Bounding self-dual L-functions: the Conrey-Iwaniec method revisited

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a joint work with
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L-functions, circle method and applications
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Content

- The subconvexity problem
- Pioneering work by Conrey–Iwaniec
- Xiaoqing Li's results revisited
- q-aspect analog: a result of Blomer's

Maass forms

Let $\mathbb{H} = \{z = x + iy : y > 0\}$. $\gamma \in \mathrm{SL}_2(\mathbb{R})$ acts on $z \in \mathbb{H}$ by Möbius transformation $\gamma \cdot z = \frac{az+b}{cz+d}$. A Maass cusp form f for $\mathrm{SL}_2(\mathbb{Z})$ is a non-zero function $f \in L^2(\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H})$, vanishes at the cusp ∞, and is an eigenfunction of the Laplace operator $\Delta = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$. It admits Fourier expansion

$$f(z) = \sum_{n \neq 0} \rho_f(n) \sqrt{y} K_{it_f}(2\pi|n|y) e(nx).$$

Assume f is an eigenfunction of all the Hecke operators T_n, $\rho_f(\pm n) = \rho_f(\pm 1)\lambda_f(n)$. For f a Hecke–Maass cusp form we define

$$L(f,s) = \sum_{n \geq 1} \frac{\lambda_f(n)}{n^s} \quad (\mathrm{Re}(s) > 1).$$

L-functions

Given F an automorphic form on $\mathrm{GL_d}$, let $\lambda_F(n)$ be its associated Hecke eigenvalues. Let

$$L(F,s) = \sum_{n\geq 1}^{\infty} \frac{\lambda_F(n)}{n^s}, \ \Re s \gg 1$$

be the L-function attached to F. Define the completed L-function

$$\Lambda(F,s) := q_F^{\frac{s}{2}} \pi^{-\frac{ds}{2}} \prod_{j=1}^{d} \Gamma(\frac{s+\kappa_j}{2}) L(F,s),$$

where q_F is the "arithmetic conductor" of $L(F,s)$, and $\kappa_j \in \mathbb{C}$ are local parameters of $L(F,s)$ at infinity.
This satisfies a functional equation

$$\Lambda(F,s) = \varepsilon(F)\,\Lambda(\overline{F}, 1-s),$$

where $\varepsilon(F)$ (of absolute value 1) is the root number.

Analytic Conductor

Associated to $L(F, 1/2)$ one defines the "*Analytic Conductor*" (Iwaniec–Sarnak, 2000)

$$Q(F) := q_F \prod_{j=1}^{d} (1 + |s + \kappa_j|),$$

measuring the "complexity" of $L(F, 1/2)$ as F varies.

Examples:

- if $F = \chi|\cdot|^{it}$, where χ : Dirichlet character modulo q and $|\cdot|^{it} : n \to n^{it}$, then

$$Q(F) = q(1 + |t|).$$

- if F is a cusp form on $\Gamma_0(q)\backslash\mathbb{H}$ of weight k_F (if F is holomorphic) or of Laplacian eigenvalue $1/4 + k_F^2$ (if F is Maass), then

$$Q(F) = q(1 + |k_F|^2).$$

The subconvexity problem

The Phragmén–Lindelöf principle $\Rightarrow$ Convexity bound:

$$L(F, 1/2) \ll Q(F)^{1/4+o(1)}.$$

GRH $\Rightarrow$ the Generalised Lindelöf hypothesis:

$$L(F, 1/2) \ll Q(F)^{\varepsilon}.$$

The subconvexity problem: Find an $\delta > 0$ such that

$$L(F, 1/2) \ll Q(F)^{1/4-\delta}.$$

Subconvexity on GL_1

- t-aspect (i.e., $F = |\cdot|^{it}$): Weyl (1922)

$$\zeta(1/2+it) \ll (1+|t|)^{\frac{1}{6}+o(1)}.$$

  Subsequent works by many authors. Bourgain ($1/6 \to 13/84$).
- q-aspect: Burgess (1963), χ mod q Dirichlet characters

$$L(\chi, 1/2+it) \ll_t q^{3/16+o(1)};$$

  and Weyl-bound

$$L(\chi, 1/2) \ll q^{1/6+o(1)}$$

  by Conrey–Iwaniec/Petrow–Young, Nelson, Balkanova–Frolenkov–Wu, etc.

In general for $F \in \mathrm{GL_d}$, the first cases to study is for F that admits factorization, e.g.,

$$F = f \times \chi, \quad f \text{ fixed}, \chi \text{ varying}.$$

The GL_2 case: twist aspect

Let χ mod q be Dirichlet characters, f be GL_2 cusp forms, and let $L(f \times \chi, s) = \sum_{n \geq 1} \lambda_f(n)\chi(n)n^{-s}$.

Theorem (Duke–Friedlander–Iwaniec, Bykovskiĭ, ...)

Let f be fixed GL_2 automorphic forms. Then

$$L(f \times \chi, 1/2) \ll_f (q^2)^{1/4-\delta+\varepsilon}, \text{ for } \delta = 1/16.$$

- The saving $\delta = 1/16$ represents the **Burgess**-type subconvex bounds:

$$L(F, 1/2) \ll Q(F)^{1/4-1/16+\varepsilon};$$

  proving ground for new methods: Blomer–Harcos–Michel, Han Wu, Munshi, Aggarwal–Holowinsky–L.–Sun, etc.
- Best: $\delta = 1/12$ (**Weyl**-type), Conrey–Iwaniec/Petrow–Young:

$$0 \leq L(F \times \chi, 1/2), L(\chi, 1/2)^2 \ll (q^2)^{1/4-1/12+\varepsilon};$$

  extended to other number fields by Nelson, Balkanova–Frolenkov–Wu.

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A fundamental work of Conrey–Iwaniec

Let χ mod q be characters. C–I established

$$\frac{1}{q}\sum_{f_j\in\mathcal{B}(q,\mathrm{triv})} h(t_j)L(f_j\times\chi,1/2)^3+\frac{1}{q}\int_{-\infty}^{\infty}h(t)|L(\chi,1/2+it)|^6\mathrm{d}t \ll_{t_j} q^{\varepsilon}.$$

Rmk: $\mathrm{Cond}(L(f_j\times\chi,1/2)^3)=q^6$, $\mathrm{size}(\mathcal{B}(q,\mathrm{triv}))=q$,
$\Rightarrow \frac{\log(\text{conductor})}{\log(\text{family})}=6.$

Restricting to $\chi=\chi_q$ *real* and appealing to J. Guo:

$$L(f_j\times\chi_q,1/2)\geq 0,$$

C–I derived

Theorem (Conrey–Iwaniec, 2000)

Let χ_q mod q be real characters with square-free conductor q.

$$L(f_j\times\chi_q,1/2)\ll_{t_j} q^{1/3+\varepsilon},\quad L(\chi_q,1/2+it)\ll_t q^{1/6+\varepsilon}.$$

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Remarks on the Weyl bound

The bound $L(\chi_q, 1/2) \ll q^{1/6+\varepsilon}$: **first** improvement of Burgess's bound $L(\chi, 1/2) \ll q^{3/16+\varepsilon}$ (1963).

The Conrey–Iwaniec bounds were extended to

- hybrid (χ_q, t_j)-aspect by M. Young;
- **all** characters χ mod q, **all** q, over $\mathbb{Q}$ by Petrow–Young;
- Hecke characters χ over number fields K cube-free conductor, by P. Nelson and by Balkanova–Frolenkov–Wu (K totally real at the moment) independently.

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A Motohashi-type formula

A spectral identity is visible from the proof of C–I/P–Y:

$$\frac{1}{q}\sum_{f_j\in\mathcal{B}(q)} h(t_j)L(f_j\times\chi,1/2)^3+\frac{1}{q}\int_{-\infty}^{\infty} h(t)|L(\chi,1/2+it)|^6\mathrm{d}t$$

$$\rightsquigarrow MT+\frac{1}{q}\sideset{}{^\star}\sum_{\psi \bmod q} g(\psi,\chi)|L(\psi,1/2)|^4\widetilde{H}(h),$$

where

$$g(\psi,\chi)=\frac{1}{q}\sum_{u,v(q)}\chi\left(u(v+1)\right)\overline{\chi}\left(v(u+1)\right)\psi(uv-1).$$

The next move of C–I is to ignore possible sign change from the arguments of $g(\psi,\chi)$:

$$RHS \ll \|g(\psi,\chi)\|_\infty\times\frac{1}{q}\sideset{}{^\star}\sum_{\psi \bmod q}|L(\psi,1/2)|^4$$

and appeal to Deligne's RH: $\|g(\psi,\chi)\|_\infty\ll 1$ (interpreted as a "trace function" modulo q, q primes).

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Remarks on the Weyl bound: II

- To improve the Weyl-bound, one wants to detect cancellation from sign changes of ψ's in $g(\psi,\chi)$:

$$\sum_{\psi \bmod q}^{\star} g(\psi,\chi)|L(\psi,1/2)|^4 = O(q^{1-\delta}).$$

- If $g(\psi,\chi) \equiv 1$,
  M. Young (2011):

$$\sum_{\psi \bmod q}^{\star} |L(\psi,1/2)|^4 - MT = O(q^{1-\delta}).$$

  Kowalski–Michel–Sawin (2017):

$$\sum_{\psi \bmod q}^{\star} L(f\times\psi,1/2)\overline{L(g\times\psi,1/2)} - MT_{f,g} = O(q^{1-\delta}).$$

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GL_3: Xiaoqing Li's work (spectral aspect)

Recall Conrey–Iwaniec ($E_{\min} = 1 \boxplus 1 \boxplus 1$):

$$\sum_{f_j \in \mathcal{B}(q,\mathrm{triv})} h(t_j) L(f_j \times \chi, 1/2)^3 + \int_{-\infty}^{\infty} h(t) |L(\chi, 1/2 + it)|^6 \mathrm{d}t$$

$$= \sum_{f_j \in \mathcal{B}(q,\mathrm{triv})} h(t_j) L(E_{\min} \times f_j \times \chi, 1/2) + \int_{\mathbb{R}} h(t) |L(E_{\min} \times \chi, 1/2 + it)|^2 \ll q^{1+\varepsilon}.$$

Li replaced $E_{\min}$ by $F \in \mathrm{GL}_3$ cuspidal and obtained (for $q = 1$):

$$\sum_{\substack{f_j \in \mathcal{B} \\ |t_j - T| \le M}} h(t_j) L(F \times f_j, 1/2) + \int_{T-M}^{T+M} h(t) |L(F, 1/2 + it)|^2 \mathrm{d}t \ll T^{1+\varepsilon} M$$

under $M > T^{3/8}$; Lindelöf-on-average, according to Weyl's law:

$$\#\{f_j : t_j \in [T - M, T + M]\} \asymp TM.$$

By appealing to Lapid's result $L(F \times f_j, 1/2) \ge 0$,

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Li obtained

Theorem (Xiaoqing Li, 2011)

Let F be self-dual.

$$L(F \times f_j, 1/2), \, |L(F, 1/2 + it)|^2 \ll_F t^{11/8+\varepsilon} = t^{3/2-1/8+\varepsilon}.$$

- First subconvexity on GL_3; many further progresses (in the past one decade) are along this line.
- Improvements of exponent by McKee–Sun–Ye, R. Nunes, etc.

Main result

We give an improvement over Li's saving.

Theorem (L.–Ramon Nunes–Zhi Qi, 2021+)

It holds true that

$$\sum_{\substack{f_j \in \mathcal{B} \\ |t_j - T| \le M}} h(t_j) L(F \times f_j, 1/2) + \int_{T-M}^{T+M} h(t) |L(F, 1/2 + it)|^2 \mathrm{d}t$$

$$\ll T^{1+\varepsilon} M (1 + T^{1/4}/M^{5/4}).$$

Taking $M = T^{1/5}$, one has

Corollary

Let F be self-dual.

$$L(F \times f_j, 1/2),\ |L(F, 1/2 + it)|^2 \ll_F t^{6/5+\varepsilon}.$$

- The exponent 6/5 is the natural limit of this method.

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Other results: without self-duality assumption

For F *not necessarily* self-dual:

- $|L(F, 1/2 + it)|^2 \ll_F t^{3/2-\delta}$, Munshi ($\delta < 1/8$), Aggarwal ($\delta < 3/20$), Aggarwal–Leung–Munshi ($\delta < 1/4$);
- $L(F \times f_j, 1/2 + it) \ll_{F,f_j} t^{3/2-\delta}$, Munshi ($\delta < 1/51$), L.–Q. Sun ($\delta < 3/20$);
- $L(F \times f_j, 1/2) \ll_F t_j^{3/2-\delta}$, Kumar ($\delta < 1/51$);
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All proofs follow the delta-method approach pioneered by Munshi.

- P. Nelson announced a spectral-aspect subconvex bound for **all** standard L-functions on $\mathrm{GL_d}$:

$$L(F, 1/2) \ll Q(F)^{1/4-\delta}$$

away from the "conductor-dropping" case (F of level 1).

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Remarks: Motohashi-type formula

An underlying spectral identity

$$\frac{1}{TM}\sum_{|t_j-T|\le M} h(t_j)L(F\times f_j,1/2)+\frac{1}{TM}\int_{T-M}^{T+M} h(t)|L(F,1/2+it)|^2\mathrm{d}t$$

$$\leftrightsquigarrow L(F,1)\widetilde{H}(h)+\int_{-T/M}^{T/M}\widetilde{h}(t)L(F,1/2+it)\zeta(1/2-it)\mathrm{d}t,$$

obtained independently (apart from localizing support of $\widetilde{h}(t)$) by

- **Chung-Hang Kwan** (2021), by period integral approach (via Poincaré series);
- **Humphries–Khan** (forthcoming), via analytic continuation of Dirichlet series.
- Motohashi's original formula (corresp. to $F=1\boxplus 1\boxplus 1$)

$$\int_{\mathbb{R}}|\zeta(1/2+it)|^4 g(t)\mathrm{d}t$$

$$\leftrightsquigarrow \sum_{t_j\in\mathcal{B}(1)}\widetilde{g}(t_j)L(f_j,1/2)^3+\text{holo.}+\int_{\mathbb{R}}\widetilde{g}(t)|\zeta(1/2+it)|^6\mathrm{d}t+\text{linear.}$$

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Some tools

We use classical approach (approximate functional equation, Kuznetsov, Voronoï, stationary phase, Cauchy). Some tools:

- *Approximate functional equation*

$$L(F,1/2) \approx \sum_{n \ll \sqrt{Q(F)}} \frac{\lambda_F(n)}{n^{1/2}} + \varepsilon(F) \sum_{n \ll \sqrt{Q(F)}} \frac{\overline{\lambda_F(n)}}{n^{1/2}}.$$

- *Kuznetsov trace formula*

$$\sum_{t_j} h(t_j)\lambda_j(n_1)\lambda_j(n_2) + (Eis)$$

$$= \delta_{n_1,n_2} H + \sum_{\pm} \sum_{c=1}^{\infty} \frac{S(n_1, \pm n_2; c)}{c} H^{\pm}\Big(\frac{4\pi\sqrt{n_1 n_2}}{c}\Big).$$

- $\mathrm{GL_d}(\mathbb{Z})$-*Voronoï summation*:

$$\sum_{n \sim N} \frac{\lambda_F(n)}{\sqrt{n}} Kl_i(an;c) w\Big(\frac{n}{N}\Big) \approx \sum_{n' \ll \frac{c^d}{N}} \frac{\overline{\lambda_F}(n')}{\sqrt{n'}} Kl_{d-i}(\bar{a}n';c)\widehat{w}\Big(\frac{n'}{c^d/N}\Big).$$

Some tools

We use classical approach (approximate functional equation, Kuznetsov, Voronoï, stationary phase, Cauchy). Some tools:

- *Approximate functional equation*

$$L(F,1/2) \approx \sum_{n \ll \sqrt{Q(F)}} \frac{\lambda_F(n)}{n^{1/2}} + \varepsilon(F) \sum_{n \ll \sqrt{Q(F)}} \frac{\overline{\lambda_F(n)}}{n^{1/2}}.$$

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Some tools (cont.)

- Stationary phase

$$\int_{\mathbb{R}} g(x)e(\phi(x))\mathrm{d}x = \frac{e(\phi(x_0))}{\sqrt{\phi''(x_0)}} g_A(x_0) + O_A\left(T^{-A}\right).$$

Here $\phi'(x_0) = 0$.

- Cauchy–Schwarz/large sieve

$$\sum_{n\leq N}\sum_{m\leq M} a_n b_m \phi(n,m) \ll \Big(\sum_{n\leq N} |a_n|^2\Big)^{1/2}\Big(\sum_{n\leq N}\Big|\sum_{m\leq M} b_m\phi(n,m)\Big|^2\Big)^{1/2}$$

expecting variations of the argument of $\phi(n, m_1), \phi(n, m_2)$ to be independent, so that (for $m_1 \neq m_2$)

$$\sum_{n\leq N} \phi(n,m_1)\overline{\phi(n,m_2)} = o\Big(\sum_{n\leq N} |\phi(n,m_1)\overline{\phi(n,m_2)}|\Big).$$

Proof sketch

Our goal:

$$\sum_{\substack{f_j\in\mathcal{B}\\ |t_j-T|\le M}} h(t_j)L(F\times f_j,1/2)+\int_{T-M}^{T+M} h(t)|L(F,1/2+it)|^2\mathrm{d}t$$
$$= TM\,L(\overline{F},1)\widetilde{H}(h)+O\big(T^{5/4+\varepsilon}/M^{1/4}\big).$$

Proof steps: approx functional eq+Kuznetsov+Voronoï+inverse Mellin+functional eq+large sieve ineq.

- AFE gives

$$\sum_{|t_j-T|\le M} h(t_j)\sum_{m^2n\le T^{3+\epsilon}}\frac{A_F(n,m)\lambda_j(n)}{(m^2n)^{1/2}}+(Eis);$$

- Kuznetsov gives

$$\sum_{m^2n\le T^{3+\epsilon}}\frac{A_F(n,m)}{(m^2n)^{1/2}}\Big(TM\delta_{n,1}+\sum_{\pm}\sum_{c\ge1}\frac{1}{c}S(n,\pm1;c)B^{\pm}\big(\frac{4\pi\sqrt{n}}{c}\big)\Big);$$

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- Voronoï transforms the off-diagonal into

$$(\text{off}) = \sum_{r\geq 1} \frac{1}{r^2} \sum_{\tilde{n}\geq 1} A_F(1,\tilde{n}) e\Big(\pm\frac{\tilde{n}}{r}\Big) \mathcal{W}^{\pm}\Big(\frac{N\tilde{n}}{r^3}; \frac{\sqrt{N}}{r}\Big).$$

Rmk: One reason our proof being much shorter is because we apply the balanced version of Voronoï due to Miller–Zhou:

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Rmk: The final subconvex quality relies on

$$\int_{|t|\leq U} |L(\overline{F}, 1/2 - it)|^2 dt \ll_F U^{3/2}. \quad (1)$$

- Any improvement of this (trivial) bound will lead to improvement of the exponent 6/5 in

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- If $F = E_{\min} \rightsquigarrow$ Ivić's work, then indeed (off) $= O(T^{1+\varepsilon})$ by Zhi Qi (c.f. his talk at the Meeting).
- It seems difficult to improve over (1), since $\frac{\log(\text{conductor})}{\log(\text{family})} = 6$.
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Let χ mod q be Dirichlet characters. Let $F \in \mathrm{GL}_3$ be a *fixed* cusp form.
By a similar method, Blomer obtained

Theorem (Blomer, 2012)

For F self-dual and χ_q real characters, we have

$$L(F \times f_j \times \chi_q, 1/2),\ L(F \times \chi_q, 1/2)^2 \ll_{F, f_j} q^{\frac{5}{4}+\varepsilon} = q^{3/2-1/4+\varepsilon}.$$

- The exponent 5/4 is better than 11/8 (Li's bound), due to the use of large sieve ineq. in place of second Voronoï in the later case.
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Comparison: Blomer vs Conrey–Iwaniec

Conrey–Iwaniec ($F = 1 \boxplus 1 \boxplus 1$):

$$\frac{1}{q} \sum_{f_j \in \mathcal{B}(q,\mathrm{triv})} h(t_j) L(f_j \times \chi_q, 1/2)^3 + \frac{1}{q} \int_{-\infty}^{\infty} h(t) |L(\chi_q, 1/2+it)|^6 \mathrm{d}t \ll q^{\varepsilon};$$

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resulting from Cauchy–Schwarz:

$$RHS \ll \frac{1}{q} \|g(\psi,\chi)\|_{\infty} \Big(\sum_{\psi \bmod q}^{\star} |L(F \times \psi, 1/2)|^2 \Big)^{1/2} \Big(\sum_{\psi \bmod q}^{\star} |L(\overline{\psi}, 1/2)|^2 \Big)^{1/2}$$

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Recall

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To improve Blomer for q primes, one can try to improve

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or (if ambitious) to improve

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seems **difficult**! e.g. We do not know how to study

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GL_3 revisited: q composite

Theorem (L.–Ramon Nunes, in progress)

Let F be self-dual. Let $q = q_1 q_2$. $\exists\, \delta = \delta(\frac{\log q_1}{\log q_2}) > 0$, s.t.

$$L(F \times f_j \times \chi, 1/2),\ |L(F \times \chi, 1/2)|^2 \ll q^{5/4-\delta}.$$

Rmk: The strongest saving is when $q_1 \asymp q^{1/5}, q_2 \asymp q^{4/5}$, then

$$L(\cdots, 1/2) \ll q^{6/5+\varepsilon},$$

consistent with the t-aspect case.

Sketch for $L(\cdots, 1/2) \ll q^{5/4-\delta}$, F self-dual

Key observation: unbalance in Blomer's bound

$$\sum_{f_j \in \mathcal{B}(q)} h(t_j) L(F \times f_j \times \chi, 1/2) + \int_{-\infty}^{\infty} h(t) |L(F \times \chi_q, 1/2 + it)|^2 \mathrm{d}t \ll q \left(q^{\varepsilon} + q^{1/4+\varepsilon} \right).$$

Basic idea: summing over a **larger** family to improve the off-diagonal (at the cost of increasing diagonal).

- Choose $q' > q$ such that $\mathrm{Cond}(L(F \times f_j \times \chi, 1/2)) = q^6$, then

$$\sum_{f_j \in \mathcal{B}(q')} h(t_j) L(F \times f_j \times \chi, 1/2) + (Eis) \ll q' \left(q^{\varepsilon} + q^{1/4-\eta+\varepsilon} \right).$$

- Balance the diagonal and off-diagonal contribution.

Sketch for $L(\cdots, 1/2) \ll q^{5/4-\delta}$, F self-dual

Key observation: unbalance in Blomer's bound

$$\sum_{f_j \in \mathcal{B}(q)} h(t_j) L(F \times f_j \times \chi, 1/2) + \int_{-\infty}^{\infty} h(t) |L(F \times \chi_q, 1/2 + it)|^2 \mathrm{d}t \ll q \left(q^{\varepsilon} + q^{1/4+\varepsilon} \right).$$

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