilon)dt
\end{aligned}
$$

$$
= 2WW_{3\epsilon,5\epsilon}[p(5\epsilon)p(0) + p(4\epsilon)p(\epsilon) + p(3\epsilon)p(2\epsilon)]dt
$$

$$
\begin{aligned}
P(3\epsilon \rightarrow 5\epsilon)dt \quad &= [w_{3\epsilon,\epsilon} + w_{4\epsilon,2\epsilon}]dtp(2\epsilon)p(\epsilon) \\
&+[w_{3\epsilon,\epsilon} + w_{4\epsilon,2\epsilon}]dtp(\epsilon)p(2\epsilon) \\
&+[w_{2\epsilon,0} + w_{5\epsilon,3\epsilon}]dtp(3\epsilon)p(0) \\
&+[w_{2\epsilon,0} + w_{5\epsilon,3\epsilon}]dtp(0)p(3\epsilon) \\
&= WW_{5\epsilon,3\epsilon}P(3\epsilon)dt
\end{aligned}
$$

$$
= 2WW_{5\epsilon,3\epsilon}[p(3\epsilon)p(0) + p(2\epsilon)p(\epsilon)]dt
$$

$$
\begin{aligned}
P(3\epsilon \to 5\epsilon)dt \quad &= [w_{3\epsilon,\epsilon} + w_{4\epsilon,2\epsilon}]dtp(2\epsilon)p(\epsilon) \\
&+[w_{3\epsilon,\epsilon} + w_{4\epsilon,2\epsilon}]dtp(\epsilon)p(2\epsilon) \\
&+[w_{2\epsilon,0} + w_{5\epsilon,3\epsilon}]dtp(3\epsilon)p(0) \\
&+[w_{2\epsilon,0} + w_{5\epsilon,3\epsilon}]dtp(0)p(3\epsilon) \\
&= WW_{5\epsilon,3\epsilon}P(3\epsilon)dt
\end{aligned}
$$

$$
= 2WW_{5\epsilon,3\epsilon}[p(3\epsilon)p(0) + p(2\epsilon)p(\epsilon)]dt
$$

$$
\begin{aligned}
w_{\epsilon,\epsilon'} \quad &= f(\epsilon - \epsilon') \\
&\quad if\ \epsilon > \epsilon'
\end{aligned}
$$

$$
\begin{aligned}
P(5\epsilon \to 3\epsilon)dt \quad &= [w_{\epsilon,3\epsilon} + w_{0,2\epsilon}]dtp(3\epsilon)p(2\epsilon) \\
&+[w_{\epsilon,3\epsilon} + w_{0,2\epsilon}]p(2\epsilon)p(3\epsilon) \\
&+w_{2\epsilon,4\epsilon}dtp(4\epsilon)p(\epsilon) + w_{2\epsilon,4\epsilon}dtp(\epsilon)p(4\epsilon) \\
&+w_{3\epsilon,5\epsilon}dtp(5\epsilon)p^{-}(0) + w_{3\epsilon,5\epsilon}dtp(0)p(5\epsilon) \\
&= WW_{3\epsilon,5\epsilon}P(5\epsilon)dt
\end{aligned}
$$

$$
= 2WW_{3\epsilon,5\epsilon}[p(5\epsilon)p(0) + p(4\epsilon)p(\epsilon) + p(3\epsilon)p(2\epsilon)]dt
$$

$$
\begin{aligned}
w_{\epsilon,\epsilon'} \quad &= f(\epsilon - \epsilon') \\
&\; if\ \epsilon' > \epsilon \\
&\; \epsilon' - \epsilon < E - \epsilon' \\
w_{\epsilon,\epsilon'} \quad &= 2f(\epsilon - \epsilon') \\
&\; if\ \epsilon' > \epsilon \\
&\; \epsilon' - \epsilon > E - \epsilon'.
\end{aligned}
$$

$$w_{\epsilon,\epsilon'} = f(\epsilon - \epsilon')$$
$$if\ \epsilon > \epsilon'$$

$$\begin{aligned} w_{\epsilon,\epsilon'} &= f(\epsilon - \epsilon') \\ &if\ \epsilon' > \epsilon \\ &\epsilon' - \epsilon < E - \epsilon' \\ w_{\epsilon,\epsilon'} &= 2f(\epsilon - \epsilon') \\ &if\ \epsilon' > \epsilon \\ &\epsilon' - \epsilon > E - \epsilon'. \end{aligned}$$

$$
\begin{aligned}
P(3\epsilon \to 5\epsilon)dt \quad &= [w_{3\epsilon,\epsilon} + w_{4\epsilon,2\epsilon}]dtp(2\epsilon)p(\epsilon) \\
&+[w_{3\epsilon,\epsilon} + w_{4\epsilon,2\epsilon}]dtp(\epsilon)p(2\epsilon) \\
&+[w_{2\epsilon,0} + w_{5\epsilon,3\epsilon}]dtp(3\epsilon)p(0) \\
&+[w_{2\epsilon,0} + w_{5\epsilon,3\epsilon}]dtp(0)p(3\epsilon) \\
&= WW_{5\epsilon,3\epsilon}P(3\epsilon)dt
\end{aligned}
$$

$$
= 2WW_{5\epsilon,3\epsilon}[p(3\epsilon)p(0) + p(2\epsilon)p(\epsilon)]dt
$$

$$
\begin{aligned}
w_{\epsilon,\epsilon'} \quad &= f(\epsilon - \epsilon') \\
&if\ \epsilon > \epsilon'
\end{aligned}
$$

Assuming Markovian Evolution

$$\begin{aligned} P(3\epsilon \to 5\epsilon)dt \quad &= [w^2_{3\epsilon,\epsilon} + w^1_{4\epsilon,2\epsilon}]dt p^1(2\epsilon)p^2(\epsilon) \\ &+[w^1_{3\epsilon,\epsilon} + w^2_{4\epsilon,2\epsilon}]dt p^1(\epsilon)p^2(2\epsilon) \\ &+[w^2_{2\epsilon,0} + w^1_{5\epsilon,3\epsilon}]dt p^1(3\epsilon)p^2(0) \\ &+[w^1_{2\epsilon,0} + w^2_{5\epsilon,3\epsilon}]dt p^1(0)p^2(3\epsilon) \\ &= WW^{1,2}_{5\epsilon,3\epsilon} P(3\epsilon)dt \end{aligned}$$

$$= WW^{1,2}_{5\epsilon,3\epsilon} dt[p^1(2\epsilon)p^2(\epsilon) + p^1(\epsilon)p^2(2\epsilon) + p^1(3\epsilon)p^2(0) + p^1(0)p^2(3\epsilon)]$$

$$\mathbf{T_1 \neq T_2}$$

$$P(3\epsilon \to 5\epsilon)dt \quad = [f^1(2\epsilon) + f^2(2\epsilon)]dtP(3\epsilon),$$

$$
\begin{aligned}
P(5\epsilon \to 3\epsilon)dt \quad &= [w_{\epsilon,3\epsilon} + w_{0,2\epsilon}]dtp(3\epsilon)p(2\epsilon) \\
&+[w_{\epsilon,3\epsilon} + w_{0,2\epsilon}]p(2\epsilon)p(3\epsilon) \\
&+w_{2\epsilon,4\epsilon}dtp(4\epsilon)p(\epsilon) + w_{2\epsilon,4\epsilon}dtp(\epsilon)p(4\epsilon) \\
&+w_{3\epsilon,5\epsilon}dtp(5\epsilon)p^{-}(0) + w_{3\epsilon,5\epsilon}dtp(0)p(5\epsilon) \\
&= WW_{3\epsilon,5\epsilon}P(5\epsilon)dt
\end{aligned}
$$

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= 2WW_{3\epsilon,5\epsilon}[p(5\epsilon)p(0) + p(4\epsilon)p(\epsilon) + p(3\epsilon)p(2\epsilon)]dt
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$$
\begin{aligned}
w_{\epsilon,\epsilon'} \quad &= f(\epsilon - \epsilon') \\
&\quad if\ \epsilon' > \epsilon \\
&\quad \epsilon' - \epsilon < E - \epsilon' \\
w_{\epsilon,\epsilon'} \quad &= 2f(\epsilon - \epsilon') \\
&\quad if\ \epsilon' > \epsilon \\
&\quad \epsilon' - \epsilon > E - \epsilon'.
\end{aligned}
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Assuming Markovian Evolution

$$
\begin{aligned}
P(5\epsilon \to 3\epsilon)dt \quad &= [w^1_{\epsilon,3\epsilon} + w^2_{0,2\epsilon}]dtp^1(3\epsilon)p^2(2\epsilon) \\
&+[w^2_{\epsilon,3\epsilon} + w^1_{0,2\epsilon}]p^1(2\epsilon)p^2(3\epsilon) \\
&+w^1_{2\epsilon,4\epsilon}dtp^1(4\epsilon)p^2(\epsilon) + w^2_{2\epsilon,4\epsilon}dtp^1(\epsilon)p^2(4\epsilon) \\
&+w^1_{3\epsilon,5\epsilon}dtp^1(5\epsilon)p^2(0) + w^2_{3\epsilon,5\epsilon}dtp^1(0)p^2(5\epsilon) \\
&= WW^{1,2}_{3\epsilon,5\epsilon}P(5\epsilon)dt
\end{aligned}
$$

$$
= WW^{1,2}_{3\epsilon,5\epsilon}dt[p^1(2\epsilon)p^2(3\epsilon) + p^1(3\epsilon)p^2(2\epsilon) + p^1(5\epsilon)p^2(0) + p^1(0)p^2(5\epsilon) + p^1(4\epsilon)p^2(\epsilon) + p^1(\epsilon)p^2(4\epsilon)]
$$

$$\mathbf{T_1 \neq T_2}$$

$$P(5\epsilon \to 3\epsilon)dt = [f^1(-2\epsilon) + f^2(-2\epsilon)]dtp^1(3\epsilon)p^2(2\epsilon)$$
$$+[f^1(-2\epsilon) + f^2(-2\epsilon)]dtp^1(2\epsilon)p^2(3\epsilon)$$
$$+2f^1(-2\epsilon)dtp^1(4\epsilon)p^2(\epsilon) + 2f^2(-2\epsilon)dtp^1(\epsilon)p^2(4\epsilon)$$
$$+2f^1(-2\epsilon)dtp^1(5\epsilon)p^2(0) + 2f^2(-2\epsilon)dtp^1(0)p^2(5\epsilon)$$

$$P(5\epsilon \to 3\epsilon)dt = [f^1(-2\epsilon) + f^2(-2\epsilon)]dtp^1(3\epsilon)p^2(2\epsilon)$$
$$+[f^1(-2\epsilon) + f^2(-2\epsilon)]dtp^1(2\epsilon)p^2(3\epsilon)$$
$$+2f^1(-2\epsilon)dtp^1(4\epsilon)p^2(\epsilon) + 2f^2(-2\epsilon)dtp^1(\epsilon)p^2(4\epsilon)$$
$$+2f^1(-2\epsilon)dtp^1(5\epsilon)p^2(0) + 2f^2(-2\epsilon)dtp^1(0)p^2(5\epsilon)$$

$$= [f^1(-2\epsilon) + f^2(-2\epsilon)]dtP(5\epsilon)$$

$$+[f^1(-2\epsilon) - f^2(-2\epsilon)]dt[p^1(4\epsilon)p^2(\epsilon) - p^1(\epsilon)p^2(4\epsilon)]$$
$$+[f^1(-2\epsilon) - f^2(-2\epsilon)]dt[p^1(5\epsilon)p^2(0) - p^1(0)p^2(5\epsilon)]$$

$$< 2[f^1(-2\epsilon) + f^2(-2\epsilon)]dtP(5\epsilon)$$

Conclusions

- Evolution of generic systems connected to multiple reservoirs is non-Markovian.
- Additional research is needed to understand fluctuation relations for these systems.