Down to calculate in EFT

ICTS Lectures 2024

Foreword

To frame these lectures in the background of the school they have been designed as workshops with a main central problem in each lecture. The students are encouraged to work through them & the solutions will be given in the lectures or provided later. This does not mean however that the motto is shut up & calculate; each problem is supposed to illustrate an aspect of EFT so that you use it to show to yourself how things work rather than being told. As such some problems will be more academic and/or present simplified version of a current-research-type computation yet all the elementary skills should be covered so that you learn not only how-to but also what & why.

Problems for you to try are marked in purple.

References

These two are useful sources for these lectures

- EFT Aneesh arxiv:1804.05863
- EFT D. B. Kaplan lectures @ ICTP-SAFIR

As well as notes from other lecturers at this ICTS-SATPP school.

In addition other material we will pull from circumstantially

- Callan, Coleman, Wess & Zumino PR 177, 1969
- Lee, Quigg & Thacker PRD 16, 1977
- TASI lectures Gherghetta 1008.2570
- Jenkins, Manohar & Trott 1308.2627

Outline

L1 – Introduction; features of EFT in simple set-ups

L2 – Model (in)dependence; Matching to different theories

L3 – Field redefinition redundancy, Unitarity

L4 – Renormalisability, Physical inputs

L5 – Anomalous dimensions & operator mixing

Conventions & Formulae

Natural units & the mostly-minus metric:

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

a quantised complex field will read

$$\phi_I = \int \frac{d^3p}{2E_p(2\pi)^3}\left(a e^{-ip\cdot x} + b^\dagger e^{ipx} \right)$$

with $[a_p, a_k^\dagger] = 2E_p(2\pi)^3 \delta^3(\vec{p}-\vec{k})$ etc

while the Fourier transform

$$\phi(x) = \int \frac{d^4q}{(2\pi)^4} \tilde{\phi}(q) e^{iqx}$$

& at times we'll use shorthand

$$\int [dl] = \int \frac{d^4l}{(2\pi)^4} \mu^{2\epsilon}$$

while in dimensional regularisation with $d = 4-2\epsilon$

$$\mu^{2\epsilon}\int \frac{d^dl}{(2\pi)^d} \frac{(K^2)^a}{(K^2-\Delta)^b} = \frac{i\mu^{2\epsilon}(-1)^{a-b}}{(4\pi)^{d/2}} \frac{\Gamma(d/2+a)\Gamma(b-a-d/2)\Delta^{\frac{d}{2}+a-b}}{\Gamma(d/2)\Gamma(b)}$$

L1 Introduction & features of EFT in simple systems

We will take a look at simple systems to illustrate features of EFT, highlighted in green below.

One has that EFTs are founded on

- Identifying relevant degrees of freedom
- Symmetries / redundancies
- An expansion parameter

While some of their properties are

- Limited range of validity (Unitarity)
- Model independence (Matching)
- Renormalisable order by order (experimental inputs)
- Predictive

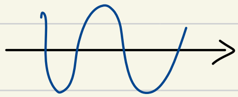
A) 1-d Scattering in QM

The following example serves to illustrate a number of features & all we have to do is solve Schrodinger's eq. & work through boundary conditions. (This is taken from Kaplan's notes)

Square well

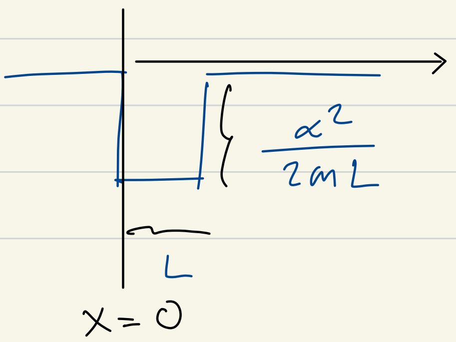
Consider a square-well potential and and an incident wave or particle

$$\psi \sim \sqrt{F}\, e^{i(kx - Et)}$$



where F is the flux of particles coming in and $E = k^2/2m$.

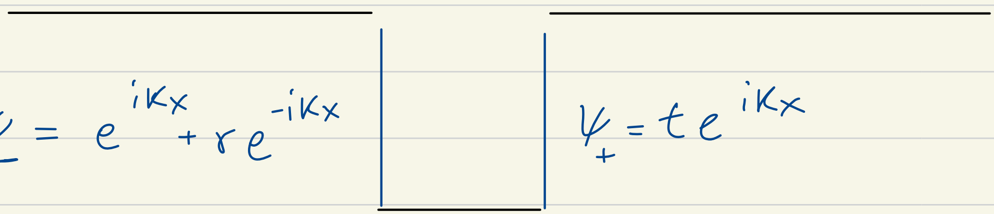
The wave encounters a square-well potential as



$$V(x) = \begin{cases} -\frac{\alpha^2}{2mL^2} & 0 \leq x \leq L \\ 0 & \text{elsewhere} \end{cases}$$

Some of it will reflect back & some will continue forwards to $x = \infty$, to determine each we solve the (time-independent) Schrodinger eq.

$$E\,\psi(x) = \left[-\frac{\partial_x^2}{2m} + V(x) \right] \psi(x)$$



$$\psi_- = e^{ikx} + r e^{-ikx} \qquad \psi_+ = t e^{ikx}$$

$$\psi_0 = a e^{iqx} + b e^{-iqx}$$

where $q^2 = k^2 + (\alpha/L)^2$.

One B.C. we have already put in since ψ_+ has one term only (nothing is coming from $+\infty$) & we have divided by $\sqrt{F}$ everywhere to make things simpler.

(I.a) Your **first assignment** is to solve Schrodinger's eq.

That is ($\partial_x^2 \psi$ is discontinuous so ψ is C^1)

from

$$\psi_-(0) = \psi_0(0), \quad \psi_0(L) = \psi_+(L),$$

$$\partial_x \psi_-(0) = \partial_x \psi_0(0), \quad \partial_x \psi_0(L) = \partial_x \psi_+(L),$$

obtain

$$\psi_- = e^{ikx} + \frac{(k^2-q^2)\, e^{-ikx}}{(k^2+q^2) - 2iqk\, \mathrm{ctg}(qL)}$$

$$\psi_+ = \frac{-2qk\, e^{ik(x-L)}}{(2qk)\cos qL + (k^2+q^2)\, i \sin qL}$$

And the reflection & transmission coefficients

$$r^2 = \frac{1}{1 + \frac{(2qk)^2}{[\sin(qL)(q^2-k^2)]^2}}, \quad t^2 = 1 - r^2$$

You can check this

why do you think this is satisfied?

You might have guessed it, flux or probability conservation demands $t^2 + r^2 = 1$.

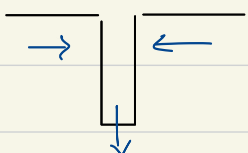
In fact we can define an S matrix in this set up

$$S_{\to\to} = t \,, \quad S_{\to\leftarrow} = r \,.$$

There are only two possible out states, backwards $\leftarrow$ or forwards $\to$ with $t = 1$ the limit of no scattering & $S = 1$. Unitarity then demands $(S S^\dagger)_{\to\to} = 1 = |t|^2 + |r|^2$.

One more thing, the most relevant for us, is that the limit $L \to 0$, that is a narrow well, one can identify the expansion parameter $\varepsilon = KL$. so that

$\varepsilon \ll 1$;



$q = \frac{\alpha}{L} + O(KL) \gg K$

and the reflection coefficient

$$r^2 = 1 - \left(\frac{2KL}{\alpha \sin\alpha}\right)^2 + O\left((KL)^4\right).$$

Delta function

We are going to do it again but easier. Replace the previous potential for

$$V(x) = -\frac{g}{2mL}\delta(x)$$

following the very same steps, the solution on either side

$$\psi_-(x) = e^{ikx} + re^{-ikx} \qquad \psi_+(x) = te^{ikx}$$

but now $\partial_x^2\psi$ contains a Dirac delta so $\partial_x\psi$ has a discontinuity. In any case (I.b) you can show that from

- $\psi_-(0) = \psi_+(0)$,
- $\partial_x\psi_-(0) - \partial_x\psi_+(0) = \frac{g}{L}\psi(0)$,

it follows

$$r = \frac{1}{1 + \frac{i2\kappa L}{g}} \quad , \quad t = \frac{1}{1 - \frac{ig}{2\kappa L}} .$$

This set-up makes it a little easier to calculate and further, in an expansion in L,

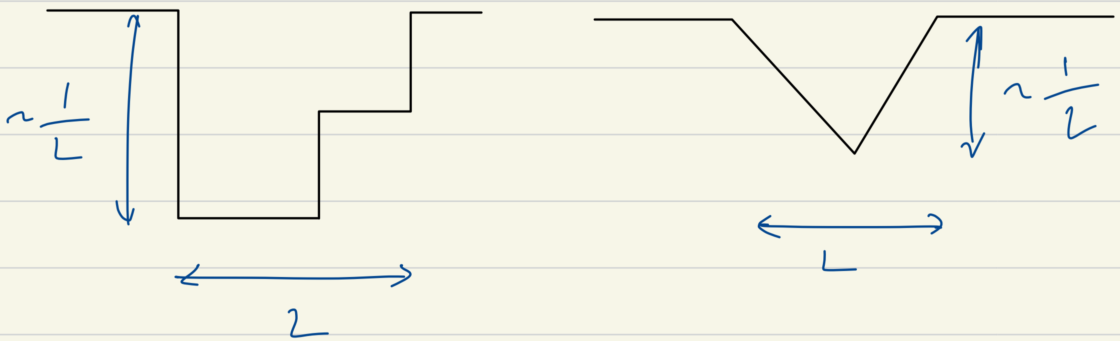
$$|r|^2 = \frac{1}{1 + \left(\frac{2\kappa L}{g}\right)^2} = 1 - \left(\frac{2\kappa L}{g}\right)^2 + \mathcal{O}\left((\kappa L)^4\right) .$$

can reproduce the square-well if we identify

$$g = \alpha \, \text{sm} \, a$$

In EFT jargon this is Matching, & it is important to realise that this only works to a certain order in κL, our expansion parameter. To reproduce higher order terms we need more terms as $\partial_x^n \delta(x)$.

One of the use of EFT is that we don't need to know $g = \alpha \sin\alpha$ to make predictions. In fact it's better to leave it free cause what if the potential looked like



We would have different matching conditions but still the EFT will hold provided $L \ll \frac{1}{k}$ & we can Taylor expand & capture the model's in the coefficients

This is what is meant by EFTs being model independent. One could be more precise to say EFT package model dependence in coefficients that can simply come along for the ride in calculations of the $\varepsilon \ll 1$ regime.

This example captures the essence of EFT; within its regime of validity $\kappa \ll \frac{1}{L}$

- The EFT captures physics of a scale not directly observable (L) in the coefficients of local terms $(\partial_x^n \delta(x))$.

- At a given order, including all possible terms will allow the EFT to describe a limit of any theory

- Even if one might be tempted to obtain $O((\kappa L)^2)$ results from $\delta(x)$, the EFT only makes sense when all quantities are expanded in ε.

- The EFT will signal its own breakdown, take

$$r^2 = 1 - 4\frac{\kappa^2 L^2}{g^2}$$

what goes wrong if I increase κ arbitrarily?

Yes, conservation of probability (Unitarity in HEP parlance) would not follow! We need the full model.

Other examples to amuse yourself

B) Nuclear shapes

One has that probing the nucleus with alpha particles

$$\frac{d\sigma}{d\Omega} = |F(\kappa^2)|^2 \frac{\alpha_{em}^2 Z_{He} Z_{Nucl}}{16 E_\kappa^2 \sin^4(\theta/2)}$$

with the form factor F

$$F(q^2) = \int d^3x \, e^{-i\vec{\kappa}\cdot\vec{x}} f(\vec{x})$$

with f the nucleus charge distribution, normalised to $\int d^3x \, f(x) = 1$ so that

$$V_{em}(\vec{y}) = \alpha_{em} Z_{He} Z_{Nd} \int d^3x \frac{f(\vec{x})}{|\vec{y} - \vec{x}|}$$

A ball-model gives $f(\vec{x}) = \dfrac{8}{4\pi L^3}\begin{cases} 1 & |\vec{x}| < L \\ 0 & |\vec{x}| > L \end{cases}$

(I,c) Using polar coordinates for the integral, show this gives

$$F(\kappa^2) = \frac{3}{(|K|L)^2}\left(\frac{\sin |K|L}{|K|L} - \cos(|K|L)\right)$$

and

$$F(\kappa^2) = 1 - \frac{|KL|^2}{10} + \mathcal{O}((|K|L)^2)$$

while if we Fourier transform back the first two terms we get

$$F(\kappa^2) = 1 + c\,\kappa^2 L^2$$

$$\hookrightarrow f(\vec{x}) = \delta^3(\vec{x}) - c L^2 \nabla_x^2 \delta^3(\vec{x})$$

This is an example of a case with a local term and a higher order term.

In fact we could have $\mathcal{O}(\nabla\delta)$ in between, can you think of a reason why the ball example doesn't have it?

Again we find an expansion in local terms which is a multipole expansion. You can show that the higher order gives

$$\bullet \; f(\vec{x}) = \left(1 - c\, L^2 \vec{\nabla}^2\right) \delta^3(\vec{x})$$

Returns a potential

$$V(r) = \frac{Z Z' \alpha_{em}}{r} \left(1 + \frac{3 c L^2}{r^2}\right)$$

C) Superconductivity

One last example illustrates that even identifying degrees of freedom is not a straightforward task. Kaplan has a neat discussion of this in section 3.4.

Some condensed matter systems have that starting from a description in terms of electrons, the theory might develop a instability (signaled as a break-down in your expansion) in which the atractive interaction mediated by phonons (the lattice of positive charge ions) gets stronger. The outcome is a theory that looks nothing like the original one where the quanta, the degrees of freedom, are bozons!

These are Cooper pairs made up of two electrons & are relevant to understand superconductivity, for example their size is much larger than the typical Δx for the original electrons so they can go through the defects that cause resistance and conduct electricity much more efficiently.