Fermionic Chern–Simons theory on $S^2 \times S^1$ in the 'temporal' gauge

Positive Geometry in Scattering Amplitudes Cosmological Correlators, ICTS, Bengaluru

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Objective

We look at a fully non-perturbative, finite-temperature solution to a non-supersymmetric, non-abelian gauge theory with vector fermions in curved space – fermionic Chern–Simons theory on $S^2 \times S^1$.

Background

- Light-cone gauge calculations demonstrate Bose–Fermi duality [AGAY12] at the level of the thermal free energy;
- Reproduced the free energy calculations for the fermionic theory on $\mathbb{R}^2 \times S^1$ using the *'temporal' gauge* [MNTV23];
- Set up the calculation amenable to general genus-g Riemann manifolds.

Motivation

To establish Bose–Fermi duality on genus-g Riemann manifolds (possibly at finite N)!

Path integral of our theory

The finite temperature partition function of $U(N)$ Chern–Simons theory coupled to fundamental fermionic matter is defined by the Euclidean path integral,

$$Z = \int \mathcal{D}\left[A_\mu\right] \mathcal{D}\left[\psi\right] \mathcal{D}\left[\bar{\psi}\right] e^{-S_E}, \tag{1}$$

where the Euclidean action S_E is given by,

$$S_E = \frac{i\kappa}{4\pi} \int_{\Sigma \times S^1} \mathrm{Tr}\left\{A \wedge \mathrm{d}A - \frac{2i}{3} A \wedge A \wedge A\right\} + \int_{\Sigma \times S^1} \bar{\psi}(\not{D} + M)\psi. \tag{2}$$

We study the path integral of this theory on $\Sigma \times S^1$, where Σ is an arbitrary Riemann surface, and set up the problem at finite N and temperature $T = \beta^{-1}$, where β is the circumference of the S^1.

Gauge conditions adopted

- The *'temporal' gauge* is defined by $\partial_3 A_3(x) = 0$, which imposes on the holonomy field $U(x)$,

$$\partial_3 U(x) = \partial_3 e^{i\beta A_3(x)} = 0; \tag{3}$$

- The remaining two-dimensional $U(N)$ is reduced to a two-dimensional $U(1)^N$ by simultaneously diagonalizing the holonomy field $U(\vec{x})$ for every point $\vec{x}$ on Σ [BT93, JMS+13];
- Complete gauge is fixed by imposing the Coulomb gauge on the time-independent (zero mode) diagonal elements of A_1, A_2, or equivalently,

$$(A_{\dot{\mu}})_\sigma{}^\sigma = \sum_{j=1}^{N(g)=2g} \alpha_j^\sigma a_{\dot{\mu},j} + \epsilon_{\dot{\mu}\dot{\nu}} \partial_{\dot{\nu}} \chi^\sigma, \qquad \dot{\mu}, \dot{\nu} = 1, 2, \tag{4}$$

where $\{a_j\}$ is a basis for the nontrivial flat one-forms on the genus g Riemann surface Σ (the number of such one-forms is $N(g)$).

Exact path integral on a general manifold Σ at finite N and volume

In summary, the exact (in N and volume) partition function down to an unfixed two-dimensional $U(1)^N$ is given by,

$$Z = \left(\prod_{\alpha=1}^{N}\sum_{n_\alpha=1}^{\infty}\right)(-1)^{(N-1)\sum_\alpha n_\alpha}\int\prod_{\alpha=1}^{N}\left(\mathcal{D}\left[A_3^\alpha(\vec{x})\right]\mathcal{D}\left[(A_{\dot\mu})_\alpha{}^\alpha(\vec{x})\right]\prod_{m\in\mathbb{Z}+\frac{1}{2}}\mathcal{D}\left[\bar\psi_m^\alpha(\vec{x})\right]\mathcal{D}\left[\psi_{\alpha,m}(\vec{x})\right]\right)e^{-S_{U(1)^N}}, \quad (5)$$

where $\dot\mu = 1,2$, and the complicated yet completely local (in two dimensions) action $S_{U(1)^N}$ is:

$$\begin{aligned} S_{U(1)^N} =& \frac{i\kappa\beta}{2\pi}\sum_\alpha\int_\Sigma A_3^\alpha (F_{12})_\alpha{}^\alpha - \frac{1}{8\pi}\int_\Sigma R\ln V(\vec{x}) + \frac{1}{\beta}\int_\Sigma\sum_{\substack{m\in\mathbb{Z}+\frac{1}{2}\\ \alpha}}\bar\psi^\alpha_{-m}(\vec{x})\left\{\tilde{\not{D}} + M + i\gamma^3\left(\frac{2\pi m}{\beta} - A_3^\alpha(\vec{x})\right)\right\}\psi_{\alpha,m}(\vec{x}) \\ & -\frac{2\pi}{\kappa\beta^3}\int_\Sigma\sum_{\substack{l\in\mathbb{Z}\\ m,n\in\mathbb{Z}+\frac{1}{2}\\ \alpha,\sigma\\ \alpha\neq\sigma \text{ at } l=0}}\frac{1}{\frac{2\pi l}{\beta} - A_3^\alpha(\vec{x}) + A_3^\sigma(\vec{x})}\ \bar\psi^\alpha_{-m}(\vec{x})\gamma^1\psi_{\sigma,m-l}(\vec{x})\bar\psi^\sigma_{-n}(\vec{x})\gamma^2\psi_{\alpha,n+l}(\vec{x}). \end{aligned} \quad (6)$$

Effectively, (5) describes two-dimensional $U(1)^N$ gauge fields interacting with N neutral scalar fields A_3^α and an infinite number of fermionic fields $\psi_{\alpha,n}(\vec{x})$ via the action (6).

Specialization to the case $\Sigma = S^2$ and simplifications at large N

Specializing to a sphere, fixing the residual two-dimensional $U(1)^N$ gauge freedom, integrating out the Abelian A_1, A_2 fields from (5), and taking the large-N limit, the Vandermonde becomes global and the A_3 fields become space-independent (expressed in terms of eigenvalues $\{\lambda_\alpha\}$):

$$Z = \left(\prod_{\alpha=1}^{N} \sum_{n_\alpha=1}^{\infty}\right) (-1)^{(N-1)\sum_\alpha n_\alpha} \int \left(\prod_{\alpha=1}^{N} \mathrm{d}\lambda_\alpha\right) \; V \; \exp\left(-i\kappa \sum_\alpha n_\alpha \lambda_\alpha - S_{\rm eff}\left(\{\lambda_\alpha\}, \{n_\alpha\}\right)\right), \tag{7}$$

where,

$$e^{-S_{\rm eff}(\{\lambda_\alpha\},\{n_\alpha\})} = \prod_{\alpha=1}^{n} \int \prod_{m\in\mathbb{Z}+\frac{1}{2}} \mathcal{D}\left[\bar{\psi}^\alpha_m(\vec{x})\right] \mathcal{D}\left[\psi_{\alpha,m}(\vec{x})\right] \; e^{-S_{\rm f}}, \tag{8}$$

$S_{\rm eff}\left(\{\lambda_\alpha\}, \{n_\alpha\}\right)$ is the renormalized effective action after integrating out the fermionic fields,

$$S_{\rm f} = \frac{1}{\beta}\int_{S^2} \sum_{\substack{m\in\mathbb{Z}+\frac{1}{2}\\ \alpha}} \bar{\psi}^\alpha_{-m}(\vec{x}) \left\{\tilde{\not{D}} + M + i\gamma^3\left(\frac{2\pi m}{\beta} - \frac{\lambda_\alpha}{\beta}\right)\right\} \psi_{\alpha,m}(\vec{x}) - \frac{2\pi}{\kappa\beta^3}\int_{S^2} \sum_{\substack{l\in\mathbb{Z}\\ m,n\in\mathbb{Z}+\frac{1}{2}\\ \alpha,\sigma\\ \alpha\neq\sigma \text{ at } l=0}} \frac{1}{\frac{2\pi l}{\beta} - \frac{\lambda_\alpha}{\beta} + \frac{\lambda_\sigma}{\beta}} \bar{\psi}^\alpha_{-m}(\vec{x})\gamma^1\psi_{\sigma,m-l}(\vec{x})\bar{\psi}^\sigma_{-n}(\vec{x})\gamma^2\psi_{\alpha,n+l}(\vec{x}). \tag{9}$$

The spatial covariant derivative $\tilde{D}$ in (9) is taken in the background of the gauge fields corresponding to the constant fluxes in the $U(1)^N$.

Spin-weighted spherical harmonics and Dirac operator on S^2

Spin-weighted spherical harmonics ${}_sY_{jm}(\theta,\phi)$ are defined on S^2 by,

$$ {}_{s+1}\bar{\eth}\,{}_s\eth({}_sY_{jm}) = (s(s+1) - j(j+1))\,{}_sY_{jm}\,, \qquad (-i\partial_\phi - s)\,{}_sY_{jm} = m\,{}_sY_{jm}\,, \tag{10} $$

where ${}_s\bar{\eth}$ and ${}_s\eth$ are the spin-weighted ladder operators.

For an appropriate choice of coordinate chart, the eigenvectors of $\tilde{\not{D}}_\alpha$ are given by,

$$ {}_{q_\alpha}Z^{\pm}_{j,m} = \frac{1}{\sqrt{2}}\begin{pmatrix} \pm i e^{i\phi/2}\,{}_{q_\alpha-\frac{1}{2}}Y_{jm} \\ e^{-i\phi/2}\,{}_{q_\alpha+\frac{1}{2}}Y_{jm} \end{pmatrix}, \qquad j = |q_\alpha| + \frac{1}{2}, |q_\alpha| + \frac{3}{2}, \dots, \tag{11} $$

such that,

$$ \tilde{\not{D}}_\alpha\,{}_{q_\alpha}Z^{\pm}_{j,m} = \pm\frac{i}{R}\sqrt{j(j+1) - \left(q_\alpha + \frac{1}{2}\right)\left(q_\alpha - \frac{1}{2}\right)}\,{}_{q_\alpha}Z^{\pm}_{j,m}, \qquad j = |q_\alpha| + \frac{1}{2}, |q_\alpha| + \frac{3}{2}, \dots. \tag{12} $$

We expand the fermions in this basis to evaluate the kinetic and the interaction terms.

Finite temperature gap equation

Following the Schwinger–Dyson procedure, the finite temperature gap equation is obtained as,

$$\Sigma_T(P_{m\alpha,3}) = -\frac{1}{2\kappa\beta R^2}\sum_{n,\sigma}\frac{1}{P_{m\alpha,3}-Q_{n\sigma,3}}\,H\left(\sum_{j=|q_\sigma|-\frac{1}{2}}^{\infty}\frac{2j+1}{i\not{Q}_{n\sigma j}+M+\Sigma_T(Q_{n\sigma,3})}\right), \tag{13}$$

where $H(C)\equiv\gamma^1 C\gamma^2-\gamma^2 C\gamma^1$, $P_{m\alpha}=\left(\vec{p},\,\frac{2\pi m-\lambda_\alpha}{\beta}\right)$, $q_\alpha=\frac{n_\alpha}{2}$, and,

$$\not{Q}_{m\alpha j} = \begin{pmatrix} Q_{m\alpha,3} & \frac{1}{R}\sqrt{j(j+1)-\left(q_\alpha+\frac{1}{2}\right)\left(q_\alpha-\frac{1}{2}\right)} \\ \frac{1}{R}\sqrt{j(j+1)-\left(q_\alpha+\frac{1}{2}\right)\left(q_\alpha-\frac{1}{2}\right)} & -Q_{m\alpha,3}\end{pmatrix}. \tag{14}$$

In close analogy, for the flat space case, we obtain,

$$\Sigma_T(P_{n\sigma,3}) = -\frac{2\pi}{\kappa\beta}\sum_{\substack{m\in\mathbb{Z}+\frac{1}{2}\\ \alpha\\ \alpha\neq\sigma\text{ at }m=n}}\int\frac{\mathrm{d}^2q}{(2\pi)^2}\frac{1}{P_{n\sigma,3}-Q_{m\alpha,3}}\,H\left(\frac{1}{i\not{Q}_{m\alpha}+M+\Sigma_T(Q_{m\alpha,3})}\right). \tag{15}$$

Solution of the finite temperature gap equation

(13) is solved component-wise,

$$\Sigma_{T,I}(P_{m\alpha,3}) = P_{m\alpha,3}\,\sin\left(\frac{4\pi}{\kappa}\Omega_T(M_T, P_{m\alpha,3})\right) + M_T\,\cos\left(\frac{4\pi}{\kappa}\Omega_T(M_T, P_{m\alpha,3})\right), \tag{16}$$

$$\Sigma_{T,3}(P_{m\alpha,3}) = P_{m\alpha,3}\left(\cos\left(\frac{4\pi}{\kappa}\Omega_T(M_T, P_{m\alpha,3})\right) - 1\right) - M_T\,\sin\left(\frac{4\pi}{\kappa}\Omega_T(M_T, P_{m\alpha,3})\right). \tag{17}$$

where,

$$\Omega_T(M_T, P_{m\alpha,3}) = \frac{1}{4\pi\beta R^2}\sum_{\substack{n\in\mathbb{Z}+\frac{1}{2}\\ \sigma\\ \sigma\neq\alpha \text{ at } n=m}}\sum_{j=|q_\sigma|-\frac{1}{2}}^{\infty}\frac{2j+1}{(Q_{n\sigma,3})^2 + \frac{j(j+1)-\left(q_\sigma-\frac{1}{2}\right)\left(q_\sigma+\frac{1}{2}\right)}{R^2} + M_T^2}\frac{1}{P_{m\alpha,3} - Q_{n\sigma,3}}, \tag{18}$$

$$\Phi_T(M_T) = \frac{1}{4\pi\beta R^2}\sum_{\substack{\sigma\\ n\in\mathbb{Z}+\frac{1}{2}}}\sum_{j=|q_\sigma|-\frac{1}{2}}^{\infty}\frac{2j+1}{(Q_{n\sigma,3})^2 + \frac{j(j+1)-\left(q_\sigma-\frac{1}{2}\right)\left(q_\sigma+\frac{1}{2}\right)}{R^2} + M_T^2}, \tag{19}$$

and the thermal mass M_T obeys the mass gap equation,

$$M_T = M - \frac{4\pi}{\kappa}\Phi_T(M_T). \tag{20}$$

Some relevant identities

These **'symmetrization' identities** help solve the gap equation and evaluate the thermal free energy in the large-N limit [MZJ15, MNTV23]:

- $$\frac{1}{\beta}\int\frac{\mathrm{d}^2q}{(2\pi)^2}\sum_{\substack{\sigma\\ l\in\mathbb{Z}+\frac{1}{2}}}\frac{1}{Q_{l\sigma}^2+M_T^2}\frac{1}{P_{m\alpha,3}-Q_{l\sigma,3}}\Omega_T(M_T,Q_{l\sigma,3})^{(n-1)}=\frac{1}{n}\Omega_T(M_T,P_{m\alpha,3})^n; \quad (21)$$

- $$\frac{1}{\beta}\int\frac{\mathrm{d}^2q}{(2\pi)^2}\sum_{\substack{\sigma\\ l\in\mathbb{Z}+\frac{1}{2}}}\frac{Q_{l\sigma,3}}{Q_{l\sigma}^2+M_T^2}\frac{1}{P_{m\alpha,3}-Q_{l\sigma,3}}\Omega_T(M_T,Q_{l\sigma,3})^{(n-1)}=\frac{P_{m\alpha,3}}{n}\Omega_T(M_T,P_{m\alpha,3})^n; \quad (22)$$

- $$\frac{1}{\beta}\int\frac{\mathrm{d}^2q}{(2\pi)^2}\sum_{\substack{\alpha\\ m\in\mathbb{Z}+\frac{1}{2}}}\frac{1}{Q_{m\alpha}^2+M_T^2}\Omega_T(M_T,Q_{m\alpha,3})^n=0, \qquad \text{for} \quad n>0; \quad (23)$$

- $$\frac{1}{\beta}\int\frac{\mathrm{d}^2q}{(2\pi)^2}\sum_{\substack{\alpha\\ m\in\mathbb{Z}+\frac{1}{2}}}\frac{Q_{m\alpha,3}^n}{Q_{m\alpha}^2+M_T^2}\Omega_T(M_T,Q_{m\alpha,3})^j=\frac{1}{n+1}\begin{cases}\Phi_T(M_T)^{n+1}, & j=n\\ 0, & j>n\end{cases}. \quad (24)$$

Evaluation of thermal free energy

Subtracting the zero point energy to remove the UV-divergences, we get the flux and holonomy-dependent thermal free energy functional S_{eff} as,

$$\begin{aligned} S_T - S_0 =& -\frac{4\pi\beta V_2}{\kappa}\left(\Phi_T(M_T)\right)^2\left(\frac{4\pi}{3\kappa}\Phi_T(M_T)+M_T\right)+\frac{4\pi\beta V_2}{\kappa}\left(\Phi_0(M_0)\right)^2\left(\frac{4\pi}{3\kappa}\Phi_0(M_0)+M_0\right)\\ &-\frac{V_2}{4\pi R^2}\sum_{\substack{\sigma\\ n\in\mathbb{Z}+\frac{1}{2}}}\sum_{j=|f_\sigma|-\frac{1}{2}}^{\infty}(2j+1)\ln\left(Q_{n\sigma,3}^2+\frac{j(j+1)-\left(f_\sigma-\frac{1}{2}\right)\left(f_\sigma+\frac{1}{2}\right)}{R^2}+M_T^2\right)\\ &+\frac{V_2\beta}{4\pi R^2}\int\frac{\mathrm{d}q_3}{2\pi}\sum_{\sigma}\sum_{j=|f_\sigma|-\frac{1}{2}}^{\infty}(2j+1)\ln\left(q_3^2+\frac{j(j+1)-\left(f_\sigma-\frac{1}{2}\right)\left(f_\sigma+\frac{1}{2}\right)}{R^2}+M_T^2\right)\\ &-\frac{V_2\beta}{4\pi R^2}\int\frac{\mathrm{d}q_3}{2\pi}\sum_{\sigma}\sum_{j=|f_\sigma|-\frac{1}{2}}^{\infty}(2j+1)\ln\left(\frac{q_3^2+\frac{j(j+1)-\left(f_\sigma-\frac{1}{2}\right)\left(f_\sigma+\frac{1}{2}\right)}{R^2}+M_T^2}{q_3^2+\frac{j(j+1)-\left(f_\sigma-\frac{1}{2}\right)\left(f_\sigma+\frac{1}{2}\right)}{R^2}+M_0^2}\right). \end{aligned} \tag{25}$$

Evaluation of $\Phi_0(M_0)$

$\Phi_T(M_T)$ is a simple explicit function for the flat space case,

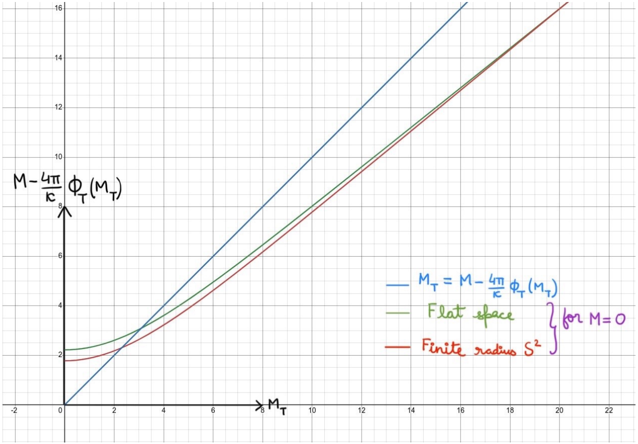
$$\Phi_T(M_T) = -\frac{1}{4\pi\beta}\sum_{\sigma} \ln\left|2(\cosh(\beta M_T) + \cos(\lambda_\sigma))\right| . \tag{26}$$

For the finite-volume case with fluxes, it becomes tedious to evaluate. At zero temperature, for fluxes $\{f_\sigma\} \equiv \{f_1, f_2, \ldots, f_N\}$, where $\{\cdot\}$ is ordered such that $f_1 \leq f_2 \leq \cdots \leq f_{\mathfrak{n}} \leq \cdots \leq f_N$,

$$\begin{aligned}
\Phi_0\left(M_0 > \frac{f_{\mathfrak{n}}}{R}\right) = &-\sum_{\sigma=1}^{\mathfrak{n}} \frac{1}{4\pi}\sqrt{M_0^2 - \frac{f_\sigma^2}{R^2}} - \sum_{\sigma=1}^{\mathfrak{n}} \frac{1}{4\pi R^2} \sum_{p=\frac{1}{2} \text{ or } 1}^{f_\sigma - 1} \frac{p}{\sqrt{M_0^2 + \frac{p^2 - f_\sigma^2}{R^2}}} - \sum_{\sigma=1}^{\mathfrak{n}} \frac{1}{8\pi R^2}\int_0^{\sqrt{M_0^2 R^2 - f_\sigma^2}} \mathrm{d}y \, \frac{2y}{\sqrt{M_0^2 - \frac{y^2 + f_\sigma^2}{R^2}}} \times \begin{cases} (\tanh \pi y - 1), & f_\sigma \in \mathbb{Z} + \frac{1}{2} \\ (\coth \pi y - 1), & f_\sigma \in \mathbb{Z} \end{cases} \\
&+ \sum_{\sigma=\mathfrak{n}+1}^{N} \frac{1}{4\pi}\int_0^{\sqrt{\frac{f_\sigma^2}{R^2} - M_0^2}} \mathrm{d}y \tan\left(\pi\sqrt{f_\sigma^2 - M_0^2 R^2 - y^2 R^2}\right) - \sum_{\sigma=\mathfrak{n}+1}^{N} \frac{1}{4\pi R^2} \sum_{p \geq \sqrt{\frac{f_\sigma^2}{R^2} - M_0^2}}^{f_\sigma - 1} \frac{p}{\sqrt{M_0^2 + \frac{p^2 - f_\sigma^2}{R^2}}} .
\end{aligned} \tag{27}$$

$\Phi_T(M_T)$ for the finite volume case

$$\Phi_T(M_T) = \frac{1}{4\pi\beta R^2} \sum_{\substack{\sigma \\ n\in\mathbb{Z}+\frac{1}{2}}} \sum_{j=|q_\sigma|-\frac{1}{2}}^{\infty} \frac{2j+1}{(Q_{n\sigma,3})^2 + \frac{j(j+1)-\left(q_\sigma-\frac{1}{2}\right)\left(q_\sigma+\frac{1}{2}\right)}{R^2} + M_T^2}. \tag{28}$$



Evaluation of $\Phi_T(M_T)$

Now, at finite temperature, for fluxes $\{f_\sigma\} = (0, 0, \ldots, 0)$,

$$\Phi_T(M_T) = -\frac{1}{4\pi\beta}\sum_{\sigma=1}^{N} \ln\left|2(\cosh(\beta M_T) + \cos(\lambda_\sigma))\right| \\ -\sum_{\sigma=1}^{N}\frac{1}{8\pi R^2}\left[\int_0^{M_T R}\mathrm{d}y\,\frac{2y(\coth\pi y - 1)}{\sqrt{M_T^2 - \frac{y^2}{R^2}}}\frac{\sinh\left(\frac{\beta}{R}\sqrt{M_T^2R^2 - y^2}\right)}{\cosh\left(\frac{\beta}{R}\sqrt{M_T^2R^2 - y^2}\right) + \cos(\lambda_\sigma)} + \int_{M_T R}^{\infty}\mathrm{d}y\,\frac{2y(\coth\pi y - 1)}{\sqrt{\frac{y^2}{R^2} - M_T^2}}\frac{\sin\left(\frac{\beta}{R}\sqrt{y^2 - M_T^2R^2}\right)}{\cos\left(\frac{\beta}{R}\sqrt{y^2 - M_T^2R^2}\right) + \cos(\lambda_\sigma)}\right]. \tag{29}$$

For $\{f_\sigma\} = \{0, 0, 0, \ldots, f_N\}$, where $f_N = \frac{1}{2}$,

$$\Phi_T(M_T) = \begin{cases} -\frac{1}{4\pi\beta}\sum_{\sigma=1}^{N-1}\ln\left|2(\cosh(\beta M_T) + \cos(\lambda_\sigma))\right| - \sum_{\sigma=1}^{N-1}\frac{1}{8\pi R}\left[\int_0^{M_T R}\mathrm{d}y\,\frac{2y(\coth\pi y - 1)}{\sqrt{M_T^2R^2 - y^2}}\frac{\sinh\left(\frac{\beta}{R}\sqrt{M_T^2R^2 - y^2}\right)}{\cosh\left(\frac{\beta}{R}\sqrt{M_T^2R^2 - y^2}\right) + \cos(\lambda_\sigma)} + \int_{M_T R}^{\infty}\mathrm{d}y\,\frac{2y(\coth\pi y - 1)}{\sqrt{y^2 - M_T^2R^2}}\frac{\sin\left(\frac{\beta}{R}\sqrt{y^2 - M_T^2R^2}\right)}{\cos\left(\frac{\beta}{R}\sqrt{y^2 - M_T^2R^2}\right) + \cos(\lambda_\sigma)}\right] \\ +\frac{1}{4\pi\beta}\ln\left|\frac{\cosh(\beta M_T) + \cos\lambda_N}{\cosh\left(\beta\sqrt{M_T^2 - \frac{f_N^2}{R^2}}\right) + \cos\lambda_N}\right| - \frac{1}{8\pi R}\left[\int_0^{\sqrt{M_T^2R^2 - f_N^2}}\mathrm{d}y\,\frac{2y(\tanh\pi y - 1)}{\sqrt{M_T^2R^2 - y^2 - f_N^2}}\frac{\sinh\left(\frac{\beta}{R}\sqrt{M_T^2R^2 - y^2 - f_N^2}\right)}{\cosh\left(\frac{\beta}{R}\sqrt{M_T^2R^2 - y^2 - f_N^2}\right) + \cos(\lambda_N)} + \int_{\sqrt{M_T^2R^2 - f_N^2}}^{\infty}\mathrm{d}y\,\frac{2y(\tanh\pi y - 1)}{\sqrt{y^2 + f_N^2 - M_T^2R^2}}\frac{\sin\left(\frac{\beta}{R}\sqrt{y^2 + f_N^2 - M_T^2R^2}\right)}{\cos\left(\frac{\beta}{R}\sqrt{y^2 + f_N^2 - M_T^2R^2}\right) + \cos(\lambda_N)}\right], & M_0 > \frac{f_N}{R} \\ -\frac{1}{4\pi\beta}\sum_{\sigma=1}^{N-1}\ln\left|2(\cosh(\beta M_T) + \cos(\lambda_\sigma))\right| - \sum_{\sigma=1}^{N-1}\frac{1}{8\pi R^2}\left[\int_0^{M_T R}\mathrm{d}y\,\frac{2y(\coth\pi y - 1)}{\sqrt{M_T^2 - \frac{y^2}{R^2}}}\frac{\sinh\left(\frac{\beta}{R}\sqrt{M_T^2R^2 - y^2}\right)}{\cosh\left(\frac{\beta}{R}\sqrt{M_T^2R^2 - y^2}\right) + \cos(\lambda_\sigma)} + \int_{M_T R}^{\infty}\mathrm{d}y\,\frac{2y(\coth\pi y - 1)}{\sqrt{\frac{y^2}{R^2} - M_T^2}}\frac{\sin\left(\frac{\beta}{R}\sqrt{y^2 - M_T^2R^2}\right)}{\cos\left(\frac{\beta}{R}\sqrt{y^2 - M_T^2R^2}\right) + \cos(\lambda_\sigma)}\right] \\ -\frac{1}{8\pi R}\int_0^{\infty}\mathrm{d}y\,\frac{2y(\tanh\pi y - 1)}{\sqrt{y^2 + f_N^2 - M_T^2R^2}}\frac{\sin\left(\frac{\beta}{R}\sqrt{y^2 + f_N^2 - M_T^2R^2}\right)}{\cos\left(\frac{\beta}{R}\sqrt{y^2 + f_N^2 - M_T^2R^2}\right) + \cos(\lambda_N)} + \frac{1}{4\pi\beta}\ln\left|\frac{\cosh(\beta M_T) + \cos\lambda_N}{\cos\left(\beta\sqrt{M_T^2 - \frac{f_N^2}{R^2}}\right) + \cos\lambda_N}\right| + \frac{1}{4\pi\beta}\sum_{n=\frac{1}{2}}^{\frac{\lambda_N}{2\pi} + \sqrt{\frac{f_N^2}{R^2} - M_T^2}}\pi\tan\left(\pi R\sqrt{\frac{f_N^2}{R^2} - M_T^2 - \left(\frac{2\pi n}{\beta} - \frac{\lambda_N}{\beta}\right)^2}\right), & M_0 < \frac{f_N}{R} \end{cases} \tag{30}$$

Discussion and future directions

- The thermal effective action is not independent of fluxes at finite volume - holonomy eigenvalues are not quantized - Bose–Fermi duality has to work in a mathematically different way (compared to flat space);

- At finite volume, the fermion determinant receives flux contributions from different sectors, and the free energy will be presented in terms of a sum over these fluxes (in addition to the summation over Kaluza–Klein momenta and the spin-weighted spherical harmonic labels), which is seemingly similar to the calculations of the superconformal index [MMPS22];

- First-principles computation of S-matrices [MMP$^+$22] may be possible with our choice of gauge that could shed new light on their unusual crossing symmetry properties;

- Computation of the free energy for bosons has some subtleties related to the $\phi^2 A^2$ coupling.

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