L3 Field redefinitions & Unitarity

A) Field redefinitions

To obtain a truly model-independent (or all-model encompassing) EFT one should write all terms allowed by the symmetry to a given order. Not all operators are independent however; fields are integration variables in the path integral & there exists a freedom to do field redefinition & obtain a theory physically indistinguishable. This redundancy in the description follows from the LSZ formula

$$S_{in \to out} = \frac{\int D\varphi \left[\prod_i \frac{p_i^2 - m^2}{\sqrt{Z}} \tilde{\varphi}(p_i) \right] e^{iS[\varphi]}}{\int D\varphi \, e^{iS[\varphi]}}$$

That says the S matrix is in the all-leg residue of correlation functions. One has that a field transform of the form

$$\phi \to \phi + \delta\phi \, , \qquad \delta\phi = \sum_{n=2} c_n \phi^n$$

will change the correlation functions but not the residue.

We say that this is because both cases excite a one particle state out of the vacuum

$$\langle 0 | \phi \, a^\dagger | 0 \rangle = \langle 0 | \Big(\phi + \sum_{n=2} c_n \phi^n \Big) a^\dagger | 0 \rangle$$

and other lectures have gone into more depth as to why this happens.

Here instead of dwelving in the theory we'll show how this happens in practise.

Consider the action $S = S_4 + S_E$ with a single higher dimensional operator

$$S_E = \int d^4x \, \frac{c_E}{\Lambda^2} \left(\phi^* \phi \, \partial^2 \phi^* \phi - 2 \phi^* \phi \, \partial_\mu \phi^* \partial^\mu \phi \right)$$

$$= \int d^4x \, \frac{c_E}{\Lambda^2} \, \phi^* \phi \left(\phi^* \partial^2 \phi + h.c. \right)$$

It is a linear combination of an operator we considered and one we didn't. Now consider the field transformation

$$\phi \to \phi + \frac{\delta \phi}{\Lambda^2}$$

and its effect on the PI

$$S_4 + S_E \longrightarrow S_4 + \frac{\delta\phi}{\Lambda^2} \frac{\delta S}{\delta \phi} + h.c. + S_E + \mathcal{O}\left(\frac{1}{\Lambda^4}\right)$$

$$= S_4 + S_E$$

$$+ \int d^4x \, \frac{\delta\phi}{\Lambda^2} \left[-\left(\partial^2 + m^2 \right) \phi^* - \frac{\lambda}{2} \phi^* \phi \, \phi^* \right] + h.c.$$

Now if one chooses $\delta\phi = c_E \phi^* \phi\, \phi$ the resulting action

$$\bar{S} \equiv \bar{S}_4 + S_6$$

$$= S_4 - \frac{c_E}{\Lambda^2} \int d^4x\, 2 \left(\phi^*\phi\, m^2\, \phi^*\phi \quad + \frac{\lambda}{2} (\phi^*\phi)^3 \right)$$

with parameters in the $\bar{S}$ case

$$\bar{\lambda} = \lambda + \frac{c_E\, 8 m^2}{\Lambda^2} \quad , \quad c_6 = \frac{-\lambda c_E}{\Lambda^2} .$$

you can derive this

(III.a) You can check that this is the same resulting action if we use the EoM

$$\partial^2 \phi = -m^2 \phi - \frac{\lambda}{2} \phi^2 \phi^* \quad \text{on}$$

show this

$$(\phi^*\phi)\, \phi^* \partial^2 \phi + \text{h.c.} = -2\, \phi^*\phi \left(m^2\, \phi^*\phi + \frac{\lambda}{2} (\phi^*\phi)^2 \right)$$

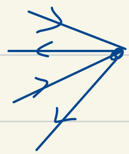
For higher order however note that the procedure "safe to use" is with $\delta\phi$.

Let us now show for a few S-matrix elements that both $\boxed{S}$ & $\boxed{\bar{S}}$ give the same result.

First let's take $2 \to 2$ scattering or equivalently 4-point where

$\boxed{S}$

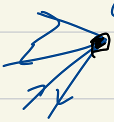
$$-\lambda(\phi^*\phi)^2/4 \rightarrow$$



$$-i\,2\cdot 2\,\frac{\lambda}{4}$$

$$\frac{c_E}{\Lambda^2}\,\phi^*\phi\left(\phi^*\Box\phi + h.c.\right)$$

$$\downarrow$$



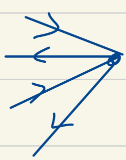
$\frac{c_E}{\Lambda^2}$

$$i\left(-2\left(p_1^2 + p_3^2\right) - 2\left(p_2^2 + p_4^2\right)\right)\frac{c_E}{\Lambda^2}$$

The sum for on-shell particles

$$-i\lambda \quad -i\,8\,\frac{m^2}{\Lambda^2}c_E$$

$\boxed{\bar{S}}$

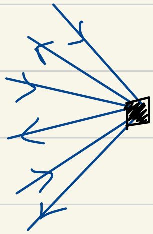


$$-i\,\bar{\lambda} = -i\left(\lambda + \frac{8m^2}{\Lambda^2}c_E\right)$$

Let's do another one, the contact term @ 6pt

$$-\frac{\lambda c_E}{\Lambda^2}(\phi^*\phi)^3$$

$\boxed{\bar{S}}$



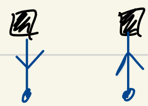
$$\Downarrow$$

$$3!\,3!\left(-\frac{i\lambda c_E}{\Lambda^2}\right)$$

+ non local terms $\dfrac{1}{q^2-m^2}$

If we compare with the contact term in S_E we have to do some combinatorics

$\boxed{S}$

 ← any of the 2 outgoing → in λ & 2 incoming in c_E

$$(4+4)(3!)^2 \leftarrow \text{external combinatorics}$$

$$\times\left(\frac{6m^2+2q^2}{4}\right)\left(\frac{-ic_E}{\Lambda^2}\right)\frac{i}{q^2-m^2}\left(\frac{-i\lambda}{4}\right)$$

$$= -i(3!)^2\lambda\frac{c_E}{\Lambda^2} + \underbrace{\frac{8m^2}{q^2-m^2}(-i)\frac{\lambda}{2}(3!)^2}_{\text{non local}}$$

✓

(IIb) If you are confused by the combinatorics the same result can be obtained from:

$$\langle 0 | e^{i\int d^4x \mathcal{L}_I} a_1^\dagger a_2^\dagger a_3^\dagger b_4^\dagger b_5^\dagger b_6^\dagger |0\rangle$$

$$= \cdots + \frac{1}{2} \langle 0 | \left(i S_{\phi^4}\, i S_E + i S_E\, i S_{\phi^4} \right) \times a_1^\dagger a_2^\dagger a_3^\dagger b_4^\dagger b_5^\dagger b_6^\dagger |0\rangle$$

with $S_{\phi^4} = \int d^4x \left[-\frac{\lambda (\phi^* \phi)^2}{4} \right]$, S_E as given, and ϕ are the quantised ϕ_I given in the introduction.

It's quite tedious but remember we are only interested in the $\frac{q^2}{q^2-m^2}$ terms, which we take as

$$\frac{q^2}{q^2-m^2} = \frac{q^2-m^2}{q^2-m^2} + \frac{m^2}{q^2-m^2} = 1 + \text{non local}$$

What have we learned? The two theories are the same & there is no need to compute twice, I can just choose one.
In our cases we defined our basis with S_6 so we say

$$S_{EFT} + S_E \sim S_4 + \int d^4x \; \frac{c_6 - c_E \lambda}{\Lambda^2} (\phi^* \phi)^3$$

$$+ S_g + S_s$$

But an unknown plus an unknown is an unknown is an unknown so there's no need to use 2 separate variables, define

$$c_6' = c_6 - c_E \lambda$$

and we are back in S_{EFT}.

B) Unitarity

As we saw for 1d scattering, the EFT will signal its limit if one knows where to look.

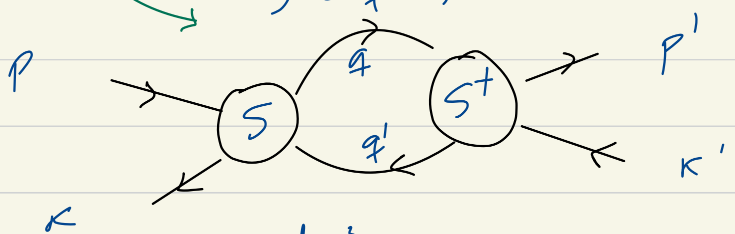
Consider $\phi(p) + \phi^*(k) \to \phi(p') + \phi^*(k')$ scattering and the exact S matrix element

$$S_{2\to 2} = (2\pi)^3 2E_p 2E_k \delta^3(p-p')\delta^3(k-k')$$
$$- i \mathcal{M} (2\pi)^4 \delta^4(p+k-p'-k')$$

Unitarity demands

$$S S^\dagger = (2\pi)^3 2E_p 2E_k \delta^3(p-p')\delta^3(k-k')$$

$$\int \frac{d^3q}{2E_q (2\pi)^3}$$



$$\int \frac{d^3q'}{2E_{q'} (2\pi)^3}$$

(III.c) If you haven't already you can show that if $M = M_0$ with M_0 scattering-angle θ independent and neglecting the mass of ϕ

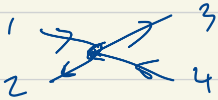
$$2\,\mathrm{Im}(M_0) + \frac{|M_0|^2}{8\pi} = 0$$

which you can show, implies

$$\left|\mathrm{Re}\,|M_0|\right| \le 8\pi .$$

We do have a few operators that would be subject to this bound. Take

$$\frac{c_s}{\Lambda^2}\,\phi^*\phi\,\partial^2\phi^*\phi$$



$$\Big\downarrow$$

$$-iM = -i\frac{c_s}{\Lambda^2}\Big[(p_1+p_2)^2 + (p_3+p_4)^2$$

$$+ (p_1-p_3)^2 + (p_2-p_4)^2\Big]$$

that is $\mathcal{M} = 2c_s \dfrac{s+t}{\Lambda^2}$.

This can be expanded in partial waves each with a bound; the one we know about is

$$\mathcal{M}_0 = \frac{1}{2}\int s\theta\, d\theta\, \mathcal{M}\, P_0(\theta) = \frac{1}{2}\int s\theta\, d\theta\, \mathcal{M}$$

You can show that this leads to

$$\left|\frac{s\, c_s}{\Lambda^2}\right| \leq 8\pi .$$

One can estimate the scale at which unitarity is not respected & we expect new physics as

$$E_{UV} \leq \sqrt{\frac{8\pi}{c_s}}\,\Lambda$$

A similar bound applies to Fermi's Theory or even to the SM without the Higgs scalar. In analogy we have

$$E_{UV} \leq \sqrt{8\pi\, G_F^{-1}} = \sqrt{8\sqrt{2}\,\pi}\; v$$

which will help you understand why the LHC was said to have a guaranteed discovery.

For the case of a model we know both sides of the equation & we can check for consistency.

(IIIc) Take the singlet scalar model & substitute $E_{UV} = M_S$ & the matching condition. Is the resulting inequality consistent for all κ?