Weights of highest weight modules over Borcherds–Kac–Moody Lie algebras

(Joint works with Apoorva Khare and Souvik Pal)

Algebraic and Combinatorial Methods in Representation Theory
(In honour of Prof. Vyjayanthi Chari's 65^{th} Birthday)
ICTS Bangalore

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Notations for BKM Lie algebras $\mathfrak{g}$

1. A = Borcherds–Kac–Moody (BKM) matrix. (We assume A is *Symmetrizable*). $A_{ii} = 2$ or $A_{ii} \in \mathbb{R}_{\leq 0}$. $A_{ii} = 2 \implies A_{ij} \in \mathbb{Z}_{\leq 0}$ for all $i, j \in [n]$ or $i, j \in \mathcal{I}$.

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2. $\mathcal{I} = \mathcal{I}^{-} \sqcup \mathcal{I}^{0} \sqcup \mathcal{I}^{+}$ the Dynkin diagram/graph nodes.
   $= \{i \in \mathcal{I} \mid A_{ii} \in \mathbb{R}_{<0}\} \sqcup \{i \in \mathcal{I} \mid A_{ii} = 0\} \sqcup \{i \in \mathcal{I} \mid A_{ii} = 2\}$.
   Ex: for $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$, $\mathcal{I} = \mathcal{I}^{+} = \{1, 2\}$; for $\mathfrak{g} = \widehat{\mathfrak{sl}_3(\mathbb{C})}$, $\mathcal{I} = \mathcal{I}^{+} = \{0, 1, 2\}$;
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   when $A = 0_{n \times n}$, $\mathcal{I} = \mathcal{I}^0 = \{1, \ldots, n\}$.
3. $\mathfrak{g} = \mathfrak{g}(A)$ semisimple/ Kac–Moody/ Borcherds–Kac–Moody Lie algebra over $\mathbb{C}$ corresponding to A.
   $e_i, f_i, \alpha_i^\vee \ \forall\, i \in \mathcal{I}$ – Chevalley generators of $\mathfrak{g}$.
   $\mathbb{C}\{e_i, f_i, \alpha_i^\vee\} \cong$ 1) $\mathfrak{sl}_2(\mathbb{C})$ if $i \in \mathcal{I}^+ \sqcup \mathcal{I}^-$, 2) 3-dim. Heisenberg Lie alg. if $i \in \mathcal{I}^0$.
   $\mathfrak{h}$ = Cartan subalgebra of $\mathfrak{g}$.
   $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ (triangular decomposition).
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4. Δ = root system of $\mathfrak{g}$. Root space decomposition: $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha\right)$.

5. $\Pi = \{\alpha_i \mid i \in \mathcal{I}\}$ simple roots, $\Pi^\vee = \{\alpha_i^\vee \mid i \in \mathcal{I}\}$ simple co-roots.
   $\Delta^+ = \Delta \cap \mathbb{Z}_{\geqslant 0}\Pi$ positive roots w.r.t. Π.

6. W = Weyl group gen. by real-simple reflections: $s_i(\lambda) := \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$, $i \in \mathcal{I}^+$.

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4. $M(\lambda) \twoheadrightarrow V$ denotes an arbitrary highest weight $\mathfrak{g}$-module V with highest weight λ.

   $V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$, the μ-weight space $V_\mu = \{v \in V \mid h \cdot v = \mu(h)v\ \forall\ h \in \mathfrak{h}\}$.
   The set weights of V is $\mathrm{wt}\, V = \{\mu \in \mathfrak{h}^* \mid V_\mu \neq 0\}$. So, $V = \bigoplus_{\mu \in \mathrm{wt}\, V} V_\mu$.

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Over general Kac–Moody $\mathfrak{g}$ ($\mathcal{I} = \mathcal{I}^+$ and $\mathcal{I}^0 = \mathcal{I}^- = \emptyset$), we know for $\lambda \in \mathcal{P}^+$:

$$L^{\max}(\lambda) := \frac{M(\lambda)}{\sum_{i \in \mathcal{I}} U(\mathfrak{g}) f_i^{\langle \lambda, \alpha_i^\vee \rangle + 1} M(\lambda)_\lambda} \longrightarrow \overset{(?)}{\cdots} \longrightarrow L(\lambda) \longrightarrow 0.$$

$L^{\max}(\lambda) = L(\lambda)$ when $\mathfrak{g}$ is symmetrizable.

Weights and characters of highest weight integrable V

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((1) and (2) hints us to look at "Heisenberg holes" (holes in $\mathcal{I}^0$) in a sense.)

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Over BKM $\mathfrak{g}$, a **weight-formula** for all integrable $L(\lambda)$:
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$J = J_1 \sqcup \cdots \sqcup J_m$ (Dynkin subdiag. comp.), *then* $\exists\, j_k \in J_k\ 1 \leq k \leq m$*, s.t.:*

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Integrable directions for any $\lambda \in \mathfrak{h}^*$: $\quad J_\lambda := \left\{ j \in \mathcal{I} \;\middle|\; \lambda(\alpha_j^\vee) \in \frac{A_{jj}}{2}\,\mathbb{Z}_{\geq 0} \right\}$.

So, $\quad f_i^{\frac{2\lambda(\alpha_i^\vee)}{A_{ii}}+1} \cdot L(\lambda)_\lambda \;=\; 0 \;=\; f_j \cdot L(\lambda)_\lambda \qquad \forall\, i \in J_\lambda \setminus \mathcal{I}^0 \ \ \&\ \ \forall\, j \in J_\lambda \cap \mathcal{I}^0$.

Parabolic Vermas & weights of all simples $L(\lambda)$ in KM case

Given a BKM Lie algebra $\mathfrak{g}$, and $J \subseteq \mathcal{I}$, "parabolic analogues":

- $\mathfrak{g}_J \ = \ \mathfrak{g}(A_{J\times J})$, with:
- Simple roots $\Pi_J = \{\alpha_j \mid j \in J\}$ and Simple co-roots $\Pi_J^\vee = \{\alpha_j^\vee \mid j \in J\}$.
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  Weyl group of $\mathfrak{g}_J \ = \$ parabolic subgroup $W_{J\cap\mathcal{I}^+} \ := \ \langle\{s_j \mid j \in J \cap \mathcal{I}^+\}\rangle$ of W.
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$\mathfrak{g}$ is a Kac–Moody algebra for below (and for the next two slides.)

Key tool in computing all $\operatorname{wt}L(\lambda)$ and all $\operatorname{wt}V$: *parabolic Verma modules*.

Recall, $J_\lambda := \{j \in \mathcal{I} \mid \lambda(\alpha_j^\vee) \in \mathbb{Z}_{\geq 0}\}$. For any $\lambda \in \mathfrak{h}^*$ and $J \subseteq J_\lambda$, we define

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Theorem (Khare [*J. Algebra* 2016], Dhillon–Khare [*J. Algebra* 2022])

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$$\mathcal{H}_V := \left\{ H \subseteq J_\lambda \;\middle|\; 1)\ H \text{ independent set},\ 2)\ \prod_{h\in H} f_h^{\langle \lambda, \alpha_h^\vee\rangle+1} \cdot V_\lambda = 0 \right\}.$$

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$$\mathcal{H}_{M(0)} = \emptyset. \qquad \mathcal{H}_{M(0,\{1\})} = \{\ \{1\},\ \{1,2\}\ \}.$$

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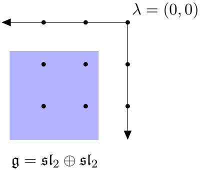

Holes of highest weight modules V over KM $\mathfrak{g}$

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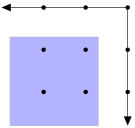

Example 2: $\mathfrak{g} = \mathfrak{sl}_6(\mathbb{C})$ and $V' := \dfrac{M(0)}{\langle \{ f_1 f_3,\ f_2,\ f_4 f_5 \} \cdot M(0)_0 \rangle}$:

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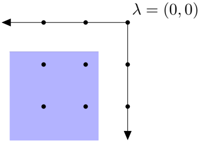

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Example 3: $\mathfrak{g}$ any KM and $J \subseteq J_\lambda$. $V = M(\lambda, J)$ then $\mathcal{H}_V^{\min} = \{\ \{j\} \mid j \in J\ \}$.

Higher order Verma modules: Given any $\lambda \in \mathfrak{h}^*$ and holes $\mathcal{H} \in \mathrm{Indep}(J_\lambda)$,

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Example 2. n=2 & $\lambda = (0,0)$: Here $\mathcal{I} = \{1,2\}$.

$$V = \frac{M(\bar{0})}{\langle\ \{ f_2^{6},\ f_1^{1}f_2^{6},\ f_1^{3}f_2^{3},\ f_1^{5}f_2^{2},\ f_1^{7}\ \} \cdot M(\bar{0})_{\bar{0}}\ \rangle}.$$

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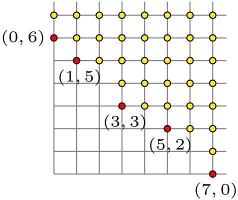

Definition and examples of holes in BKM setting

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- $H \in \mathrm{Indep}(J_\lambda)$. (Recall $J_\lambda := \left\{ j \in \mathcal{I} \;\middle|\; \lambda(\alpha_j^\vee) \in \frac{A_{jj}}{2}\mathbb{Z}_{\geq 0} \right\}$).
- Power sequence/map $m_H : H \to \mathbb{Z}_{\geq 0}$ s.t.
  $$m_H(h) = \frac{2}{A_{hh}} \lambda(\alpha_h^\vee) + 1 \;\; \forall\, h \in H \cap (\mathcal{I}^+ \sqcup \mathcal{I}^-).$$
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Example 2: For (any) $\lambda \in \mathfrak{h}^*$, minimal holes $\mathcal{H}^{\min}_{L(\lambda)}$ of $L(\lambda)$ are all singletons: $\mathcal{H}^{\min}_{L(\lambda)} = \left\{ \left(\{j\},\ \frac{2}{A_{ii}}\lambda(\alpha_j^\vee) + 1\right) \mid j \in J_\lambda \cap (\mathcal{I}^- \sqcup \mathcal{I}^+) \right\} \bigsqcup \left\{ \left(\{i\},\ 1\right) \mid i \in J_\lambda \cap \mathcal{I}^0 \right\}$.

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Partial order on holes: $(H, m_H) \leq (K, m_K)$ iff $H \subseteq K$ & $m_H(h) \leq m_K(h) \; \forall \, h \in H$.

1) For any hole-set $\mathcal{H} \in \mathrm{INDEP}(J_\lambda)$: the minimal holes $\mathcal{H}^{\min}$ and upper closure $\bar{\mathcal{H}}$.

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Weight formula for V over $\mathfrak{g}$ generated by Heisenbergs

Theorem (Pal–T.)

Given any $\mathfrak{g} = \mathfrak{g}(A)$ *with* $A_{ii} = 0\ \forall\ i$, *and* $\lambda \in \mathfrak{h}^*$ *and* $M(\lambda) \twoheadrightarrow V \neq 0$,

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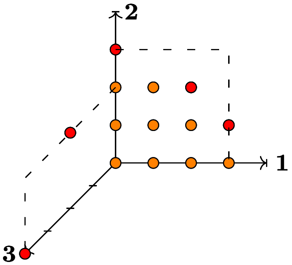

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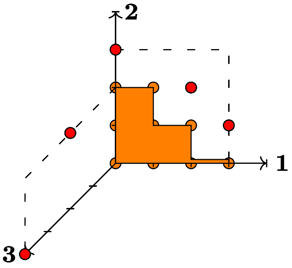

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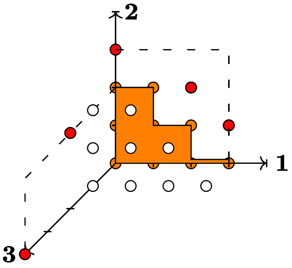

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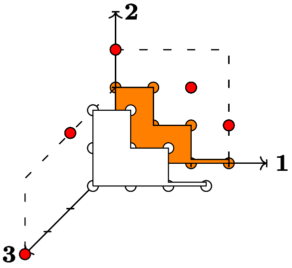

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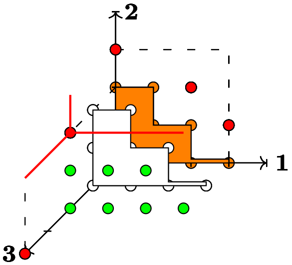

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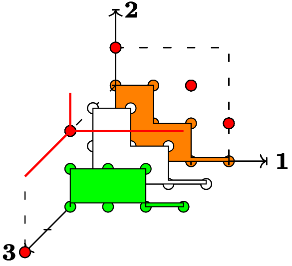

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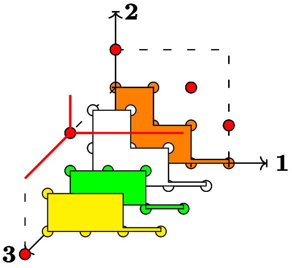

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$$\operatorname{wt} V = \bigcup_{I,J \text{ satisfying } (1)-(3)} \left[\operatorname{wt} V \cap \left(\lambda - \mathbb{Z}_{\geq 0}\Pi_J\right)\right] - \mathbb{Z}_{>0}\Pi_I. \qquad (0.1)$$

(1) $J \in \operatorname{Indep}(J_\lambda)$. *(2)* $J = \emptyset \implies I = \emptyset$.
(3) Every Dynkin subdiagram component of I *has a node sharing an edge with some node in* J.

For the first term in (0.1): determine $\mu \in \operatorname{wt} V$ with $\operatorname{supp}(\lambda - \mu) \in \operatorname{Indep}(J_\lambda)$.

Example: $n = 3$ and $\mathfrak{g} = \mathfrak{g}(0_{3\times 3})$

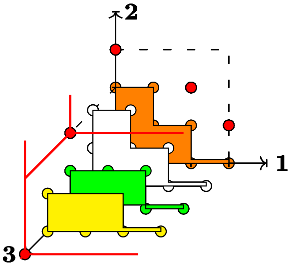
$$V = \frac{M(0) \;\cong\; \mathbb{C}[f_1,\ f_2,\ f_3]}{\langle\{\ f_1^3 f_2,\ \ f_1^2 f_2^2,\ \ f_2^3,\ \ f_2^2 f_3^2,\ \ f_3^4\ \} \cdot M(0)_0\rangle}.$$

$$\mathcal{H}_V^{\min} = \{\ (\{1,2\},\ (3,1)),\ (\{1,2\},\ (2,2)),\ (\{2\},\ 3), (\{2,3\},\ (2,2)),\ (\{3\},\ 4)\}.$$

Weights of V? More generally, the complement of an upper-closed subset in $\mathbb{Z}^n_{\geq 0}$?

Algorithm:

Start for e.g. with $\{1,2\} \subsetneq \mathcal{I}$ and observe geometrically:



Main result 1: Weight-sets of Parabolic higher order Vermas

Nice holes-sets $\mathcal{H}$ (w.r.t real directions) in our pursuit of weights:

$$(H, m_H) \in \mathcal{H}^{\min} \quad \Longrightarrow \begin{cases} \text{either } H \subseteq \mathcal{I}^- \sqcup \mathcal{I}^0, \\ \text{or } H \subseteq \mathcal{I}^+ \text{ and } |H| = 1. \end{cases}$$

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Example: $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$ and $\mathcal{I} = \{1, 2\}$.

$\big\{ \ (\{1\},\ 2),\ (\{1,2\},\ (3,4))\big\}$ is nice, $\big\{ \ (\{1,2\},\ (3,4)) \ \big\}$ is not nice.

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Parabolic higher order Verma modules: given $\lambda \in \mathfrak{h}^*$ and nice $\mathcal{H} \in \mathrm{INDEP}(J_\lambda)$

$$M(\lambda, \mathcal{H}) := \frac{M(\lambda)}{\sum\limits_{(H,\ m_H)\ \in\ \mathcal{H}^{\min}} \left(U(\mathfrak{n}^-) \cdot \prod\limits_{h \in H} f_h^{m_H(h)} \right) \cdot M(\lambda)_\lambda}.$$

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Given any BKM $\mathfrak{g}$, $\lambda \in \mathfrak{h}^*$ and nice hole-set $\mathcal{H} \in \mathrm{INDEP}(J_\lambda)$,

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(C1) $\mu(\alpha_i^\vee) \in \mathbb{Z}_{\geq 0}\ \forall\, i \in \mathcal{H}^+$.

(C2) *If* $\mu = \lambda - \sum_{j \in J} c_j \alpha_j \precneqq \lambda$ *for* $c_j \in \mathbb{Z}_{>0}\ \forall\, j \in J$, *and* $J = J_1 \sqcup \cdots \sqcup J_m$ *(Dynkin subdiag. comp.), then* $\exists\, j_k \in J_k$ $1 \leq k \leq m$, *s.t.:*

$$\mu_\circ = \lambda - \sum_{k \text{ s.t. } |J_k| = 1} c_{j_k} \alpha_{j_k} - \sum_{k \text{ s.t. } |J_k| > 1} \alpha_{j_k} \in \mathrm{wt}\, M(\lambda, \mathcal{H}).$$

Main result 2: Weight-set of every highest weight $\mathfrak{g}$-module

Higher order Verma modules $\mathbb{M}(\lambda, \mathcal{H})$: given $\lambda \in \mathfrak{h}^*$ and $\mathcal{H} \in \mathrm{INDEP}(J_\lambda)$,

$$\mathbb{M}(\lambda, \mathcal{H}) := \frac{M(\lambda)}{\sum\limits_{H \in \mathcal{H}} \left(U(\mathfrak{n}^-) \cdot \prod_{h \in H} f_h^{\lambda(\alpha_h^\vee)+1} \right) \cdot M(\lambda)_\lambda}.$$

$\mathbb{M}(\lambda,\ \mathcal{H}) \ = \ M(\lambda,\ \mathcal{H}) \iff \mathcal{H}$ is nice. $\qquad \mathbb{M}(\lambda,\ \mathcal{H}_V) \twoheadrightarrow V$ always.

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A uniform, explicit, non-recursive and cancellation-free weight-formula over all BKM $\mathfrak{g}$:

Theorem (Pal–T.)

For any general BKM $\mathfrak{g}$, $\lambda \in \mathfrak{h}^*$, and any $\mathfrak{g}$-module $M(\lambda) \twoheadrightarrow V \neq 0$:

$$\mathrm{wt}\, V \ = \ \mathbb{M}(\lambda, \mathcal{H}_V) \ = \bigcup_{\substack{\mathcal{H} \text{ is nice} \\ \mathcal{H}_V \subseteq \mathcal{H} \subseteq \mathrm{INDEP}(J_\lambda)}} \mathrm{wt}\, M(\lambda, \mathcal{H}).$$

Example: $\mathcal{I} \ = \ \mathcal{I}^+ \sqcup \mathcal{I}^0$, and $\mathcal{I}^+ \ = \ \{1, 2\}$ is an independent set and $\mathcal{I}^0 \ = \ \{3\}$.

$$V \ = \ \frac{M(0)}{\langle \ \{ \ f_1 f_2, \ f_3^2 \ \} \cdot M(0)_0 \rangle} \ \longleftrightarrow \ \{ \ (\{1,2\}, \ (1,1)), \ \ (\{3\}, \ 2) \ \}.$$

$$\mathrm{wt}\, V \ = \ \mathrm{wt}\, M\big(0, \{ \ (\{1\}, \ 1), \ (\{3\}, \ 2) \ \}\big) \bigsqcup \mathrm{wt}\, M\big(0, \{ \ (\{2\}, \ 1), \ (\{3\}, \ 2) \ \}\big)$$
$$\bigsqcup \mathrm{wt}\, M\big(0, \{ \ (\{1\}, \ 1), \ (\{2\}, \ 1), \ (\{3\}, \ 2) \ \}\big).$$

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