Towards a conformal field theory for critical planar interfaces

Eveliina Peltola

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Women @ Intersection of Math & Th. Phys, ICTS, New Year 2026

FINNISH CENTRE OF EXCELLENCE IN RANDOMNESS AND STRUCTURES 2022–2029

ACADEMY OF FINLAND

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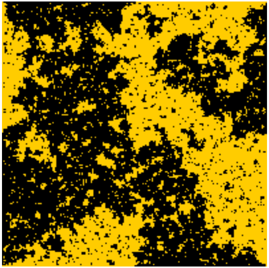
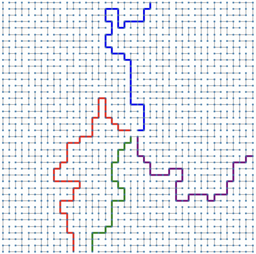
Lattice models in 2D statistical physics

- phase transitions, scaling limits
- interfaces: random interacting curves
- **conformal invariance** at criticality (?)





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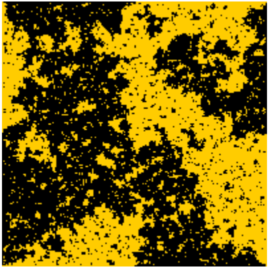
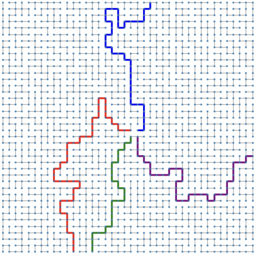
- operator algebra, fusion rules, central charge c
- singular vectors and null fields

  $\Rightarrow$ BPZ PDEs for correlation functions





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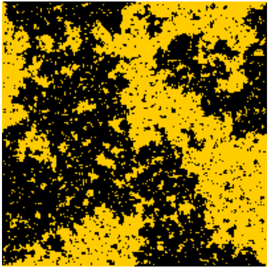

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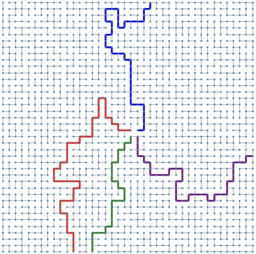
Interaction of interfaces

- probabilities of topological events
- interface "partition functions"
- relation to **correlation functions** in CFT / QFT



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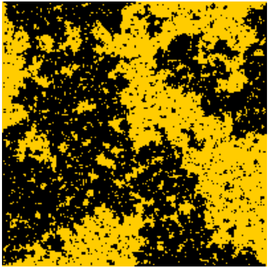

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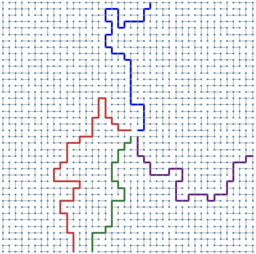
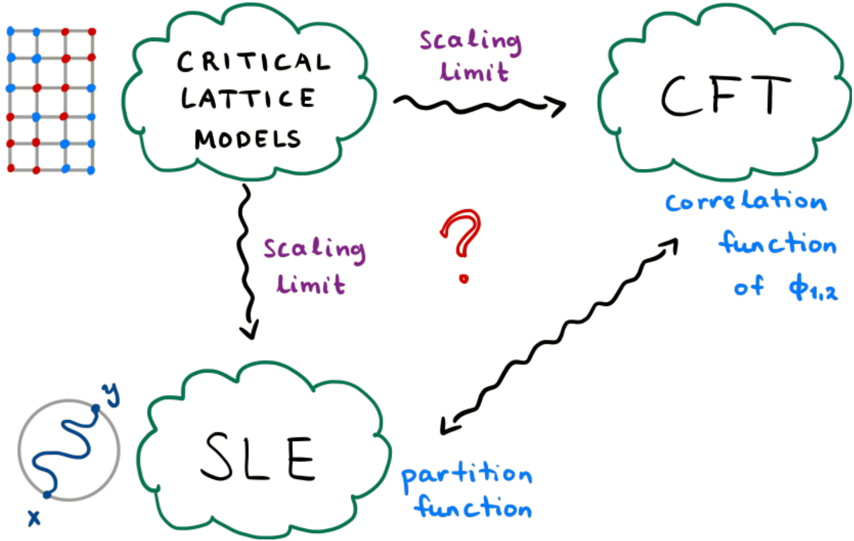


Various prototypical models... c :

	c
critical **percolation** (trivial? logarithmic?)	0
critical **Ising model** (free fermion?)	$\frac{1}{2}$
Gaussian free field (free boson?)	1
uniform spanning trees (sympl. fermion?)	-2

2D critical models: emergent conformal symmetry



Conformal invariance "conjecture" Polyakov '70s, BPZ '80s, ...

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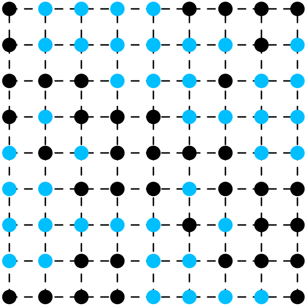
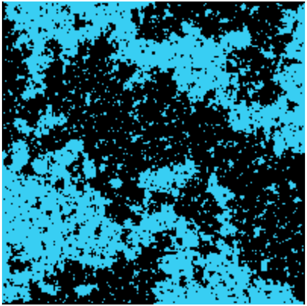


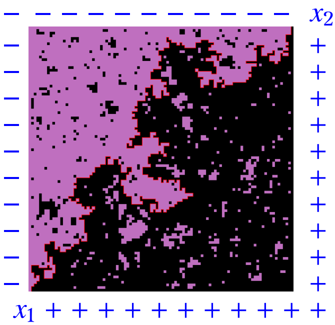


CFT: *conformally invariant* quantum field theory...

What is this supposed to mean?

CONFORMAL INVARIANCE IN TERMS OF **OBSERVABLES**

Interfaces: SLE(κ) curves

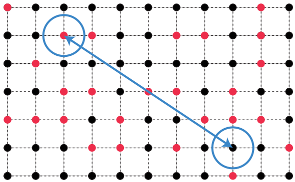


Scaling limit $\delta \to 0$

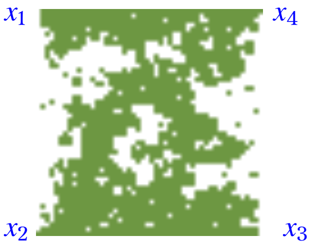
at criticality ($T = T_c$)

$\Longrightarrow$ **conformal invariance?**

$\mathbb{E}[\sigma_{x_1} \sigma_{x_2}]$



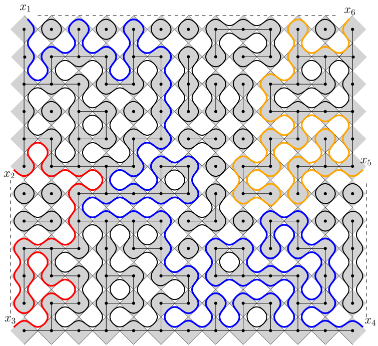
Correlations (e.g. between spins)

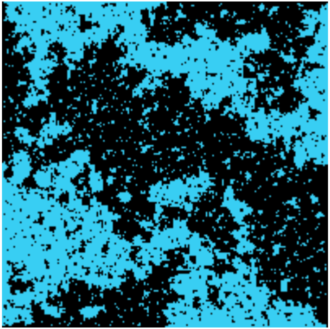


Probabilities of topological events

Prototypical examples:

Percolation / spin models





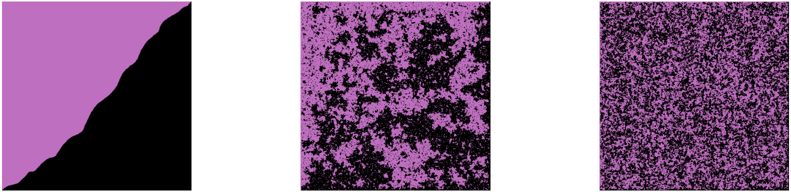
Ising model: ferromagnetic phase transition

[Lenz & Ising '20s, Peierls 30's, Kramers & Wannier 40's, Onsager 40's →]

- random **spins** $\sigma_x = \pm 1$ at vertices x of a graph
- nearest neighbor interaction: $\mathbb{P}\,[\text{config.}] \propto \exp\left(\frac{1}{T}\sum_{x\sim y}\sigma_x\sigma_y\right)$
- **continuous phase transition** at **critical** temperature $T = T_c$

look at *correlation* of a pair of spins at x and y

$C(x,y) = \mathbb{E}\,[\sigma_x\sigma_y] - \mathbb{E}\,[\sigma_x]\,\mathbb{E}\,[\sigma_y]$ when $|x-y| >> 1$:



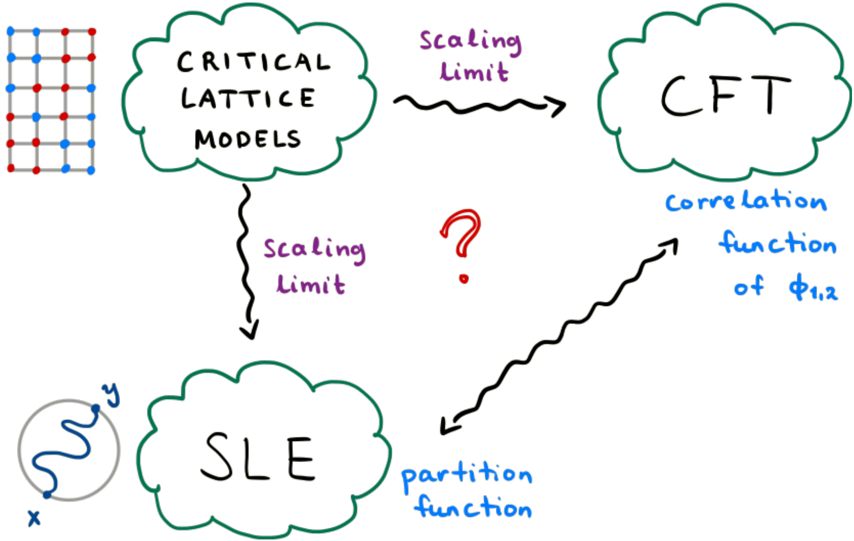
$T < T_c$ — $C(x,y) \sim \text{const.}$

$T = T_c$ — $C(x,y) \sim |x-y|^{-\beta}$

$T_c < T$ — $C(x,y) \sim e^{-\frac{1}{\xi}|x-y|}$

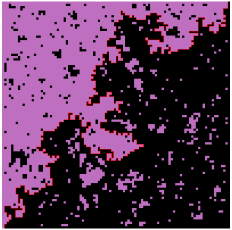
- scaling limit at critical temperature T_c: **conformal invariance**?

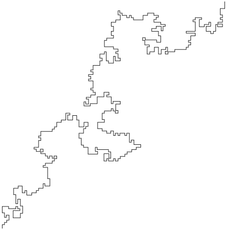
Critical planar models and Conformal Field Theory



Scaling limits of critical interfaces — SLE curves

- $\kappa > 0$ labels *universality class* (and $c(\kappa)$) (e.g. $q(\kappa) := 4\cos^2(4\pi/\kappa)$)
- convergence weakly for probability measures on curves





(critical) interface $\xrightarrow{\delta \to 0}$ **Schramm-Loewner evolution**, SLE(κ)

Usual proof strategy:

1. *tightness* (e.g. control via crossing estimates, RSW etc.)

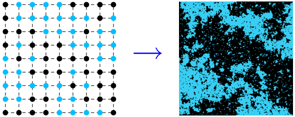
   [Aizenman & Burchard '99, Kemppainen & Smirnov '17, ...]

2. *identification* of the limit (e.g. via discrete holomorphic observable)

   [Kenyon '00, Chelkak & Smirnov '01–'11, ...]

Conformal invariance "conjecture" in SLE context

Boundary conditions:
"*disorder*" fields in CFT?



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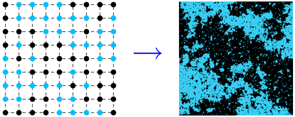

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- [Cardy 84'; Bauer & Bernard '02]: SLE(κ) curve started at x should be generated by *CFT primary field* $\Phi_{1,2}(x)$ (degenerate at level 2).
- **Why?** Correlation functions give rise to (local) SLE(κ)-mgles:

$$M_t(x; z_1, \ldots, z_n) = \Big(\prod_{j=1}^{n} g_t'(z_j)^{\Delta_j} \, g_t'(\bar{z}_j)^{\bar{\Delta}_j} \Big) \, \mathscr{Z}(W_t; g_t(z_1), \ldots, g_t(z_n))$$

has zero drift iff $\mathscr{Z}$ solves 2nd order "BPZ PDE" related to $\Phi_{1,2}(x)$:

$$\Big\{ \frac{\kappa}{2} \frac{\partial^2}{\partial x^2} + \sum_{j=1}^{n} \Big(\frac{2}{z_j - x} \frac{\partial}{\partial z_j} - \frac{2\Delta_j}{(z_j - x)^2} \Big) \Big\} \mathscr{Z}(x; z_1, \ldots, z_n) = 0$$

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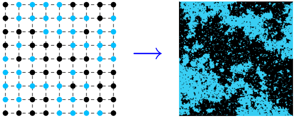

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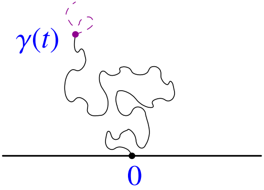
  has zero drift iff $\mathscr{Z}$ solves 2nd order "BPZ PDE" related to $\Phi_{1,2}$
- Generally, $s-1$ curves at x conjecturally related to $\Phi_{1,s}(x)$ (fusion)
  $\Longrightarrow$ Higher order PDEs [e.g. Duplantier & Saleur '87; Bauer & Saleur '89]



Thm. [Schramm 2000]

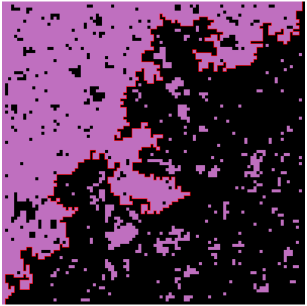
$\exists!$ one-parameter family $(\mathrm{SLE}(\kappa))_{\kappa \geq 0}$ of probability measures on chordal curves with **conformal invariance** and **domain Markov property**

$g_t : \mathbb{H} \setminus \gamma[0,t] \to \mathbb{H}$

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}$$

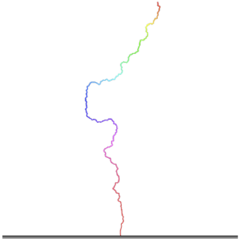
$$g_0(z) = z$$

$g_t(\gamma(t)) = \sqrt{\kappa} B_t$

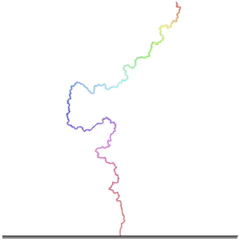


Loewner driving process: **Brownian motion** B **of “speed”** $\kappa \geq 0$

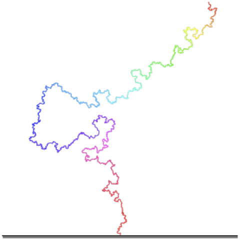
One-parameter family of random fractal curves



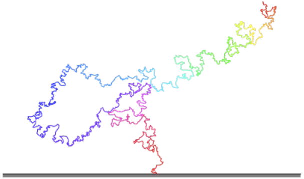
(a) SLE($\frac{1}{2}$)



(b) SLE(1)



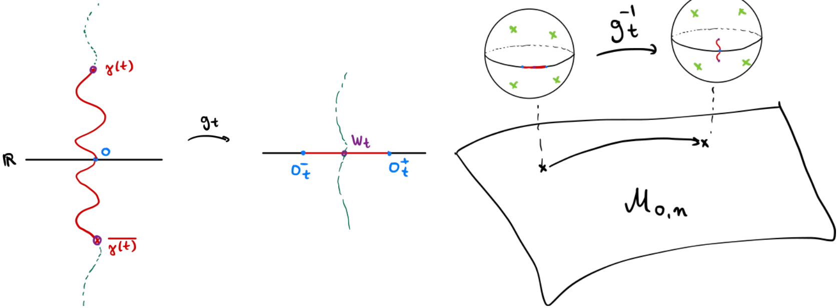
(c) SLE(2)



(d) SLE(4)

Idea: follow time-evolution of the curve

- curve's time-evolution encoded into maps $g_t \colon \mathbb{H} \setminus \gamma[0,t] \to \mathbb{H}$
- marked points $z_1, z_2, \ldots$ represent observable locations
- $\Delta_1, \Delta_2, \ldots$ are "scaling dimensions"



Our observable: *martingale* (mgle) $\Longrightarrow$ **useful PDEs**

$$M_t(x=0; z_1, \ldots) = \Big(\prod_j g_t'(z_j)^{\Delta_j}\, g_t'(\bar{z}_j)^{\bar{\Delta}_j} \Big)\, \mathscr{Z}(W_t; g_t(z_1), \ldots)$$

Conformal invariance "conjecture" in SLE context

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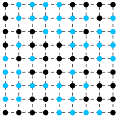
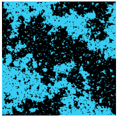


$\longrightarrow$



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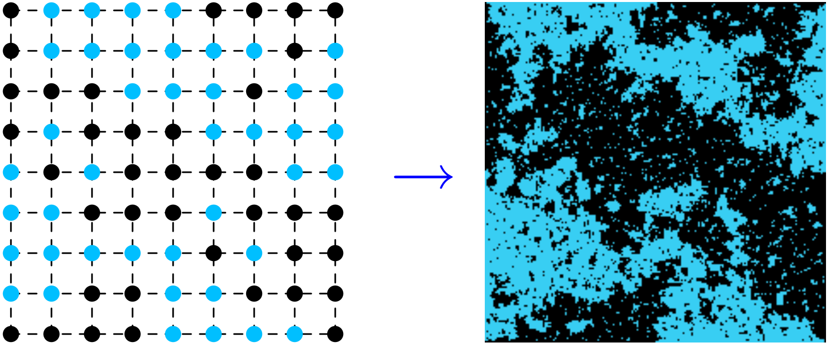
Example result



for Ising model

Crossing probabilities as CFT correlations

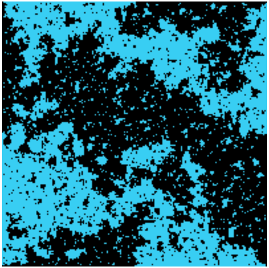
$(\Omega^\delta; x_1^\delta, \ldots, x_{2N}^\delta) \xrightarrow{\delta \to 0} (\Omega; x_1, \ldots, x_{2N})$ (in close-Carathéodory sense)

Thm. [Cardy 1980s; Bauer, Bernard & Kytölä 2005; Izyurov 2011–; P. & Wu 2018]

For critical **Ising model** on Ω^δ with *alternating* boundary cond.,

$$\lim_{\delta \to 0} \mathbb{P}^\delta [\text{crossing} = \alpha] \; \propto \; \mathcal{Z}_\alpha^{(\kappa=3)}(\Omega; x_1, \ldots, x_{2N})$$

- $\mathcal{Z}_\alpha^{(\kappa=3)}(\Omega; x_1, \ldots, x_{2N})$ thought of as **CFT correlation functions**
- **Key point:** *interaction* of the random curves



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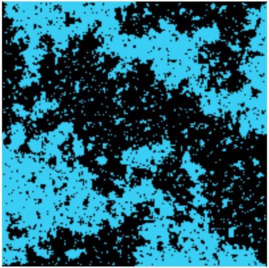

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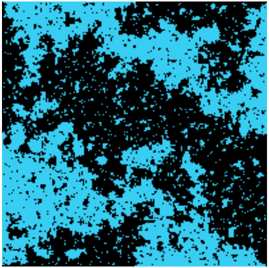
  - conf. invariance $\rightsquigarrow$ conf. field theory (CFT)
  - BPZ PDEs [Belavin, Polyakov & Zamolodchikov 1984]
  - fusion rules/asy. $\rightsquigarrow$ "selection rules"
- **NB:** non-local log-CFT
- Other (critical) models!?! Special cases.

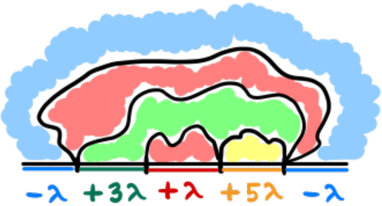
  [... ; Liu, P. & Wu 2021; Feng, P. & Wu 2022; ...]

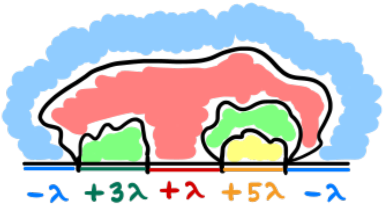


Moral:

Cardy's b.c.c. operators

- $\Phi_{1,2}$ = one-leg operator; conformal weight $h_{1,2}$ in Kac table
- $\Phi_{1,s} = (s-1)$-leg operator; conformal weight $h_{1,s}$ in Kac table





(This figure illustrates a height model instead of Ising spin model.)

Algebraic content of conformal symmetry

Virasoro algebra $\mathfrak{Vir}$: Lie algebra generated by $(L_n)_{n\in\mathbb{Z}}$ and central C

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{C}{12}n(n^2-1)\,\delta_{n+m,0}, \qquad [C, L_n] = 0$$

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$$[L_n, L_m] = (n-m)L_{n+m} + \frac{C}{12} n(n^2-1)\, \delta_{n+m,0}, \qquad [C, L_n] = 0$$

- *primary field* Φ should generate $\mathfrak{Vir}$-module $\mathrm{M}_\Phi \cong \mathfrak{Vir}.v_\Phi$

  (slightly more precisely, $v_\Phi := \lim_{z \to 0} \Phi(z)\,|\,0\rangle$ via "state-field correspondence")

  $$L_0 v_\Phi = h v_\Phi, \qquad L_n v_\Phi = 0 \;\text{ for } n \geq 1, \qquad C.v_\Phi = c\, v_\Phi,$$

  where $c \in \mathbb{C}$ *central charge* and $h = h(\Phi) \in \mathbb{C}$ *weight* of Φ

Algebraic content of conformal symmetry

Virasoro algebra $\mathfrak{Vir}$: Lie algebra generated by $(L_n)_{n\in\mathbb{Z}}$ and central C

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$$h = h_{r,s}(\kappa) := \frac{(r^2-1)}{4}\frac{\kappa}{4} + \frac{(s^2-1)}{4}\frac{4}{\kappa} + \frac{(1-rs)}{2}, \qquad c = c(\kappa) := 13 - 6\left(\frac{\kappa}{4} + \frac{4}{\kappa}\right)$$

for some $r, s \in \mathbb{Z}_{>0}$ and $\kappa \in \mathbb{C} \setminus \{0\}$.

Singular vectors $\implies$ BPZ PDEs

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- each *primary field* Φ generates $\mathfrak{Vir}$-module $\mathrm{M}_\Phi \cong \mathrm{V}_h^c/\mathrm{N}$
- N non-trivial? Generated by *singular vector*

$$w = P(L_{-m} : m \in \mathbb{N}).v_\Phi \in \mathrm{N} \qquad \implies \qquad [w] = [0] \in \mathrm{V}_h^c/\mathrm{N}$$

where P polynomial in $\mathfrak{Vir}$-generators L_{-m}

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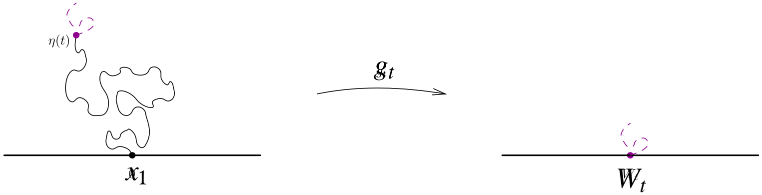
Upshot. **PDEs for correlation functions of Φ:**

$$\mathcal{D}^{(z)} \langle \Phi(z)\, \Phi_1(z_1) \cdots \Phi_d(z_d) \rangle = 0$$

with $\mathcal{D}^{(z)} = P(\mathcal{L}_{-m}^{(z)} : m \in \mathbb{N})$ and $\mathcal{L}_{-m}^{(z)} = -\sum_{j=1}^n \left(\frac{1}{(z_j - z)^{m-1}} \frac{\partial}{\partial z_j} + \frac{(1-m)\, h_j}{(z_j - z)^m} \right)$ and $h_j = h(\Phi_j)$

Example: "SLE(κ) field $\Phi_{1,2}$" of weight $h_{1,2}(\kappa) = \frac{6-\kappa}{2\kappa}$

"insert" $\Phi_{1,2}$ at points $x_1 < x_2 < \cdots < x_{2n}$ [Cardy '84; Bauer-Bernard '02]

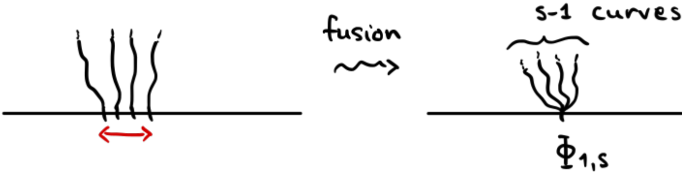


$$\mathrm{d}W_t = \sqrt{\kappa}\,\mathrm{d}B_t + \kappa\,\partial_1 \log \mathscr{Z}(W_t, g_t(x_2), g_t(x_3), \ldots, g_t(x_{2n}))\,\mathrm{d}t$$

- parameter $\kappa > 0$, central charge $c = \frac{1}{2\kappa}(3\kappa - 8)(6-\kappa) = 13 - 6\left(\frac{\kappa}{4} + \frac{4}{\kappa}\right)$
- singular vector $\left(L_{-2} - \frac{3}{2(2h_{1,2}(\kappa)+1)} L_{-1}^2\right) v_{1,2}$ for **Virasoro module**
- (together with translation invariance) gives rise to **BPZ PDE system** $\forall\ j$

$$\left\{\frac{\kappa}{2}\frac{\partial^2}{\partial x_j^2} + \sum_{i \neq j}\left(\frac{2}{x_i - x_j}\frac{\partial}{\partial x_i} - \frac{2h_{1,2}(\kappa)}{(x_i - x_j)^2}\right)\right\} \underbrace{\langle \Phi_{1,2}(x_1) \cdots \Phi_{1,2}(x_{2n}) \rangle}_{\mathscr{Z}(x_1, x_2, \ldots, x_{2n})} = 0$$

Fusion of "SLE(κ) fields $\Phi_{1,2}$" of weight $h_{1,2} = \frac{6-\kappa}{2\kappa}$

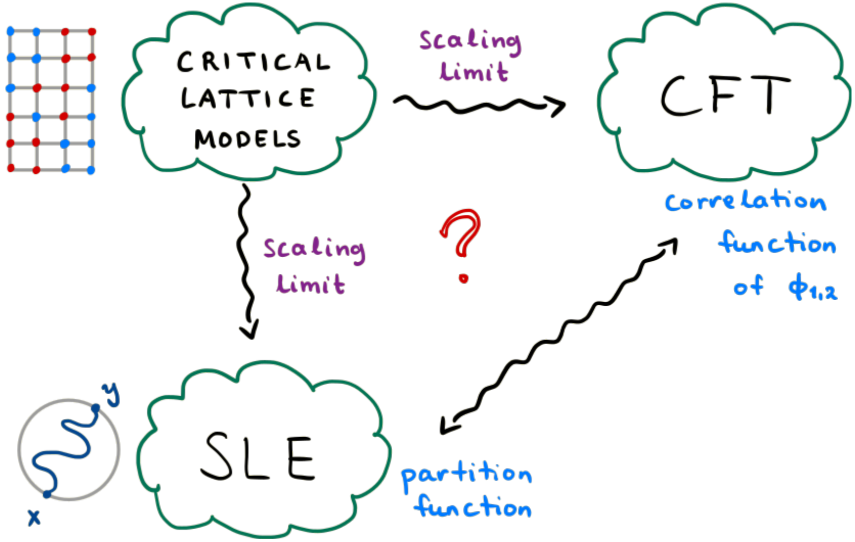


Generally, $s-1$ curves at x conjecturally related to $\Phi_{1,s}(x)$ (fusion)

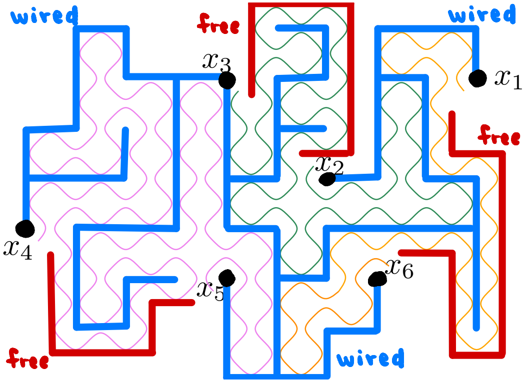
Higher order BPZ PDEs? [e.g. Duplantier & Saleur '87; Bauer & Saleur '89]

$$\Bigg\{ \frac{\partial^3}{\partial \xi^3} - \frac{16}{\kappa} \sum_{i=3}^{2n} \left(\frac{h_{1,2}(\kappa)}{(x_i-\xi)^2} - \frac{1}{x_i - \xi}\frac{\partial}{\partial x_i} \right) \frac{\partial}{\partial \xi} + \frac{8(8-\kappa)}{\kappa^2} \sum_{i=3}^{2n} \left(\frac{2h_{1,2}(\kappa)}{(x_i-\xi)^3} - \frac{1}{(x_i-\xi)^2}\frac{\partial}{\partial x_i} \right) \Bigg\} \underbrace{\langle \Phi_{1,3}(\xi)\, \Phi_{1,2}(x_3) \cdots \Phi_{1,2}(x_{2n}) \rangle}_{\mathscr{Z}(\xi, x_3, \ldots, x_{2n})} = 0$$

- $\kappa \in (0,8) \setminus \mathbb{Q}$: [Dubédat '15, P. '20]
- $\kappa = 4$: [Liu & Wu '21; Lafay, P. & Roussillon '24]
- $\kappa = 2$: [Karrila, Kytölä, P. '17; Karrila, Lafay, P. & Roussillon '24]
- $\kappa \in (8/3, 8)$, special case of 3rd order PDEs: [Feng, Liu, P. & Wu '24]


CRITICAL
LATTICE
MODELS
Scaling
Limit
CFT
Scaling
Limit
?
correlation
function
of $\phi_{1,2}$
y
x
SLE
partition
function

These are not minimal models! $\rightsquigarrow$ log-CFT ?!?



CROSSING PROBABILITIES FOR UST ON $\Omega^\delta \subset \delta\mathbb{Z}^2$

$(\Omega^{\delta,\diamond}; x_1^{\delta\diamond}, \ldots, x_{2N}^{\delta\diamond}) \xrightarrow{\delta\to 0} (\Omega; x_1, \ldots, x_{2N})$ (assuming C^1-Jordan domain)

Thm. [Liu & P. & Wu '25]

Connectivities of UST Peano curves on $(\Omega^{\delta,\diamond}; x_1^{\delta\diamond}, \ldots, x_{2N}^{\delta\diamond})$ with b.c. β converge: for each possible $\alpha \in \mathrm{LP}_N$,

$$\lim_{\delta\to 0} \mathbb{P}^\delta_\beta[\text{ connectivity} = \alpha] \quad = \quad \frac{\mathcal{Z}_\alpha^{(\kappa=8)}(\Omega; x_1, \ldots, x_{2N})}{\mathcal{F}_\beta^{(\kappa=8)}(\Omega; x_1, \ldots, x_{2N})}.$$

- $\{\mathcal{Z}_\alpha^{(\kappa=8)} : \alpha \in \mathrm{LP}_N\}$ "pure partition functions"
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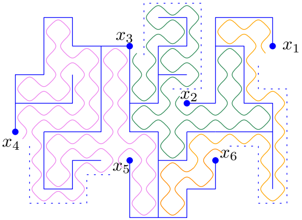

also [Dubédat '07]

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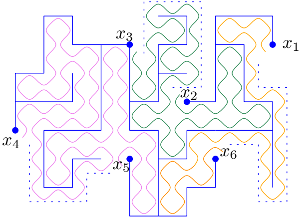


Both are explicit. **Interesting properties:**

- Möbius cov. of $\mathcal{Z}_\alpha$ & $\mathcal{F}_\beta$: *conformal invariance* of $\mathbb{P}_\beta[\alpha]$
- BPZ PDEs (martingales / $\Phi_{1,2}$-field in CFT)
- explicit *logarithmic* asymptotics (fusion in $c = -2$ log-CFT)
- simultaneous *positivity*

also [Dubédat '07]

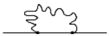
LOGARITHMIC FUSION FOR SLE(8) BOUNDARY FIELD $\Phi_{1,2}$

Thm. [Han & Liu & Wu '20; Liu & P. & Wu '21]
Explicit fusion rules from SLE(8) *pure partition functions* $\mathcal{Z}_\alpha^{(\kappa=8)}$:

- "zero-leg channel"

$$\frac{\mathcal{Z}_\alpha(x_1, \ldots, x_{2N})}{|x_{j+1} - x_j|^{1/4} \log |x_{j+1} - x_j|}$$

$$\xrightarrow{x_j, x_{j+1} \to \xi} \quad \mathcal{Z}_{\alpha \setminus \{j,j+1\}}(x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{2N}) \qquad \text{if } \{j, j+1\} \in \alpha$$



Logarithmic fusion for SLE(8) boundary field $\Phi_{1,2}$

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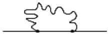

- "two-leg channel" (NB: limit is **independent of** ξ)

$$\frac{\mathcal{Z}_\alpha(x_1,\ldots,x_{2N})}{|x_{j+1}-x_j|^{1/4}} \xrightarrow{x_j,x_{j+1}\to\xi} \pi\,\mathcal{Z}_{\wp(\alpha)\setminus\{j,j+1\}}(x_1,\ldots,x_{j-1},x_{j+2}\ldots,x_{2N}) \qquad \text{if } \{j,j+1\}\notin\alpha$$

Logarithmic fusion for SLE(8) boundary field $\Phi_{1,2}$

Consequence.

For *any* CFT boundary fields describing SLE(8) curves,

(whatever that'd mean...)

OPE product has *explicit form*

$$\Phi_{1,2}(z)\,\Phi_{1,2}(w) \;\sim\; (z-w)^{-1/4}\Big(\pi\,\Phi_{1,1}(z) - \log(z-w)\,\widetilde{\Phi}_{1,3}(z)\Big).$$

Logarithmic fusion for $\mathrm{SLE}(8)$ boundary field $\Phi_{1,2}$

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Compare with fusion of two simple Virasoro modules with $c=-2$:

- for simple module $\mathrm{S}_{1,2}$ (corresponding to $\Phi_{1,2}$ with $\kappa = 8$):

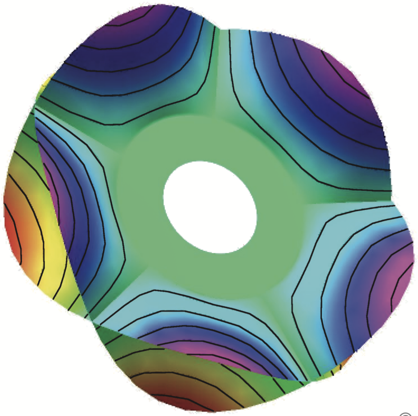
$$0 \longrightarrow \mathrm{S}_{1,1} \xrightarrow{\iota} \mathrm{S}_{1,2} \boxtimes \mathrm{S}_{1,2} \xrightarrow{\pi} \mathrm{S}_{1,3} \longrightarrow 0$$

where $\mathrm{S}_{1,2} \boxtimes \mathrm{S}_{1,2}$ is so-called *staggered module* (not semisimple)

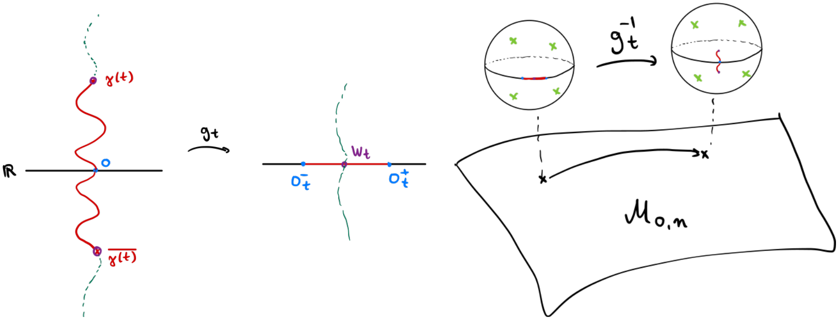
- $\mathrm{S}_{1,1}$ corresponding to $\Phi_{1,1}$
- $\mathrm{S}_{1,3}$ corresponding to its "log-partner" $\widetilde{\Phi}_{1,3}$

[Gurarie '93; Gaberdiel & Kausch '96; Rohsiepe '96; Kytölä & Ridout '09]

Beyond regular singularities?



Beyond regular singularities?



$$M_t(x=0;z_1,\ldots) = \Big(\prod_j g_t'(z_j)^{\Delta_j}\, g_t'(\bar z_j)^{\bar\Delta_j}\Big)\,\mathscr{Z}(W_t;g_t(z_1),\ldots)$$

Partition function with irregular singularities: $\kappa = 4$

Thm. [Desiraju, Korzhenkova & P. 2025+]
We find a matrix-valued (local) martingale for the SLE(4) curve:

$$M_t = \Big(\prod_{i=1}^{n} g_t'(z_i)^{\Delta_i} \exp\Big(\frac{s_i^2}{6} \mathscr{S}(g_t)(z_i) + s_i \sqrt{\Delta_i} \mathscr{A}(g_t)(z_i) \Big) \Big) \mathscr{Z}(W_t, g_t(\boldsymbol{z}); g_t(\boldsymbol{s}))$$

- $W_t = 2B_t \in \mathbb{R}$: Loewner driving function of the SLE(4) curve,
- $g_t(z_i)$ time-evol. of punctures $\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$
- $g_t(s_i)$ time-evol. of *Birkhoff spectral invariants* $\boldsymbol{s} = (s_1, \ldots, s_n) \in \mathbb{C}^n$

  (here, eigenvalues of matrices $A_{i,1}$ in non-Fuchsian system)
- $\mathscr{A}(g) := g''/g'$ is the pre-Schwarzian
- $\mathscr{S}(g) := (g''/g')' - \frac{1}{2}(g''/g')^2$ is the Schwarzian
- $\mathscr{Z}(W_t, g_t(\boldsymbol{z}); g_t(\boldsymbol{s})) = \tau(g_t(\boldsymbol{z}); g_t(\boldsymbol{s}))\, Y(W_t, g_t(\boldsymbol{z}); g_t(\boldsymbol{s}))$,

  where $Y(x, \boldsymbol{z}; \boldsymbol{s}) \sim (x - z_i)^{-2\Delta_i} \exp\big(- \frac{s_i}{x - z_i} \big)$

Partition function with irregular singularities: $\kappa = 4$

Thm. [Desiraju, Korzhenkova & P. 2025+]
The quantity

$$M_t = \left(\prod_{i=1}^{n} g_t'(z_i)^{\Delta_i} \exp\left(\frac{s_i^2}{6} \mathscr{S}(g_t)(z_i) + s_i \sqrt{\Delta_i} \mathscr{A}(g_t)(z_i) \right) \right) \mathscr{Z}(W_t, g_t(\boldsymbol{z}); g_t(\boldsymbol{s})),$$

is a martingale iff $\mathscr{Z}$ solves the **confluent BPZ PDEs**

$$\left[2\frac{\partial^2}{\partial x^2} - \sum_{i=1}^{n} \left(\frac{2}{x - z_i} \frac{\partial}{\partial z_i} + \frac{2s_i}{(x - z_i)^2} \frac{\partial}{\partial s_i} \right.\right.$$

$$\left.\left. + \frac{2\Delta_i}{(x - z_i)^2} + \frac{4s_i \sqrt{\Delta_i}}{(x - z_i)^3} + \frac{2s_i^2}{(x - z_i)^4} \right) \right] \mathscr{Z}(x, \boldsymbol{z}; \boldsymbol{s}) = 0$$

Underlying (non-Fuchsian) system

If someone is familiar... here it is:

$$\frac{\partial}{\partial x} Y(x; \boldsymbol{z}, \boldsymbol{s}) = A(x, \boldsymbol{z})\, Y(x; \boldsymbol{z}; \boldsymbol{s})$$

$$A(x, \boldsymbol{z}) := \sum_{i=1}^{n} \frac{A_{i,0}(\boldsymbol{z})}{x - z_i} + \frac{A_{i,1}(\boldsymbol{z})}{(x - z_i)^2}, \qquad A_{i,j}(\boldsymbol{z}) \in \mathfrak{sl}_2(\mathbb{C})$$

- $\boldsymbol{z} = (z_1, \ldots, z_n)$ and $\boldsymbol{s} = (s_1, \ldots, s_n)$
- $Y(x, \boldsymbol{z}; \boldsymbol{s})$ is the local solution in coordinate x
- $\tau(\boldsymbol{z}; \boldsymbol{s})$: tau-function of the related integrable system, defined via

$$(\partial_\bullet \log \tau(\boldsymbol{z}, \boldsymbol{s}))\, \mathrm{d}\bullet := H_\bullet, \qquad \bullet = \{s_i, z_i\},$$

where the Hamiltonians are $\det\left(A(y, \boldsymbol{z}) - \sigma(y, \boldsymbol{z}, \boldsymbol{s})\mathbf{1}\right) = 0$ and

$$H_{z_i} := \frac{1}{2} \operatorname*{res}_{y=z_i} \mathrm{Tr}(A(y, \boldsymbol{z}))^2 \mathrm{d}z, \qquad H_{s_i} := -2 \operatorname*{res}_{y=z_i} \frac{\sigma(y, \boldsymbol{z}, \boldsymbol{s})}{(y - z_i)} \mathrm{d}z$$

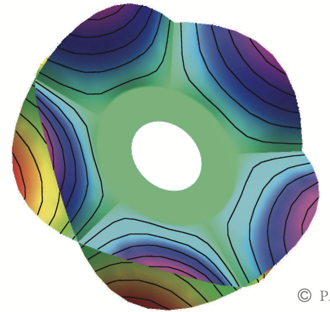
Partition function with irregular singularities: $\kappa = 4$

Puzzle. **What is the interpretation of the observable**

$$\left(\prod_{i=1}^{n} g_t'(z_i)^{\Delta_i} e^{\frac{s_i^2}{6}\mathscr{S}(g_t)(z_i) + s_i\sqrt{\Delta_i}\mathscr{A}(g_t)(z_i)} \tau(g_t(\boldsymbol{z}); g_t(\boldsymbol{s}))\, Y(W_t, g_t(\boldsymbol{z}); g_t(\boldsymbol{s})) \right. ???$$

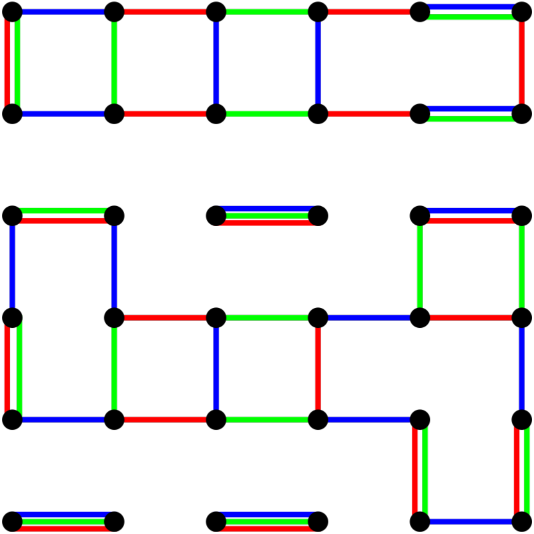
- dependence of *curve tip* x is in $Y(x, \boldsymbol{z}; \boldsymbol{s}) \sim (x - z_i)^{-2\Delta_i} \exp\left(- \frac{s_i}{x - z_i} \right)$
- Δ_i: monodromy when SLE curve winds around puncture z_i
- ... but the irregular nature of the singularities also introduces

Stokes phenomenon...

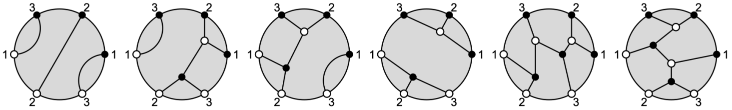


Beyond curves?

Triple dimers ($c = 2$) and W_3-Conformal blocks?



Beyond curves?



Thm. [Lafay, Le & Roussillon 2025]

$$\mathbb{P}^{\delta}_{\beta}[\text{triple dimer web} = \alpha] \xrightarrow{\delta \to 0} \mathcal{K}_{\alpha,\beta} \frac{\mathcal{Z}_{\alpha}(x_1, \ldots, x_d)}{\mathcal{U}_{\beta}(x_1, \ldots, x_d)}$$

- where $\mathcal{U}_{\beta}$ is the conformal block function (fused Specht polynomial)

$$\mathcal{U}_{\beta}(x_1, \ldots, x_d) = \prod_{i<j} (x_j - x_i)^{-s_i s_j/3} \, \mathcal{P}_{\beta^t}(x_1, \ldots, x_d)$$

- $\mathcal{Z}_{\alpha} := \sum_{\beta} \mathcal{K}^{-1}_{\alpha,\beta} \mathcal{U}_{\beta}$ and $\mathcal{K}_{\alpha,\beta}$, $\mathcal{K}^{-1}_{\alpha,\beta}$ *combinatorial* numbers

Upshot: Triple-dimer connection probabilities are given by specific CFT correlation functions with W_3 algebra symmetry!


Happy New Year !