Non-Normalizable Wavefunctions in Extreme Gravitational Fields: A Mathematical Perspective

(A Mathematical Perspective on Quantum Confinement and Gravity)

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Introduction – Choosing the Right Frame

By using a non-relativistic framework, where we use the Laplacian ▽^2

$$\nabla^2\Psi = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu}\left(\sqrt{|g|}g^{\mu\nu}\frac{\partial\Psi}{\partial x^\nu}\right)$$

I did introduce relativistic conditions when I placed the boundary near the event horizon. Earlier, I started with a non-relativistic 4D quantum well model to establish the fundamental structure of confined states. However, once I considered the Schwarzschild metric near the horizon, I had to transition to a relativistic framework.(What Boundary?)

The Need for a 4D Schrödinger Equation

Schrödinger Equation in Flat 4D Space

In a flat 4D space (ignoring relativistic effects), the time-dependent Schrödinger eq uation is

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}+\frac{\partial^2\Psi}{\partial z^2}+\frac{\partial^2\Psi}{\partial w^2}\right)$$

Schrödinger Equation in Non-Euclidean 4D Space

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu}\left(\sqrt{|g|}g^{\mu\nu}\frac{\partial\Psi}{\partial x^\nu}\right)\right)\Psi$$

The Schwarzschild metric (for a spherically symmetric non-rotating mass M) in four-dimensional curved spacetime is given by:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) c^2 dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

Here;

$$|g| = r^4 \sin^2\theta$$

Thus, the Schrödinger equation in curved Schwarzschild space is:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \left(1 - \frac{2GM}{r}\right) \frac{\partial \Psi}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \Psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \Psi}{\partial \phi^2} \right]$$

This equation incorporates the effects of gravitational curvature on a quantum wavefunction.

Solving the Wavefunction Using Schwarzschild Metric

energy eigenvalues for a 4D quantum well in Flat Space

$$E_{n_x,n_y,n_z,n_w} = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_x^2}{L_x^2}+\frac{n_y^2}{L_y^2}+\frac{n_z^2}{L_z^2}+\frac{n_w^2}{L_w^2}\right)$$

Now , lets assume a reflective boundary at r=2GM+ϵ , meaning:

$$\Psi(2GM+\epsilon)=0.$$

Energy Eigenvalue

(How??)

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2\pi^2 n^2}{2m\epsilon^2} - \frac{m}{2}$$

(what is actually ϵ ?)

Updated Energy Eigenvalue

$$E_{n_r,\ell,m,n_w} = \frac{\hbar^2}{2m}\left(\frac{x_{\ell n}}{R}\right)^2 + \frac{2mL^2}{\hbar^2}\pi^2 n_w^2 - \frac{GMm}{r} + \frac{2mr^2\ell(\ell+1)}{\hbar^2} + \frac{am}{r^3} - \frac{m}{2}$$

Understanding Epsilon (ε)

- The probability density of finding a particle at that boundary is zero.
- Instead of escaping, the wavefunction is reflected back into the allowed region.

Why Use a Reflective Boundary for Normalization?

In Schwarzschild spacetime, the wavefunction often diverges at the event horizon ($r=2GM$), making it non-normalizable. By imposing a reflective boundary at a small distance away from the horizon ($r=2GM+\epsilon$),
we can:

- Prevent divergence: The wavefunction is forced to remain finite and well-behaved.
- Create a quantum well-like system: The wavefunction gets trapped between the reflective boundary and another potential barrier.

Expression for ϵ

$$\epsilon = \alpha \frac{\hbar}{m} + 4GM\beta^2 + \frac{8\pi GM\hbar}{c^3}.$$

- Quantum penetration depth ($\hbar/m$).
- Gravitational blueshift effects (GM-dependent term).
- Thermal quantum effects (Hawking temperature scale).

A Blind Approach!!!

1). The 4D Quantum Schrödinger Equation in curved spacetime:

$$\left(g^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{V_{\text{eff}}}{\hbar}\right)\Psi = i\frac{\partial\Psi}{\partial t}$$

- After some re-arrangement

$$\left(\nabla^2 - 4\frac{V_{\text{eff}}}{\hbar}\right)\Psi = 4i\frac{\partial\Psi}{\partial t}$$

- EFE

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

- The structure of both equations has some resemblance (a differential term, a modifying term, and a source term).
- It can be seen as a formal analogy, but it should not be interpreted as a direct equivalence.

2) Gravitational Wave Echo Prediction

Since the quantum well imposes a reflective boundary at $r=2GM+\epsilon$, we modify the boundary condition:

[Classical GR Prediction: $A_{ref}=0$ (full absorption)

$$h_{\mu\nu}(r = 2GM + \epsilon) = A_{\text{in}} + A_{\text{ref}}$$

introduce a reflection coefficient R dependent on the energy eigenvalues of the quantum well:

$$\mathcal{R}(\omega) = e^{i\delta(\omega)}$$

where $\delta(\omega)$ is the phase shift caused by quantum confinement.

$$\Delta t_{\text{echo}} = 2\int_{r_p}^{2GM+\epsilon} \frac{dr}{1 - \frac{2GM}{r}}$$

Approximating for small ϵ:

$$\Delta t_{\text{echo}} \approx 4GM \log\left(\frac{1}{\epsilon}\right)$$

Is this logarithmic scaling matches predictions from LQG and String Theory?

Conclusion & Seeking Guidance

- **Equivalence to EFE:** No direct mapping due to imaginary terms and effective potential.
- **Event Horizon Boundary:** Reflective conditions may not fully capture quantum effects.
- **Non-Normalizability:** Gravity modifies quantum states, affecting probability interpretation.
- **Relativistic Framework:** Transition from non-relativistic to relativistic needs better justification.
- **Mathematical Consistency:** Can we fully relativize the 4D quantum well equation?

further clarification and guidance needed

- Refining the transition to a relativistic framework while keeping solvability.
- Exploring quantum gravity or effective field theories for justification.
- Defining reflective boundary conditions rigorously near the horizon.
- Understanding if non-normalizability reflects real information leakage.

Note: This is a mathematical framing only; physical validity needs further refinemen
t.

THANKYOU