L5 Experimental data and Anomalous dimensions

A) Experimental data

Too much calculating indoors is not recommended by doctors so let's go out & compare with experiment.

Say an experimentalist deduced the value for the local term in the 6-point amplitude by measuring the differential cross section

$$\phi^* + \phi \to \phi^* + \phi + \phi^* + \phi$$

to find a value $\mathcal{M}_{local}$ and the energy at which their experiment operated is K_{IR}.

Then our computation gives

$$\mathcal{M}_{local}(K_{IR}) = \frac{(3!)^2 c_6^R}{\Lambda^2}\left[1 - \frac{\gamma}{2}\log\left(\frac{\mu^2}{K_{IR}^2}\right)\right]$$

where $\gamma = \frac{4(3!)\lambda}{(4\pi)^2}$ as we found out.

We want to write predictions in term of observed parameters so define

$$\frac{C_6^{IR}}{\Lambda^2} \equiv \frac{M_{local}(\kappa_{IR})}{(3!)^2} = \frac{C_6^{\kappa}}{\Lambda^2}\left(1 - \frac{\gamma}{2}\log\left(\frac{m^2}{\kappa_{IR}^2}\right)\right)$$

Now use C_6^{IR} to write the amplitude for any κ

$$M_{local}(\kappa) = \frac{C_6^{IR}}{\Lambda^2}\frac{\left(1 - \gamma/2\log\left(m^2/\kappa^2\right)\right)}{\left(1 - \gamma/2\log\left(m^2/\kappa_{IR}^2\right)\right)}$$

$$\simeq \frac{C_6^{IR}}{\Lambda^2}\left(1 + \frac{\gamma}{2}\log\left(\frac{\kappa^2}{\kappa_{IR}^2}\right) + O(\gamma^2)\right)$$

So given $\gamma > 0$ we have that if we repeat the experiment @ higher energies the effective coupling increases!

B) Anomalous dimensions

The example we looked at showed that after making contact with experiment the effect is a logarithmic variation with the scale of your experiment & μ disappears

The un-physicalness of μ can be further exploited to obtain the renormalisation group evolution.

Let's revisit the example & demand that the final result have no μ dependence

$$\mu \frac{d}{d\mu} \left(\frac{c_6}{\Lambda^2} \left(1 - \gamma \log\left(\frac{\mu}{\kappa}\right)\right)\right) = 0$$

$$\mu \frac{d}{d\mu} c_6 = \frac{c_6 \gamma}{\left(1 - \gamma \cancel{\log\left(\mu/\kappa\right)}\right)} = \gamma c_6$$

subleading

We are in a loop expansion, so solving the equation perturbatively

$$c_6 \simeq \int_{\mu_0}^{\mu} \gamma c_6^0(\mu_0) \frac{d\mu'}{\mu'} = c_6(\mu_0) + \gamma c_6(\mu_0) \log\left(\frac{\mu}{\mu_0}\right)$$

whose effect on the observable

$$c_6(\mu_0) + \gamma c_6^0 \log\left(\frac{K}{\mu_0}\right) = \int_{\mu_0}^{K} dc_6(\mu)$$

So we can obtain the physical result if we turn μ into a dummy variable integrating between two scales we observe.

So far it was all equivalent to 1 loop, but now we can integrate without expanding on γ

$$\int \frac{dc_6}{c_6} = \gamma \int \frac{d\mu}{\mu}$$

Let's use now the matching value of c_6 in the UV as an input

$$\int_{c_6^{UV}}^{c^6} \frac{dc_6'}{c_6'} = \gamma \int_{\Lambda}^{\kappa} \frac{d\mu}{\mu} \Rightarrow c_6(\kappa) = c_6^{UV}\left(\frac{\kappa}{\Lambda}\right)^{\gamma}$$

So that the prediction for e.g. the ~~AdS~~-CFT model is

$$\mu_{local} = \frac{(3!)^2 (\kappa)^{\gamma}}{\Lambda^2 \Lambda^{\gamma}} \frac{a_4(\alpha)\, \lambda^2 \beta}{4 \nu(\nu+1)}$$

Can you now see why γ is referred to as anomalous dimension?

In our perturbative theory γ is small so the dimension is not far from 2. Strongly coupled systems can have scaling far from the usual integer one. An example is the AdS/CFT completion which in the limit $\alpha \to \infty$ gives

$$d_4 \to \frac{-1}{8\nu(\nu+1)}\left(\frac{\Lambda}{\Lambda_P}\right)^{2\nu} \quad \text{with} \quad \nu > 0$$

but otherwise arbitrary.

(V a) We have approximated γ to a constant, which is not quite right since $\lambda(\mu)$ has its own RGE eq

$$\mu \frac{d\lambda}{d\mu} = b_0 \lambda^2 \quad \text{with } b_0 \text{ constant}$$

Use the chain rule to write

$$\mu \frac{dc_6}{d\mu} = \mu \frac{d\lambda}{d\mu}\frac{dc_6}{d\lambda}$$

substitute the RGE for λ and solve for initial conditions

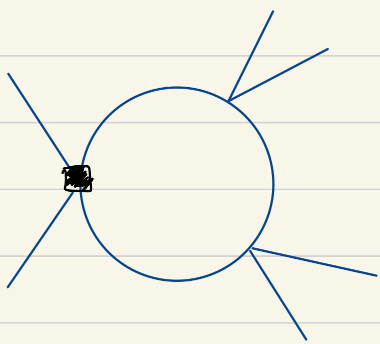
$$c_6(\lambda_{UV}) = c_6^{UV}$$

Extra $b_0 = 5/(4\pi)^2$ for 2 fields, reproduce this result.

c) Operator mixing

We have considered only divergences proportional to c_6. What about the others? They would also require renormalisation of their own constants but they also bring a new feature, mixing.

An example we can see diagrammatically is



Since it will contribute to $(\phi^*\phi)^3$ but it comes from c_5 or c_5 so it will look like

$$\mathcal{L}_{eff} \subset \int d^4x \left(\frac{c_5 \#}{(4\pi)^2} (\phi^*\phi)^3 \right)$$

That is it requires the renormalisation of c_6 proportional to c_5.

$$c_6 = c_6^R + \# c_5 \frac{1}{\varepsilon}$$

By now we know to do better than put a # symbol, we can compute.

One can show that c_8 will not contribute which I leave as an exercise.

For c_5 we need the second order variation:

$$(\delta\varphi)^2 \delta^2(\varphi^T\varphi)\partial^2\varphi^T\varphi = 2\,\delta\varphi^T\delta\varphi\,\partial^2\varphi^T\varphi + 2\,\varphi^T\varphi\,\partial^2\delta\varphi^T\delta\varphi$$
$$+ 8\,\delta\varphi^T\varphi\,\partial^2\varphi\,\delta\varphi^T$$
$$= \delta\varphi\left[4\left[\partial^2(\varphi^T\varphi)\right]\mathbb{1} + 8\,\varphi\,\overleftarrow{\partial}^2\varphi^T\right]\delta\varphi$$

Next we need the third term in

$$\log(1+x) = 1 + x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4)$$

To put it in

$iS_{1\text{-loop}}$

$$C - \frac{1}{2}\frac{1}{3}\mathrm{Tr}\left[\frac{-\delta^2 S_5}{m^2-(\ell-i\partial)^2}\,\frac{-\delta^2 S_{\phi^4}}{m^2-(\ell-i\partial)^2}\,\frac{-\delta^2 S_{\phi^4}}{m^2-(\ell-i\partial)^2}\right] +$$

$$+ \quad S_{\phi^4} \quad S_5 \quad S_{\phi^4}$$

$$S_{\phi^4} \quad S_{\phi^4} \quad S_5$$

$$= +\frac{1}{2}\left(\frac{\lambda}{16}\right)^2\left(\frac{c_5}{4\Lambda^2}\right)$$

$$\cdot \int d^4x\,[dl]\, \mathrm{tr}\Big[4\left(2\varphi\varphi^T + \mathbb{I}\,\varphi^T\varphi\right) 4\left(2\varphi\varphi^T + \mathbb{I}\,\varphi^T\varphi\right)$$

$$\left(4\,\mathbb{I}\left(\partial^2\varphi^T\varphi\right) + 8\,\varphi(\partial - il)^2\varphi^T\right)\Big]\frac{1}{(m^2-l^2)^3}$$

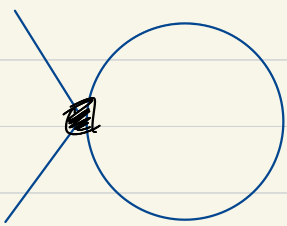
$$= \frac{1}{2}\left(\frac{\lambda}{4}\right)^2\left(\frac{c_5}{4\Lambda^2}\right)\int d^4x\,\frac{i}{(4\pi)^2}\left(\frac{1}{\varepsilon} + \log\left(\frac{\mu^2}{k^2}\right)\right)$$

$$\cdot\, 8\varphi^T\varphi\, \mathrm{tr}\left[\left((4+2)\,\varphi\varphi^T + \mathbb{I}\,\varphi^T\varphi\right)\varphi\varphi^T\right]$$

$$= i\int d^4x\, \underbrace{\frac{\lambda^2}{2}\,\frac{(6+N)}{(4\pi)^2}}_{\equiv\, -\frac{\bar{\gamma}_{65}}{2}}\,\frac{c_5}{\Lambda^2}\,\frac{(\varphi^T\varphi)^3}{8}\left[\frac{1}{\varepsilon} + \log\left(\frac{\mu^2}{k^2}\right)\right]$$

This will contribute to $c_5 - c_6$ mixing & implies that even if c_5 only was produced in the UV, by the time we get to the IR, c_6 would appear.

This is not quite the full story because we have two diagrams left. Let's not compute them but look at their structure



$$\sim \int \frac{p^2+\ell^2}{(\ell^2-m^2)}[d\ell]$$

$$\sim \frac{1}{(4\pi)^2}\frac{1}{\Lambda^2}\left(p^2m^2+m^4\right)$$

These would renormalise fields & mass. Field we take of by $\phi=\sqrt{Z}\,\phi_R$ with

$$Z = 1+\frac{\#\ m^2}{(4\pi)^2\Lambda^2}$$

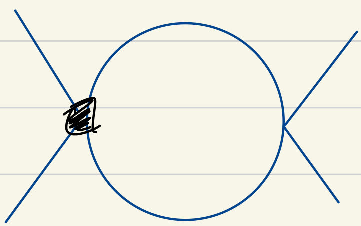
which would affect Z running at the same order as the very first contribution we computed proportional to c_6.

For the running of the c_i

coefficients themselves this however is a second order effect

$$\frac{c}{\Lambda^2}(\phi^*\phi)^2 = \frac{c}{\Lambda^2} Z^2 (\phi_R^*\phi_R)^2 = \frac{c}{\Lambda^2}(\phi_R^*\phi_R)^2 + O(\Lambda^{-4})$$

This leaves us with



$$\sim \int \frac{l^2+p^2}{(l^2+\kappa^2)^2}[dl]$$

$$\frac{1}{(4\pi)^4}(p^2+\kappa^2)$$

where now we cannot neglect external momentum in the denominator (called κ to distinguish from numerator p)

The last point I want to make is to do with these contributions & mixing.

The outcome of the loop computation need not be in the form of the operators of our basis, in particular

$$\frac{\#}{(4\pi)^2} c_5 \; (\phi^*\phi)(\phi^* \Box \phi) + \text{h.c.}$$

might show up & then we have to use field-redefinitions again!

These will, as before, produce c_6 and "secretly" give us more mixing

$$\mu \frac{d}{d\mu} \begin{pmatrix} c_6 \\ c_5 \\ c_g \end{pmatrix} = \begin{pmatrix} \gamma_{66} & \gamma_{65} + \gamma_{65}^{EoM} & 0 + \gamma_{6g}^{EoM} \\ 0 & \gamma_{55} & \gamma_{5g} \\ 0 & \gamma_{g5} & \gamma_{gg} \end{pmatrix} \begin{pmatrix} c_6 \\ c_5 \\ c_g \end{pmatrix}$$

It is only the final sum that is physical. One would expect any operators to mix unless there's a symmetry reason, e.g. if ϕ is a true Goldstone, $m, \lambda, c_5, c_6 \to 0$

and the symmetry guarantees they won't appear at loop order. Other times however one finds unexpected cancellations, see 1409.0868. These were explained with amplitude methods & helicity selection rules unveiling new properties of QFTs 1505.01844.

Let us summarise the procedure then to connect the UV (Λ) and the IR (k) when $\Lambda \gg k$.

- We match at tree level the UV & EFT to find the operators coefficients @ Λ, c^{UV}

- We solve the RGE with initial condition $c(\mu=\Lambda) = c^{UV}$ to evolve down to the scale of our experiment.

Precision can be improved systematically, next would be matching at 1 loop (setting $\mu = \Lambda$ in our diagrams) then evolving the 2-loop RGE which can be computed in the EFT etc.

There is also the case of multiple Λ_i's in the S.M. we have M_W, m_b, m_c to "cross" if we're going to do Kaon physics let's say; at each energy regime we repeat the procedure

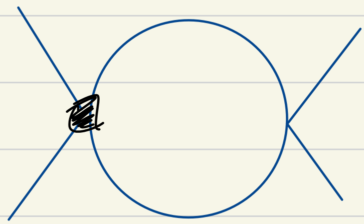
Summary

The aim of these lectures was to give you problems to teach yourselves about key features of EFTs, so it is not up to me to write the summary.

Go ahead & write here whatever you learned that you didn't know

Final Problem

Compute the c_5 column of the anomalous dimension matrix by integrating



Either with diagrams or as sketched here with functional methods.

$$i\Gamma_{eff} \subset -\frac{1}{2}\int d^4x\,[d\ell]\,(-1)\,\mathrm{tr}\left(\left(-\delta^2 S_{\phi^4}\right)\frac{1}{m^2-(\ell-i\partial)^2}\left(-\delta^2 S_{c_5}\right)\frac{1}{m^2-(\ell-i\partial)^2}\right)$$

$$= +\frac{1}{2}\int [d\ell][dp_1][dp_2][dp_3][dp_4]\,(2\pi)^4\delta^4(p_1+p_2+p_3+p_4)$$

$$\mathrm{tr}\left[\left(-\frac{\lambda}{16}\right)4\left(\mathbb{1}\,\tilde{\varphi}^T(p_1)\tilde{\varphi}(p_2)+2\,\tilde{\varphi}(p_1)\tilde{\varphi}^T(p_2)\right)\frac{1}{(\ell-p_3-p_4)^2-m^2}\right.$$

$$\left.\cdot\left(-\frac{c_5}{4\Lambda^2}\right)\frac{\left[4\left[(p_3+p_4)^2\left(\tilde{\varphi}^T(p_3)\tilde{\varphi}(p_4)\right)\right]\mathbb{1}+8\,\tilde{\varphi}(p_3)(\ell+p_4)^2\tilde{\varphi}^T(p_4)\right]}{\ell^2-m^2}\right]$$

where now we cannot neglect the p's in the denominator & we have gone to Fourier transformed fields.

The roadmap.

- Use Feynman parameters to write the denominator as

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{(xA + (1-x)B)^2}$$

- Shift the loop momentum to leave a denominator as

$$\int [d\ell'] \frac{\ell'^2 , \ell' \cdot p , p^2}{(\ell'^2 - \Delta)^2}$$

- Do the loop integrals & keep only the $\frac{1}{\varepsilon}$ + $\log \mu^2$ terms
- Do the x integral
- Go back to $\int d^4x \, \mathcal{L}(x)$ representation

and identify the operators generated

- If $\varphi^T\varphi\,\varphi^T(D\varphi)$ is produced use the EoM to remove it in favor of S_6

- Collect all your results to extract γ_{is}

The End.

Unless you want to do it again for S_8!