Asymptotic for the Cubic Moment of Maass Form L-Functions

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What are Maass forms?

- Consider $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, where $\mathbb{H}$ is the hyperbolic upper half plane

  $$\mathbb{H} = \{z = x + iy : y > 0\}$$

  with $\mathrm{SL}_2(\mathbb{R})$ action by the Möbius transform

  $$gz = \frac{az+b}{cz+d}, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

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- $\mathrm{SL}_2(\mathbb{R})$-invariant metric, measure, and Laplacian on $\mathbb{H}$:

$$\mathrm{d}^2 s = \frac{\mathrm{d}^2 x + \mathrm{d}^2 y}{y^2}, \quad \mathrm{d}\mu(z) = \frac{\mathrm{d}x\mathrm{d}y}{y^2}, \quad \Delta = y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2).$$

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- Maass cusp forms f are eigenfunctions in $L^2(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$ of the Laplacian Δ that decay (exponentially) at the cusp ∞. Let $\lambda_f = \frac{1}{4} + t_f^2$ ($t_f \geqslant 0$) be the Laplacian eigenvalue of f.

What are Maass forms?

Fourier expansion:

$$f(z) = \sum_{n \neq 0} \rho_f(n) \sqrt{y} K_{it_f}(2\pi|n|y) e(nx),$$

where $e(x) = \exp(2\pi i x)$ and $K_\nu(y)$ is the K-Bessel function.

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Assumptions:

Hecke: f is an eigenfunction of all the Hecke operators T_n ($n \geqslant 1$).

$$T_n f(z) := \frac{1}{\sqrt{n}} \sum_{ad=n} \sum_{b (\operatorname{mod} d)} f\left(\begin{pmatrix} a & b \\ & d \end{pmatrix} z\right)$$

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- It is known that

$$\rho_f(\pm n) = \rho_f(\pm 1)\lambda_f(n), \qquad \rho_f(-1) = \epsilon_f \rho_f(1).$$

What are Maass form L-functions?

Let $\mathscr{B}$ be an orthonormal basis of (even) Hecke–Maass cusp forms.
For $f \in \mathscr{B}$ define the L-function $L(s, f)$ (or Hecke series $H_f(s)$ (in Ivić's notation))

$$L(s,f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}, \qquad \mathrm{Re}(s) > 1.$$

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$$\Lambda(1-s,f) = \epsilon_f \Lambda(s,f),$$

with $\Lambda(s,f) = \gamma(s,t_f)L(s,f)$ for $\gamma(s,t) = \pi^{-s}\Gamma((s-it)/2)\Gamma((s+it)/2)$.

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We have

- $L(\frac{1}{2},f) = 0$ if $\epsilon_f = -1$ (trivially);
- $L(\frac{1}{2},f) \geqslant 0$ (Katok–Sarnak).

Moments of Maass-form L-functions

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and for $T^\varepsilon \leqslant H \leqslant T/3$

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The moment integrals of Riemann ζ function correspond to Eisenstein series (continuous spectrum).

Aleksandar Ivić

Aleksandar Ivić (1949–2020)

Ivić's moment conjectures

A. Ivić. On the moments of Hecke series at central points. *Funct. Approx. Comment. Math.*, 30:49–82, 2002.

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Ivić's (weak) moment conjecture:

$$\mathcal{M}_k(T) = T^2 P_{(k^2-k)/2}(\log T) + O_\varepsilon(T^{1+c_k+\varepsilon}),$$

where $P_{(k^2-k)/2}$ is an explicit polynomial of degree $(k^2-k)/2$, and $0 \leqslant c_k < 1$.

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Ivić's (strong) moment conjecture: $c_k = 0$, or

$$\mathcal{M}_k(T) = T^2 P_{(k^2-k)/2}(\log T) + O_\varepsilon(T^{1+\varepsilon}).$$

Note: "square-root rule".

The moment conjectures of Conrey, Farmer, Keating, Rubinstein, and Snaith (2005)

Asymptotic formulae for the following example families:

- unitary: $\zeta\left(\frac{1}{2}+it\right)$, ordered by t.
- symplectic: $L\left(\frac{1}{2},\chi_d\right)$, $\chi_d(n)=(d|n)$, ordered by $|d|$.
- orthogonal: $L\left(\frac{1}{2},f\right), f\in S_k(\Gamma_0(N))$, ordered by k (N fixed) or by N (k fixed).

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The main term: Random Matrix Theory.

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The strong moment conjecture of Ivić is the Maass-form analogue of the moment conjectures of Conrey, Farmer, Keating, Rubinstein, and Snaith.

The moment conjecture for $\zeta\left(\frac{1}{2}+it\right)$

Define

$$\mathcal{I}_k(T) = \int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^{2k} \mathrm{d}t,$$

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then its moment conjecture reads

$$\mathcal{I}_k(T) = TP_{k^2}(\log T) + E_k(T),$$

with P_{k^2} a certain polynomial of degree k^2 and

$$E_k(T) = O_\varepsilon(T^{1/2+\varepsilon}).$$

The moment conjecture for $\zeta\left(\frac{1}{2}+it\right)$

- Case $k = 1$. $E_1(T) = O(T^{1515/4816+\varepsilon}) = O(T^{0.31457\dots})$ (Littlewood, 1922; ... Bourgain–Watt, 2018).

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- Case $k = 2$. $E_2(T) = O(T^{2/3}\log^8 T)$ (Ingham, 1926; Heath-Brown, 1977; Zavorotnyĭ, 1989; Ivić–Motohashi, 1995).

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- Case $k \geqslant 3$. Even $\mathfrak{Z}_k(T) = O(T^{1+\varepsilon})$ is wide open.
- Case $k = 6$. $\mathfrak{Z}_k(T) = O(T^{2+\varepsilon})$ (Heath-Brown, 1978).

Note: $\mathfrak{Z}_k(T) = O(T^{1+\varepsilon})$ would imply that for Ivić's conjectures on $\mathscr{M}_k(T)$ the moment integral of ζ could be removed and absorbed into the error term—Maass forms should dominate in $\mathscr{M}_k(T)$.

Known results on Ivić's conjecture

Recall that Ivić's moment conjecture reads:

$$\sum_{t_f \leqslant T} \alpha_f L\left(\tfrac{1}{2}, f\right)^k + \frac{2}{\pi} \int_0^T \frac{\left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k}}{|\zeta(1 + 2it)|^2} \mathrm{d}t = T^2 P_{(k^2-k)/2}(\log T) + O_\varepsilon(T^{1+c_k+\varepsilon}).$$

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The best error bounds to date towards Ivić's conjecture:

- Case $k = 0$ (weighted Weyl law (for both even and odd forms)): $O(T/\log T)$ (Kuznetsov, 1981; Ivić–Jutila, 2003; Xiaoqing Li, 2011).
- Case $k = 1$: $O(T \log^\varepsilon T)$ (Ivić–Jutila, 2003).
- Case $k = 2$: $O(T \log^6 T)$ (Motohashi, 1992).

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- Case $k = 3$: $O(T^{8/7+\varepsilon})$ (Ivić, 2002) $c_3 = 1/7$.
- Case $k = 4$: $O(T^{4/3+\varepsilon})$ (Kuznetsov, 1999; Ivić, 2002) $c_4 = 1/3$.

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- Case $k \geqslant 5$: wide open (Kıral–Young, Blomer–Khan, Khan).

Ivić's Lindelöf-on-average bound

A. Ivić. On sums of Hecke series in short intervals. *J. Théor. Nombres Bordeaux*, 13(2):453–468, 2001.

For any $T^{\varepsilon} \leqslant H \leqslant T$,

$$\sum_{|t_f - T| \leqslant H} \alpha_f L\left(\tfrac{1}{2}, f\right)^3 + \frac{2}{\pi} \int_{T-H}^{T+H} \frac{\left|\zeta\left(\frac{1}{2} + it\right)\right|^6}{|\zeta(1+2it)|^2} \mathrm{d}t \ll HT^{1+\varepsilon}.$$

Recall that the left hand side was denoted by $\mathscr{M}_3(T, H)$.

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For any $T^{\varepsilon} \leqslant H \leqslant T$,

$$\sum_{|t_f - T| \leqslant H} \alpha_f L\left(\tfrac{1}{2}, f\right)^3 \ll HT^{1+\varepsilon}.$$

Consequence: the Weyl-type subconvex bound $L\left(\tfrac{1}{2}, f\right) \ll t_f^{1/3+\varepsilon}$ (choose $H = T^{\varepsilon}$ and use the non-negativity of $L\left(\tfrac{1}{2}, f\right)$ and $t_f^{-\varepsilon} \ll \alpha_f \ll t_f^{\varepsilon}$).

Asymptotic formula for $\mathscr{M}_3(T,H)$

Theorem 1 (Q.)

Define

$$\mathscr{M}_3(T,H) = \sum_{|t_f - T| \leqslant H} \alpha_f L\left(\tfrac{1}{2}, f\right)^3 + \frac{2}{\pi}\int_{T-H}^{T+H} \frac{\left|\zeta\left(\frac{1}{2}+it\right)\right|^6}{|\zeta(1+2it)|^2}\mathrm{d}t.$$

Then for any $T^{\varepsilon} \leqslant H \leqslant T/3$ we have

$$\mathscr{M}_3(T,H) = \int_{T-H}^{T+H} K P_3^{\natural}(\log K)\mathrm{d}K + O_{\varepsilon}(T^{1+\varepsilon}),$$

where $P_3^{\natural}$ is an explicit cubic polynomial.

This is a refinement of Ivić's Lindelöf-on-average bound.

Asymptotic formula for $\mathscr{M}_3^{\natural}(T, M)$ (smoothed variant)

Theorem 2 (Q.)

Define

$$\mathscr{M}_3^{\natural}(T, M) = \sum_{f \in \mathscr{B}} \alpha_f L\left(\tfrac{1}{2}, f\right)^3 k_{T,M}(t_f) + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\left|\zeta\left(\frac{1}{2} + it\right)\right|^6}{|\zeta(1 + 2it)|^2} k_{T,M}(t) \mathrm{d}t,$$

where

$$k_{T,M}(t) = e^{-(t-T)^2/M^2} + e^{-(t+T)^2/M^2}.$$

Then for any $T^{\varepsilon} \leqslant M \leqslant T^{1-\varepsilon}$ we have

$$\mathscr{M}_3^{\natural}(T, M) = \sqrt{\pi} M T P_3^{\natural}(\log T) + O_{\varepsilon}(T^{1+\varepsilon}).$$

This also implies Ivić's Lindelöf-on-average bound.

Asymptotic formula for $\mathscr{M}_3(T)$

Corollary (Q.)

Define

$$\mathscr{M}_3(T) = \sum_{t_f \leqslant T} \alpha_f L\left(\tfrac{1}{2}, f\right)^3 + \frac{2}{\pi} \int_0^T \frac{\left|\zeta\left(\tfrac{1}{2} + it\right)\right|^6}{|\zeta(1 + 2it)|^2} \mathrm{d}t.$$

Then we have

$$\mathscr{M}_3(T) = T^2 P_3(\log T) + O_\varepsilon(T^{1+\varepsilon}),$$

for an explicit cubic polynomial P_3.

This verifies Ivić's strong conjecture in the case $k = 3$.

Asymptotic formula for $\mathcal{M}_3(T)$

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$$\frac{\mathrm{d}(K^2 P_3(\log K))}{\mathrm{d}K} = K P_3^\natural(\log K).$$

A connection between $\mathscr{M}_3(T,H)$ and $\mathscr{M}_3^\natural(T,M)$

The unsmoothing lemma below is essentially due to Motohashi, Ivić, and Jutila (they used a $\log T$ version).

Lemma

For $T^\varepsilon \leqslant M^{1+\varepsilon} \leqslant H \leqslant T/3$ *we have*

$$\mathscr{M}_3(T,H) = \frac{1}{\sqrt{\pi}M}\int_{T-H}^{T+H} \mathscr{M}_3^\natural(K,M)\mathrm{d}K + O_\varepsilon(MT^{1+\varepsilon}).$$

A connection between $\mathscr{M}_3(T,H)$ and $\mathscr{M}_3^\natural(T,M)$

The unsmoothing lemma below is essentially due to Motohashi, Ivić, and Jutila (they used a $\log T$ version).

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$$\mathscr{M}_3(T,H) = \frac{1}{\sqrt{\pi}M}\int_{T-H}^{T+H} \mathscr{M}_3^\natural(K,M)\mathrm{d}K + O_\varepsilon(MT^{1+\varepsilon}).$$

- The optimal choice of M is T^ε. (The case $M = T^\varepsilon$ is usually hard.)
- The error term can not be better than $O(T^{1+\varepsilon})$. (Xiaoqing Li used another method to achieve $O(T/\log T)$ in her weighted Weyl law.)

Ivić's approach: Kuznetsov–Motohashi formula

In Ivić's works, he used an explicit formula of Kuznetsov and Motohashi for the twisted second moment of central L-values.

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Define

$$\mathcal{M}_2(n;h) = \sum_{f\in\mathcal{B}} \alpha_f L\left(\tfrac{1}{2}, f\right)^2 \lambda_f(n) h(t_f) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\left|\zeta\left(\frac{1}{2}+it\right)\right|^4}{|\zeta(1+2it)|^2} \tau_{it}(n) h(t)\,\mathrm{d}t,$$

with

$$\tau_s(n) = \tau_{-s}(n) = \sum_{ab=n} (a/b)^s.$$

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$$\tau_s(n) = \tau_{-s}(n) = \sum_{ab=n} (a/b)^s.$$

The Kuznetsov–Motohashi formula reads

$$\mathcal{M}_2(n;h) = \mathcal{H}_1(n;h) + \cdots + \mathcal{H}_6(n;h),$$

where $\mathcal{H}_\nu(n;h)$ are explicit in terms of the divisor function $\tau = \tau_0$ and certain integral transforms of h (roughly speaking, $\mathcal{H}_2(n;h)$, $\mathcal{H}_3(n;h)$, and $\mathcal{H}_4(n;h)$ are shifted convolution sums of τ).

Kuznetsov–Motohashi formula

- It follows from the Kuznetsov trace formula with K-Bessel function.
- It is very explicit (no need for the approximate functional equation).
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The Kuznetsov–Motohashi formula is very effective in Motohashi's treatment of the (twisted) second moment.

There however seem to be losses when Ivić applied it to the cubic moment sum by averaging $\mathscr{M}_2(n;h)$ over n (using AFE (approximate functional equation)).

Kuznetsov–Voronoï approach à la Conrey–Iwaniec

J. B. Conrey and H. Iwaniec. The cubic moment of central values of automorphic L-functions. *Ann. of Math. (2)*, 151(3):1175–1216, 2000.

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Now, the work of Conrey–Iwaniec may be interpreted as the inverse of the Motohashi formula or a reciprocity formula (Young, Petrow, Nelson, Han Wu, Blomer, Khan, Humphries, and Chung-Hang Kwan, etc.).

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X. Li. Bounds for GL(3) × GL(2) *L*-functions and GL(3) *L*-functions. *Ann. of Math. (2)*, 173(1):301–336, 2011.

M. P. Young. Weyl-type hybrid subconvexity bounds for twisted *L*-functions and Heegner points on shrinking sets. *J. Eur. Math. Soc. (JEMS)*, 19(5):1545–1576, 2017.

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- Conrey and Iwaniec treat the q-aspect.
- We treat the spectral aspect and use the analysis of Xiaoqing Li and Young (second Voronoï and large sieve).

Motivation: Nonvanishing problem for Maass forms

Need asymptotic formulae for the twisted (mollified) first and second moment.

- Shenhui Liu, Balkanova–Bingrong Huang–Södergren (Kuznetsov–Motohashi),
- Sheng-chi Liu–Q. (Kuznetsov–Voronoï).

Sheng-chi Liu and I also solved the nonvanishing problem over imaginary quadratic fields by the Kuznetsov–Voronoï approach.

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Note: the main term comes from the diagonal and the zero frequency.

Bessel kernel

For real $x > 0$ and complex s define Bessel kernel

$$B_s(x) = \frac{\pi}{\sin(\pi s)}\Big(J_{-2s}(4\pi\sqrt{x}) - J_{2s}(4\pi\sqrt{x})\Big), \quad B_s(-x) = 4\cos(\pi s)K_{2s}(4\pi\sqrt{x}).$$

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- Bessel integral in Kuznetsov:

$$\mathcal{H}(x) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} h(t)B_{it}(x)\tanh(\pi t)t\,\mathrm{d}t.$$

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- Hankel transform in Voronoï:

$$\widetilde{w}_0(y) = \int_0^{\infty} w(x)B_0(xy)\mathrm{d}x.$$

$$B_0(x) = \sum_{\pm}\frac{e(\pm(2\sqrt{x}+1/8))}{\sqrt[4]{x}}W_0(\pm\sqrt{x}), \quad B_0(-x) = O\Big(\frac{\exp(-4\pi\sqrt{x})}{\sqrt[4]{x}}\Big).$$

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$$B_s(x) = \frac{\pi}{\sin(\pi s)}\Big(J_{-2s}(4\pi\sqrt{x}) - J_{2s}(4\pi\sqrt{x})\Big), \quad B_s(-x) = 4\cos(\pi s)K_{2s}(4\pi\sqrt{x}).$$

- Plancherel integral in Kuznetsov (diagonal):

$$\mathcal{H} = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} h(t)\tanh(\pi t)t\,\mathrm{d}t.$$

- Mellin integrals in Voronoï (zero frequency):

$$\widetilde{w}_0(0) = \int_0^\infty w(x)\mathrm{d}x, \qquad \widetilde{w}_0'(0) = \int_0^\infty w(x)\log x\,\mathrm{d}x$$

Kuznetsov trace formula

For $n_1, n_2 \geqslant 1$ we have

$$\sum_{f \in \mathscr{B}} h(t_f)\alpha_f \lambda_f(n_1)\lambda_f(n_2) + \frac{1}{4\pi}\int_{-\infty}^{\infty} h(t)\alpha(t)\tau_{it}(n_1)\tau_{it}(n_2)\mathrm{d}t$$

$$= \delta_{n_1,n_2}\mathscr{H} + \sum_{\pm}\sum_{c=1}^{\infty} \frac{S(n_1, \pm n_2; c)}{c}\mathscr{H}\Big(\pm \frac{n_1 n_2}{c^2}\Big),$$

with

$$\alpha_f = \frac{|\rho_f(1)|^2}{\cosh(\pi t_f)} = \frac{2}{L(1, \mathrm{Sym}^2 f)}, \qquad \alpha(t) = \frac{4}{|\zeta(1+2it)|^2}.$$

Poisson and Voronoï summation formula

Poisson:

$$\sum_{n} e\left(-\frac{an}{c}\right) w(n) = \sum_{m \equiv a (\operatorname{mod} c)} \widehat{w}\left(\frac{m}{c}\right),$$

where $\widehat{w}$ is the Fourier transform of w.

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where $\widehat{w}$ is the Fourier transform of w.

Voronoï:

$$c \sum_{n} \tau(n) e\left(-\frac{an}{c}\right) w(n) = 2(\gamma - \log c)\widetilde{w}_0(0) + \widetilde{w}_0'(0) + \sum_{m \neq 0} \tau(m) e\left(\frac{\bar{a}m}{c}\right) \widetilde{w}_0\left(\frac{m}{c^2}\right).$$

AFE

$$L(s,f)^3 = L(s,f)^2 \cdot L(s,f).$$

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AFE

$$\zeta(s+it)\zeta(s-it) = \sum \frac{\tau_{it}(n)}{n^s}, \quad \zeta(s+it)^2\zeta(s-it)^2 = \zeta(2s) \sum \frac{\tau_{it}(n)\tau(n)}{n^s}.$$

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$$L(s,f) = \sum \frac{\lambda_f(n)}{n^s}, \quad L(s,f)^2 = \zeta(2s) \sum \frac{\lambda_f(n)\tau(n)}{n^s}.$$

$$L(\tfrac{1}{2},f) \approx 2 \sum_{n \leqslant t_f^{1+\varepsilon}} \frac{\lambda_f(n)}{\sqrt{n}}, \qquad L(\tfrac{1}{2},f)^2 \approx 2 \sum_{n \leqslant t_f^{2+\varepsilon}} \frac{\lambda_f(n)\tau(n)}{\sqrt{n}}.$$

AFE

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$$L(s,f) = \sum \frac{\lambda_f(n)}{n^s}, \quad L(s,f)^2 = \zeta(2s) \sum \frac{\lambda_f(n)\tau(n)}{n^s}.$$

$$\frac{1}{\sqrt{N_1}} \sum_{n_1} \lambda_f(n_1) w_1\left(\frac{n_1}{N_1}\right), \quad \frac{1}{\sqrt{N_2}} \sum_{n_2} \lambda_f(n_2)\tau(n_2) w_2\left(\frac{n_2}{N_2}\right),$$

for $N_1 \leqslant T^{1+\varepsilon}$ and $N_2 \leqslant T^{2+\varepsilon}$.
It is most convenient to use a version of AFE of Blomer.

Setup

- Apply AFE to $\mathscr{M}_3^{\natural} = \mathscr{M}_3^{\natural}(T, M)$:

$$\frac{1}{\sqrt{N_1 N_2}} \sum_{n_1, n_2} \sum \tau(n_2) \sum_{f \in \mathscr{B}} \alpha_f \lambda_f(n_1) \lambda_f(n_2) w_1\left(\frac{n_1}{N_1}\right) w_2\left(\frac{n_2}{N_2}\right) k(t_f) + \cdots ,$$

with

$$k(t) = k_{T,M}(t) = e^{-(t-T)^2/M^2} + e^{-(t+T)^2/M^2}.$$

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- The off-diagonal sum after Kuznetsov:

$$\frac{1}{\sqrt{N_1 N_2}} \sum_{\pm} \sum_{c} \sum_{n_1, n_2} \tau(n_2) \frac{S(n_1, \pm n_2; c)}{c} w_1\left(\frac{n_1}{N_1}\right) w_2\left(\frac{n_2}{N_2}\right) \mathcal{H}\left(\pm \frac{n_1 n_2}{c^2}\right),$$

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$$\mathcal{H}(x) \approx \frac{1}{2\pi^2} \int_{-\infty}^{\infty} k(t) B_{it}(x) \tanh(\pi t) t \, \mathrm{d}t.$$

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with

$$w^{\pm}(x_1, x_2; \Lambda) = w_1(x_1) w_2(x_2) \mathscr{H}(\pm \Lambda x_1 x_2),$$
$$\mathscr{H}(x) \approx \frac{1}{2\pi^2} \int_{-\infty}^{\infty} k(t) B_{it}(x) \tanh(\pi t) t \,\mathrm{d} t.$$

The main term

The main term becomes transparent after the Voronoï—half is from the diagonal term in Kuznetsov and half is from the zero frequency after Voronoï.

The error term

- Arithmetic part:

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  Exponential factor $e(\pm m_1 m_2/c)$ + condition $(m_1, c) = 1$.

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  Exponential factor $e(\pm m_1 m_2/c)$ + condition $(m_1, c) = 1$.
- Analytic part:
  Fourier–Hankel transform of Bessel integrals.

Analysis for the Bessel integrals

$$\mathcal{H}(x) \approx \mathcal{H}_{+}(x) + \mathcal{H}_{-}(x),$$

with

$$\mathcal{H}_{\pm}(x^2) \approx MT \int_{-M^{\varepsilon}/M}^{M^{\varepsilon}/M} g(Mr) e(Tr/\pi \mp 2x \cosh r) \mathrm{d}r,$$

$$\mathcal{H}_{\pm}(-x^2) \approx MT \int_{-M^{\varepsilon}/M}^{M^{\varepsilon}/M} g(Mr) e(Tr/\pi \pm 2x \sinh r) \mathrm{d}r,$$

$$g(r) = \frac{2}{\pi^{3/2}} e^{-r^2}.$$

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$$g(r) = \frac{2}{\pi^{3/2}} e^{-r^2}.$$

- $\mathcal{H}(x^2)$ is negligibly small unless $x > M^{1-\varepsilon}T$,
- $\mathcal{H}(-x^2)$ is negligibly small unless $x \asymp T$.

Analysis for the Fourier–Hankel transform

$$w^{\pm}(x_1, x_2; \Lambda) = w_1(x_1) w_2(x_2) \mathcal{H}(\pm \Lambda x_1 x_2).$$

$$\widehat{\widetilde{w}}^{\pm}(y_1, y_2; \Lambda) = \int_0^\infty \int_0^\infty w^{\pm}(x_1, x_2; \Lambda) e(-x_1 y_1) B_0(x_2 y_2) \mathrm{d}x_1 \mathrm{d}x_2.$$

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By stationary phase analysis (after Young), we get

$$e(\pm y_1 y_2/\Lambda) \widehat{\widetilde{w}}^{\pm}(y_1, y_2; \Lambda) \approx \frac{MT}{\sqrt{|y_1 y_2|}} \Phi^{\pm}(y_1 y_2/\Lambda),$$

for

$$|y_1| \asymp \sqrt{y_2}.$$

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Let $x = y_1 y_2/\Lambda$ and $x \asymp X$.

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$$\Phi^{+}(x) = \int e(Tr/\pi - x \tanh^2 r) V^{+}(r) \mathrm{d}r,$$

$$\Phi^{-}(x) = \int e(Tr/\pi - x \coth^2 r) V^{-}(r) \mathrm{d}r,$$

where $V^{+}(r)$, $V^{-}(r)$ are supported in

$$r \asymp U^{+}/T, \qquad |r| \asymp U^{-}/T,$$

with

$$U^{+} = T^2/|X|, \qquad U^{-} = \sqrt[3]{|X| T^2}.$$

Analysis for the Fourier–Hankel transform

Conditions:

$$|X| \asymp \sqrt{\Lambda}, \qquad T < \sqrt{\Lambda}/M^{1-\varepsilon},$$

or

$$T \asymp \sqrt{\Lambda}, \qquad |X| < \sqrt{\Lambda}/M^{3-\varepsilon},$$

in the ±-case.

Analysis for the Fourier–Hankel transform

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in the $\pm$-case.

Note: $U^{\pm} < T/M^{1-\varepsilon}$.

Aside: The average process

Recall that

$$\mathscr{M}_3(T,H) = \frac{1}{\sqrt{\pi}M}\int_{T-H}^{T+H} \mathscr{M}_3^{\natural}(K,M)\mathrm{d}K + O_\varepsilon(MT^{1+\varepsilon}).$$

Need to change T into K and average over $[T-H, T+H]$ (it is clear that $K \asymp T$).

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This is easy.

$$\frac{1}{M}\int_{T-H}^{T+H} Ke(Kr/\pi)\mathrm{d}K = \frac{Ke(Kr/\pi)}{2irM}\bigg|_{T-H}^{T+H} - \frac{1}{2irM}\int_{T-H}^{T+H} e(Kr/\pi)\mathrm{d}K.$$

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Essentially, we have the same kind of Φ-integrals but we lose $T/MU^{\pm}$:

$$V^{\pm}(r) \longrightarrow V^{\pm}(r)/rM \approx T/MU^{\pm}\cdot V^{\pm}(r).$$

Analysis for the Fourier–Hankel transform

By Mellin technique (after Young), we have

$$\Phi^{\pm}(x) = \frac{1}{T} \int_{|t| \asymp U^{\pm}} \lambda^{\pm}_{X,T}(t) |x|^{it} \mathrm{d}t,$$

for $\lambda^{\pm}_{X,T}(t) \ll 1$ (with implied constant independent on the parameters X and T). Again

$$U^{+} = T^2/|X|, \qquad U^{-} = \sqrt[3]{|X|T^2}.$$

Conclusion

We arrive at:

$$MT \sum_{c>0} \frac{1}{\sqrt{c}} \sum_{\substack{|m_1|, m_2 > 0 \\ (m_1, c) = 1}} \frac{\tau(m_2)}{\sqrt{|m_1 m_2|}} \Phi^{\pm}\left(\frac{m_1 m_2}{c}\right),$$

which is bounded by the supremum of

$$MT^{\varepsilon} \int_{|t| \asymp U^{\pm}} \left| \overline{S_{it}(C^{\pm})} S_{it}(L_1^{\pm}) S_{it}^{\natural}(L_2^{\pm}) \right| \mathrm{d}t,$$

$$\overline{S_{it}(C)} = \sum_{c \sim C} \frac{1}{c^{1/2+it}}, \quad S_{it}(L) = \sum_{m \sim L} \frac{1}{m^{1/2-it}}, \quad S_{it}^{\natural}(L) = \sum_{m \sim L} \frac{\tau(m)}{m^{1/2-it}},$$

Conclusion

$$MT^{\varepsilon}\int_{|t|\asymp U^{\pm}}\left|\overline{S_{it}(C^{\pm})}S_{it}(L_1^{\pm})S_{it}^{\natural}(L_2^{\pm})\right|\mathrm{d}t,$$

$$\overline{S_{it}(C)}=\sum_{c\sim C}\frac{1}{c^{1/2+it}},\quad S_{it}(L)=\sum_{m\sim L}\frac{1}{m^{1/2-it}},\quad S_{it}^{\natural}(L)=\sum_{m\sim L}\frac{\tau(m)}{m^{1/2-it}},$$

for dyadic parameters $C^{\pm}$, $L_1^{\pm}$, and $L_2^{\pm}$ in the ranges

$$C^{+}<\frac{\sqrt{N_1N_2}}{M^{1-\varepsilon}T},\qquad L_1^{+}\ll\frac{\sqrt{L_2^{+}N_2}}{N_1},\qquad L_2^{+}\asymp N_1,$$

$$C^{-}\ll\frac{\sqrt{N_1N_2}}{T},\qquad L_1^{-}\ll\frac{\sqrt{L_2^{-}N_2}}{N_1},\qquad L_2^{-}<\frac{N_1}{M^{2-\varepsilon}},$$

and for

$$U^{+}=\frac{C^{+}T^2}{L_1^{+}L_2^{+}},\qquad U^{-}=\frac{\sqrt[3]{L_1^{-}L_2^{-}T^2}}{\sqrt[3]{C^{-}}}.$$

Cauchy–Schwarz

By Cauchy–Schwarz:

$$M T^{\varepsilon} \left(\int_{|t| \asymp U^{\pm}} \left| S_{it}(C^{\pm}) S_{it}(L_1^{\pm}) \right|^2 \mathrm{d}t \right)^{1/2} \left(\int_{|t| \asymp U^{\pm}} \left| S_{it}^{\natural}(L_2^{\pm}) \right|^2 \mathrm{d}t \right)^{1/2} .$$

Another Voronoï

- By
$$\sum_{n \asymp N} \frac{\tau(n)}{n^{1/2-it}} \approx \sum_{m \asymp t^2/N} \frac{\tau(m)}{m^{1/2+it}},$$
we can always reduce the length of summation of $S^{\natural}_{it}(L^{\pm}_{2})$ to at most $(|t| \asymp)U^{\pm}$.

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we can always reduce the length of summation of $S^{\natural}_{it}(L^{\pm}_2)$ to at most $(|t| \asymp) U^{\pm}$.
- We check that the length of summation of $S_{it}(C^{\pm}) S_{it}(L^{\pm}_1)$ is already less than $U^{\pm}$. ($C^{\pm} L^{\pm}_1 \ll U^{\pm} T^{\varepsilon}$.)

Gallagher's large sieve

By a special case of Gallagher's large sieve:

$$\int_{-T}^{T} \left| \sum_n a_n n^{it} \right|^2 dt \ll \sum_n (T+n)|a_n|^2,$$

we infer that

$$MT^{\varepsilon} \left(\int_{|t| \asymp U^{\pm}} \left| S_{it}(C^{\pm}) S_{it}(L_1^{\pm}) \right|^2 \mathrm{d}t \right)^{1/2} \left(\int_{|t| \asymp U^{\pm}} \left| S_{it}^{\natural}(L_2^{\pm}) \right|^2 \mathrm{d}t \right)^{1/2}$$

is bounded by

$$MT^{\varepsilon} \cdot \sqrt{U^{\pm}} \cdot \sqrt{U^{\pm}} = MU^{\pm} T^{\varepsilon}.$$

Conclusion

For $\mathscr{M}_3^\natural$ note that $U^\pm < T/M^{1-\varepsilon}$ and hence

$$MU^\pm T^\varepsilon < T^{1+\varepsilon}.$$

For $\mathscr{M}_3$ recall that we lost $T/MU^\pm$ when applying the average process, but we still have (no matter how we choose M)

$$MU^\pm T^\varepsilon \cdot T/MU^\pm = T^{1+\varepsilon}.$$

Thank You!