L4 Deep in the EFT
Renormalisability

The past lectures aimed at showing that, keeping all terms to a given order, EFT can describe any model & in finding all terms we can use field redefinitions to reduce their number. In this lecture we leave the margins & models behind to dive into the EFT sea.

What we call EFT now used to get a bad reputation before Wilson came along. It was thought that they were not predictive since at the quantum level they required infinitely many parameters. The key realisation is that at a given order in $1/\Lambda^2$ we only need a finite number of parameters & the theory is predictive.

The origin of this discussion is ultraviolet (or local in space) divergences and renormalisation so here we will look at this problem at the first non-trivial order: one loop but still to first order in $1/\Lambda^2$.

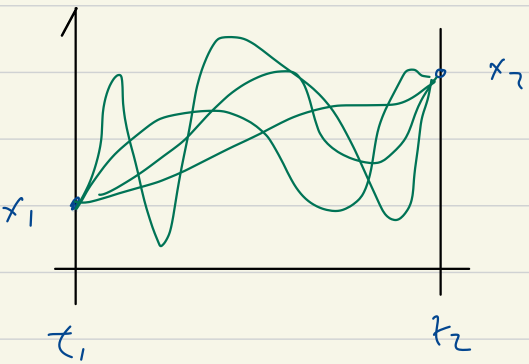
I assume the reader knows how to write down loop diagrams using Feynman diagrams but I want to use both that & another method to compute corrections. This is the functional method and I'll give a quick & unrigorous introduction first.

A) Loop corrections with path integral

The QM amplitude for a particle to go from x_1 at t_1 to x_2 at t_2 is

$$\langle x_2(t) | x_1(t) \rangle = \int_{x_1(t)}^{x_2(t)} Dx(t)\, e^{i \int dt\, L(x)}$$

The sum with a weighting phase over all paths



A sum which we cannot do in general, but if the system is not too quantum we can use an approximation around the classical path $x = x_{EoM} + \delta x$

$$\int D(\delta x)\, Exp\left(i S(x_{EoM}) + (\delta x)^2 \delta^2 S(x_{EoM}) + \mathcal{O}(\delta x^3)\right)$$

the first term goes outside the integral & the second is like a Gaussian integral.

In fact it is a Gaussian integral in Euclidean $i\,dt = -d\tau_E$ & I remind you

$$\int dx\, e^{-\frac{\mathcal{O}x^2}{2}} \equiv \int dx\, e^{-x^2/2\sigma^2} = \sqrt{2\pi}\;\sigma = \sqrt{\frac{2\pi}{\mathcal{O}}}$$

or, for N variables:

eigenvalues

$$\int d\vec{x}\; e^{-\frac{\vec{x}\,\mathcal{O}\,\vec{x}}{2}} = \int d\vec{x}'\; e^{-x_i^2\mathcal{O}_i/2}$$

$$= \prod_i \sqrt{\frac{2\pi}{\mathcal{O}_i}} = \left(\sqrt{2\pi}\right)^N e^{-\frac{1}{2}\sum_i \log \mathcal{O}_i}$$

$$= \left(\sqrt{2\pi}\right)^N e^{-\frac{1}{2}\mathrm{Tr}\left(\log(\mathcal{O})\right)}$$

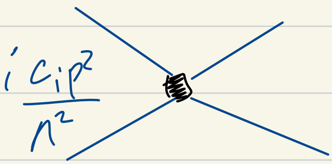
Translating this to field theory

$$i\,S_{1\,\mathrm{loop}} = -\frac{1}{2}\mathrm{Tr}\left[\log\left(-\delta^2 S\right)\right]$$

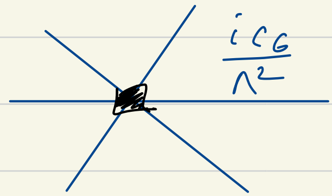
we'll use this momentarily.

B) Finite number of divergences

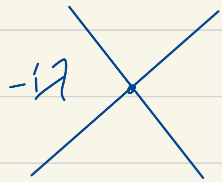
Let's first see diagrammatically that there is a finite # of divergent terms. Because we work @ $1/\Lambda^2$ we only have one vertex



or



but a priori however many



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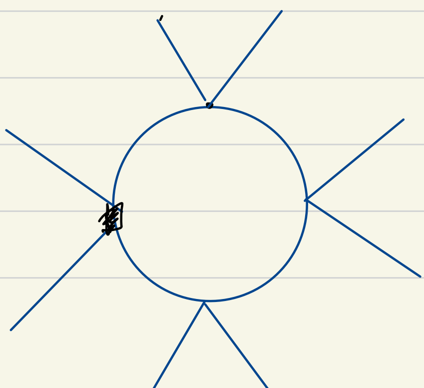
Propagators will bring $\dfrac{1}{\ell^2 - M^2}$

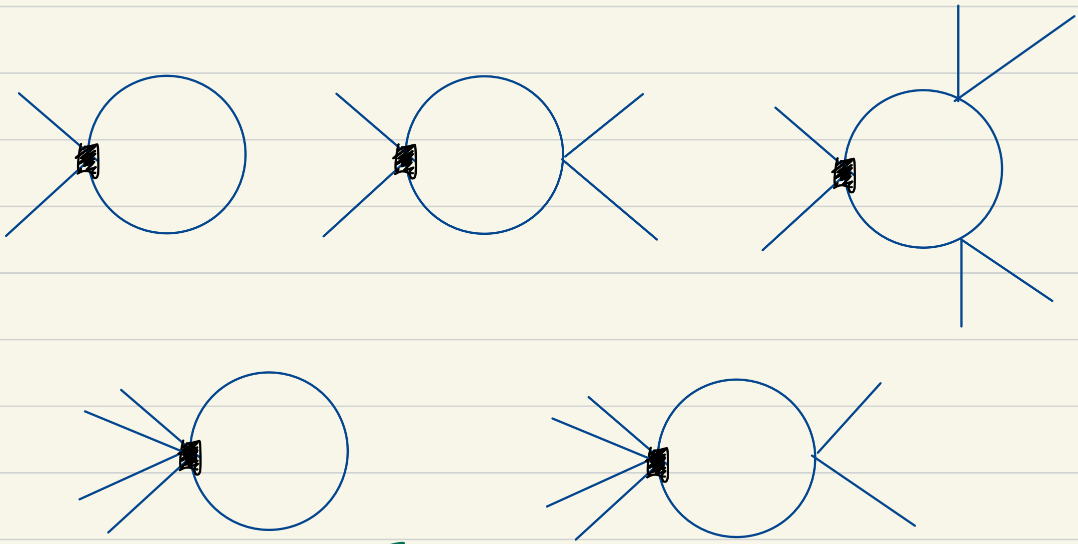
while c_g, c_6 insertions can bring as much as ℓ^2/Λ^2. Consider the diagram



does it diverge as $\ell \to \infty$?

No it doesn't $\int d^4\ell \, \frac{\ell^2}{\Lambda^2} \frac{1}{(\ell^2 - m^2)^4} \xrightarrow[\ell \to \infty]{} 0$

You can convince yourselves that more external legs will give even more convergent terms so the only diagrams to consider for renormalisation are:

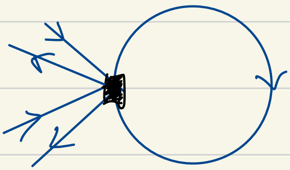


(IV.a) show that for c_6 these are the only two UV divergent diagrams

c) C_6 proportional UV divergences

Let's start with one of the simplest cases & track divergences induced by c_6.

$$-i\mathcal{M}^4_{c_6}$$



$$= \int [dl]\, 3!\,3!\, \frac{i c_6}{\Lambda^2} \frac{i}{l^2 - m^2}$$

with dim-reg and $\overline{MS}$ you can show

$$-i\mathcal{M}^4_{c_6} = -i \frac{3!^2 c_6}{\Lambda^2} \frac{m^2}{(4\pi)^2}\left(\frac{1}{\varepsilon} + \log\left(\frac{\mu^2}{m^2}\right)\right)$$

There's a divergence (oh horror 😮) but we should remember there's a tree level contribution

$$-i\mathcal{M} = -i\lambda - i\frac{c_6 m^2 (3!)^2}{(4\pi)^2 \Lambda^2}\left(\frac{1}{\varepsilon} + \log\left(\frac{\mu^2}{m^2}\right)\right)$$

and we can cancel the divergence by

$$\lambda = \lambda_R - (3!)^2 \frac{m^2 c_6}{\Lambda^2} \frac{1}{(4\pi)^2} \frac{1}{\varepsilon}$$

while introducing a logarithmic dependence in $2 \to 2$ scattering

$$-i\mathcal{M} = -i\lambda_R - i \frac{c_6 m^2 (3!)^2}{(4\pi)^2 \Lambda^2} \left(\log\left(\frac{\mu^2}{m^2} \right) \right).$$

Which is captured by the effective action

$$\Gamma_{eff} \subset \int d^4x \left(-\frac{1}{4}\right) \left[\lambda_R + \frac{c_6 m^2 (3!)^2}{(4\pi)^2 \Lambda^2} \left(\log\left(\frac{\mu^2}{m^2} \right) \right) \right] (\phi^* \phi)^2$$

You might not be familiar with Γ_{eff}, it is built to capture the local effects including loops & in fact we have a formula for it.

$$e^{i\Gamma_{eff}} = e^{iS[\phi] - \frac{1}{2} \operatorname{tr}\left[\log\left(-\delta^2 S[\phi]\right)\right]}$$

Did we get those factorials right? Let's do it again with functional methods.

We'll have to make that trace more precise, $\delta^2 S$ is a 2-pt function:

$$\frac{\delta^2 S}{\delta\phi(x)\,\delta\phi(y)} \equiv \mathcal{O}_{xy} \quad \text{so} \quad \mathrm{Tr}(\mathcal{O}) = \int d^4x\, \mathcal{O}_{xx}$$

but $\mathcal{O}$ also contains open derivatives from $\partial_\mu \delta\phi$ what happens to those?

$$\int d^4x\, \mathcal{O}_{xx} = \int \mathrm{Tr}\left(|y\rangle\langle y| \mathcal{O} |x\rangle\langle x| \right) dx\,dy$$

$$= \int \mathrm{Tr}\left(|y\rangle\, \mathcal{O}_{yx} \langle x| \right) dx\,dy$$

$$= \int \mathrm{Tr}\left(|q\rangle\langle q|y\rangle\, \mathcal{O}_{yx}(\partial) \langle x|\ell\rangle\langle \ell| \right) dx\,d\ell\,dq\,dy$$

$$= \int \mathrm{Tr}\left(|q\rangle e^{-iqy}\, \mathcal{O}_{xy}(\partial)\, e^{ixp} \langle \ell| \right) dx\,d\ell\,dq\,dy$$

$$= \int \mathrm{Tr}\left(|q\rangle e^{ix\ell - qy}\, \mathcal{O}_{xy}(\partial + i\ell) \langle \ell| \right) dx\,d\ell\,dq\,dy$$

$$= \int [d\ell] \int d^4x\, e^{i\ell(x-y)}\, \mathcal{O}_{xy}(\partial + i\ell)\, dx\,d\ell\,dy\, \cancel{dq}\, \cancel{\langle q|\ell\rangle}$$

Luckily the term we are thinking of has no derivatives

$$iS_{1\text{loop}} = -\frac{1}{2}\,\mathrm{Tr}\left(\log\left(M^2+(\partial-i\ell)^2 - s^2 S_6\right)\right)$$

$$= -\frac{1}{2}\,\mathrm{Tr}\left[\log\left(1+\frac{-s^2 S_6}{m^2+(\partial-i\ell)^2}\right)\right] + \text{field independent}$$

$$= -\frac{1}{2}\,\mathrm{Tr}\left(-\, s^2 S_6\,\frac{1}{m^2+(\cancel{\partial}-i\ell)^2}\right)$$

Let's break it down.

$$S_6 = \int d^4x\,\frac{c_6}{\Lambda^2}(\phi^*\phi)^3 = \int d^4x\,\frac{c_6}{8\Lambda^2}(\varphi^T\varphi)^3$$

So that

$$\frac{\delta}{\delta\varphi^i(x)}\frac{\delta}{\delta\varphi^j(y)}\int d^4z\,(\varphi^T\varphi)^3$$

$$= \frac{\delta}{\delta\varphi^i(x)}\,6\int \cancel{dz}\,\delta(\cancel{z-}y)\,\varphi_j(z)\,(\varphi^T\varphi)^2$$

Cancels $\int dy$ integration

$$= 6\left(\delta_{ij}(\varphi^T\varphi) + 4\varphi_i\varphi_j\right)(\varphi^T\varphi)\,\delta(x-y)$$

$$= 6\left(\mathbb{I}(\varphi^T\varphi)+4\varphi\varphi^T\right)(\varphi^T\varphi)\,\delta(x-y)$$ Matrix form

So putting it back

$$iS_{loop} = \frac{1}{2}\,\frac{c_6}{8\Lambda^2}\int \underbrace{\frac{(d\ell)}{m^2-\ell^2}}_{-1/(\ell^2-m^2)} \int d^4x \;\; 6\times 6\;(\varphi^T\varphi)^2 + \dots$$

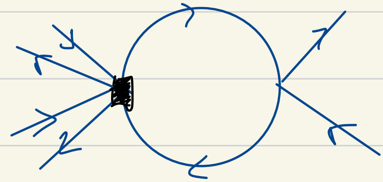
$$= i\int d^4x\left[-\frac{m^2 c_6}{(4\pi)^2\Lambda^2}\,\frac{(3!)^2}{4}\left(\frac{1}{\varepsilon}+\log\left(\frac{\mu^2}{m^2}\right)\right)\overset{(\phi^*\phi)^2}{\boxed{\frac{(\varphi^T\varphi)^2}{4}}}\right]$$

which again can be made finite by the relation given for λ, λ_R.

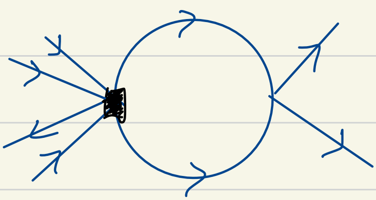
So both methods give the same answer, as they should, but it is in the spirit of these lectures to show it explicitly

We will not dwell on this effect since there is a correction of more relevance to c_6 itself:

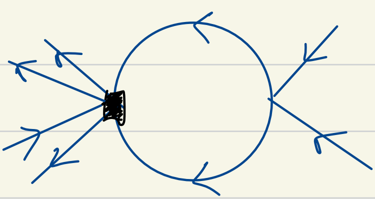
$$\frac{i c_6}{\Lambda^2}\,\frac{-i\lambda}{4}\int \frac{[d\ell]\; i^2}{(\ell^2-m^2)\left((\ell+p_{1,1'})^2-m^2\right)} \times \Bigg[$$



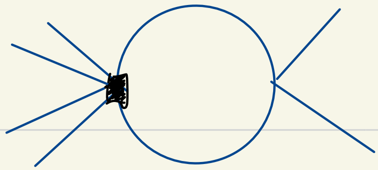
$$\underbrace{3\cdot 2\;\; 3\cdot 2}_{\text{internal}}\;\;\underbrace{3!\;\; 3!}_{\text{external}}$$



$$+\ (3\cdot 2)\;\; 3!\;\; 3!$$



$$+\ (3\cdot 2)\;\; 3!\;\; 3!\ \Bigg]$$

The result 

$$-\frac{2\,(3!)^3\, i\, c_6\, \lambda}{(4\pi)^2\,\Lambda^2}\left(\frac{1}{\varepsilon}+\log\left(\frac{\mu^2}{K^2}\right)\right)$$

translated to Γ_{1PI}

$$\int d^4x\,\frac{c_6}{\Lambda^2}\left(1-\frac{2(3!)\,\lambda}{(4\pi)^2}\left[\frac{1}{\varepsilon}+\log\left(\frac{\mu^2}{K^2}\right)\right]\right)\frac{(\varphi^T\varphi)^3}{8}$$

so

$$c_6 = c_6^R\left(1+\frac{2(3!)\,\lambda}{(4\pi)^2}\,\frac{1}{\varepsilon}\right)$$

will cancel the divergence.

Here is an exercise for you to do and check one of the results in the literature 1308.2627.

(IV b) Reproduce the result above with functional methods but with φ having N components $\varphi=\begin{pmatrix}\varphi_1\\ \varphi_2\\ \vdots\\ \varphi_N\end{pmatrix}$

Start from the expression

$$-\frac{1}{2}\,\mathrm{Tr}\left(\log\left(1+\frac{1}{m^2-(\ell-i\partial)^2}\left(-\delta^2 S_6-\delta^2 S_{\phi 4}\right)\right)\right)$$

expanding $\log(1+x) = 1+x-\frac{x^2}{2}$ we get

$$=+\frac{1}{4}\int d^4x\,[d\ell]\left(2\cdot\frac{1}{m^2-(\ell-i\partial)^2}\,\delta^2 S_6\,\frac{1}{m^2-(\ell-i\partial)^2}\,\delta^2 S_{\phi 4}\right)\Bigg)+\ldots$$

Breaking it down into:

- trace of operators; we already derived

$$\delta^2 S_6 = \frac{c_6}{8\Lambda^2}\,6\left(\mathbb{I}(\varphi^T\varphi)+4\varphi\varphi^T\right)(\varphi^T\varphi)$$

$$\delta^2 S_{\phi 4} = -\frac{\lambda}{4}\,\frac{1}{4}\,\delta^2(\varphi^T\varphi)^2 = \frac{-\lambda}{16}\,4\left(\mathbb{I}\,\varphi^T\varphi+2\varphi\varphi^T\right)$$

Show that then the trace:

$$\mathrm{Tr}\left((+\delta^2 S_6)(+\delta^2 S_{\phi 4})\right) = -\frac{c_6\lambda}{8\Lambda^2 16}\,\mathrm{tr}\left(+\delta^2(\varphi^T\varphi)^3\,\delta^2(\varphi^T\varphi)^2\right)$$

$$= -\frac{c_6\lambda}{8\Lambda^2 16}\,24(\varphi^T\varphi)\,\underbrace{\mathrm{tr}\left(\left(4\varphi\varphi^T+\mathbb{I}(\varphi^T\varphi)\right)\left(2\varphi\varphi^T+\mathbb{I}(\varphi^T\varphi)\right)\right)}_{(8+4+2+N)(\varphi^T\varphi)^2}$$

- loop integral. We can do the substitution

$$\frac{1}{m^2-(\ell-i\partial)^2}\,\mathcal{O}\quad\frac{1}{m^2-(\ell-i\partial)^2}\,\mathcal{O}'$$

$$=\frac{\mathcal{O}\mathcal{O}'}{(\ell^2-\kappa^2)^2}$$

The rest is a loop integral

$$\int[d\ell]\,\frac{1}{(\ell^2-\kappa^2)^2}=\frac{i}{(4\pi)^2}\left[\frac{1}{\varepsilon}+\log\left(\frac{m^2}{\kappa^2}\right)\right]$$

To arrive at a result

$$i\Gamma_{\text{eff}}\supset iS_6-\frac{1}{2}\operatorname{Tr}\left[\log\left(-\delta^2S\right)\right]$$

$$\supset i\int d^4x\,\frac{c_6}{8\Lambda^2}\left(1-\frac{\gamma}{2}c_6\left(\frac{1}{\varepsilon}+\log\left(\frac{m^2}{\kappa^2}\right)\right)\right)$$

With γ a function of λ, N i.e. $\gamma(\lambda, N)$.

From 1308.2627 we get, identifying φ with the Higgs doublet $\gamma=\frac{108}{4}\lambda$

- Does it agree with your result?
- For which value of N?