# An Introduction to Percolation Theory -IV

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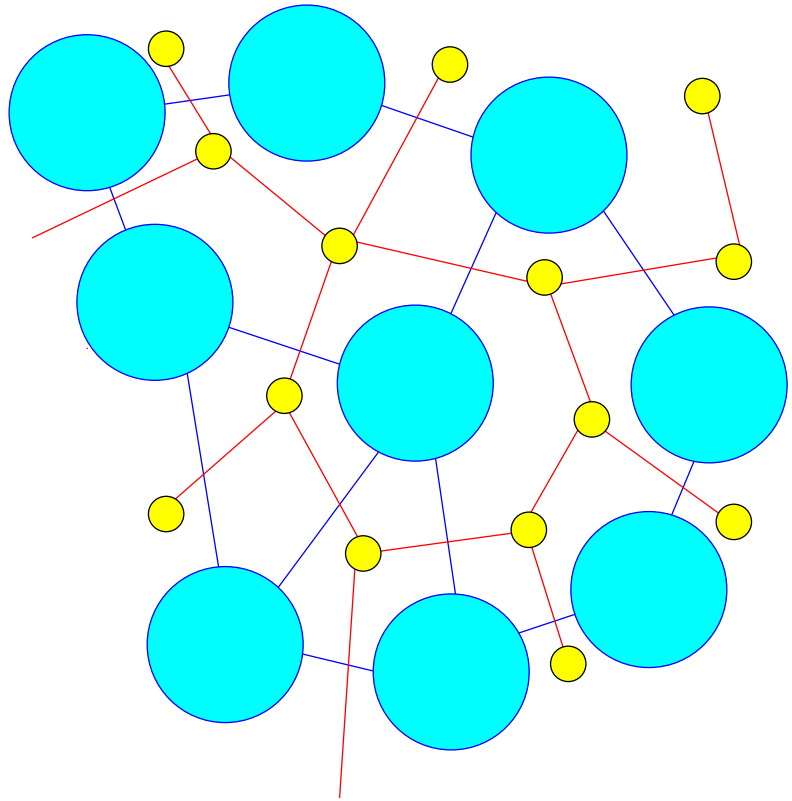
# Duality in percolation and resistance problems

In theoretical physics, there are several different notions of duality. For example, there is a strong-coupling /weak coupling duality (e.g. in the self duality of the Ising model on a square lattice.)

We will start with the notion of graph duality on planar graphs.

Consider a planar graph. The bonds may be denoted by line segments. Then the two-dimensional may be divided into disjoint polygonal areas called faces. The boundaries of polygons have vertices, and edges. In the figure, the yellow vertices and red edges form such a graph $G$.



We define a dual graph $\tilde{G}$ of these as follows: Each polygonal area of $G$ is a vertex in $\tilde{G}$ ( shown as big light blue blobs). Draw a blue edge in $\tilde{G}$ between vertices $A_1$ and $A_2$ of $\tilde{G}$, iff the corresponding faces in $G$ share a common edge.

In the figure the graph drawn between blue vertices with blue edges is the dual graph $\tilde{G}$.
It easy to see that the dual of $\tilde{G}$ is $g$.

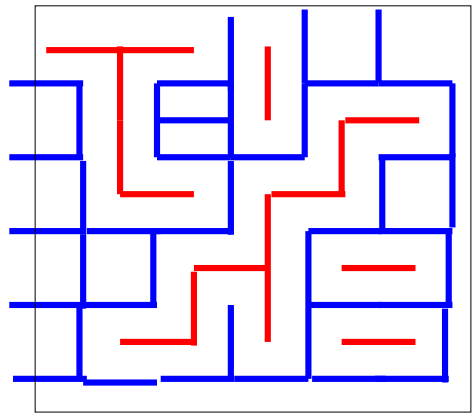
This notion may be generalized to higher dimensions.

Say, in 3-dimensions, we can start with a decomposition of the 3-d manifold into non-overlapping 3-dimensional cells. Then, the intersection of adjoing cells are 2-dimensional 'surfaces', and intersection of adjacent surfaces are 'edges', and two different edges meet at 0-dimensional 'vertices'.

We will define this set of the vertices and edges as constituting the graph $G$. The condition that the graph is embedded in 3-d manifold , is the generalization of "planarity" condition.

In the dual graph, $\tilde{G}$, there is a vertex for each "cell" in $G$. We draw an edge in $\tilde{G}$ if the volumes corresponding to the end points of this edge have a common surface in $G$. A surface in $\tilde{G}$ corresponds to bond in $G$, and each "cell" in $\tilde{G}$ corresponds to a vertex in $G$.

# Self-duality of bond percolation on the square lattice

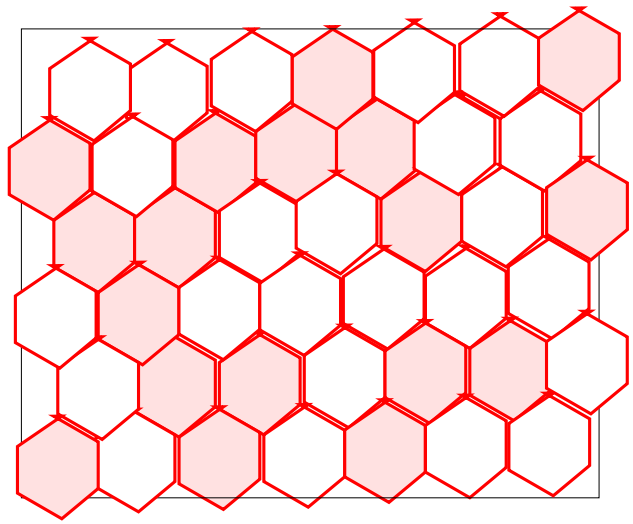


A picture of a configuration of bond percolation on a finite portion of the square lattice ( red bonds), and its dual (blue bonds) .
Clearly, the bond percolation problem on the square lattice is self-dual.
There is an obvious geometrical property: either there is a connected path of red bonds from left to right boundary, OR there is a path of blue bonds from top to bottom.

It can be proved that for $p_{red} > p^*$, for large $L$, with a probability near 1 there is a red path from left to right. Then, there is no blue path. Hence the concentration of blue bonds must be less than $p^*$.

This implies that for the square lattice bond percolation, $p^* = 1/2$.

# Self-duality for site percolation on the triangular lattice

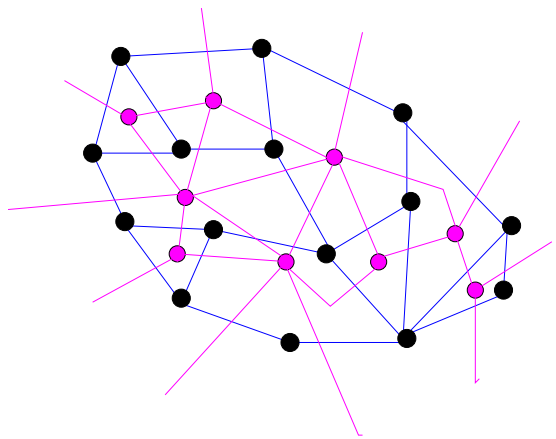


A picture of a random configuration of white and pink hexagons. The centers of hexagons form a Triangular lattice .

Again, there can either left-right crossing using only white tiles, or top to down using the pink tiles.
Hence, we have $p_c[\text{site}, \text{triangular lattice}] = 1/2$.

# Duality transformation for planar resistor networks

Consider a network of resistors forming planar graph. We are given the values of resistances of each link, and one wants to determine the current distribution.



This is solved using the Kirchhoff's laws. One assigns a voltage variable $V_i$ to each node $i$. This determines the current in each link in terms of $\{V_i\}$. Then, one imposes the conditions that net current at each (non-source) node is zero, to get linear coupled equations for $\{V_i\}$.

The alternative is to define loop current variables $\psi_j$ for each loop. Then individual link currents are differences of $\psi$'s, and the current conservation laws are automatically satisfied. Then, one gets linear equations for $\{\psi_j\}$ by requiring that net volatge drop across a loop on the dual lattice (magenta colored) is zero.

It is easily seen that the variables $\{\psi_j\}$ is just like the potentials for the dual resistor network graph, with

$$\tilde{\sigma}_j = 1/\sigma_j,$$

for the dual link $\tilde{j}$ of the link $j$ of the original graph.

This shows that resistor network problem on the planar garphs is self-dual.

# Directed Percolation

If the flow of fluid along bonds is only possible in prespecified directions, that affects qualitative behavior strongly. This problem has been studied a lot, and continues to be.

The percolation clusters are strongly anisotropic.

One may think of the preferred direction as time. This becomes a prototypical model to study non-equilibrium steady states.

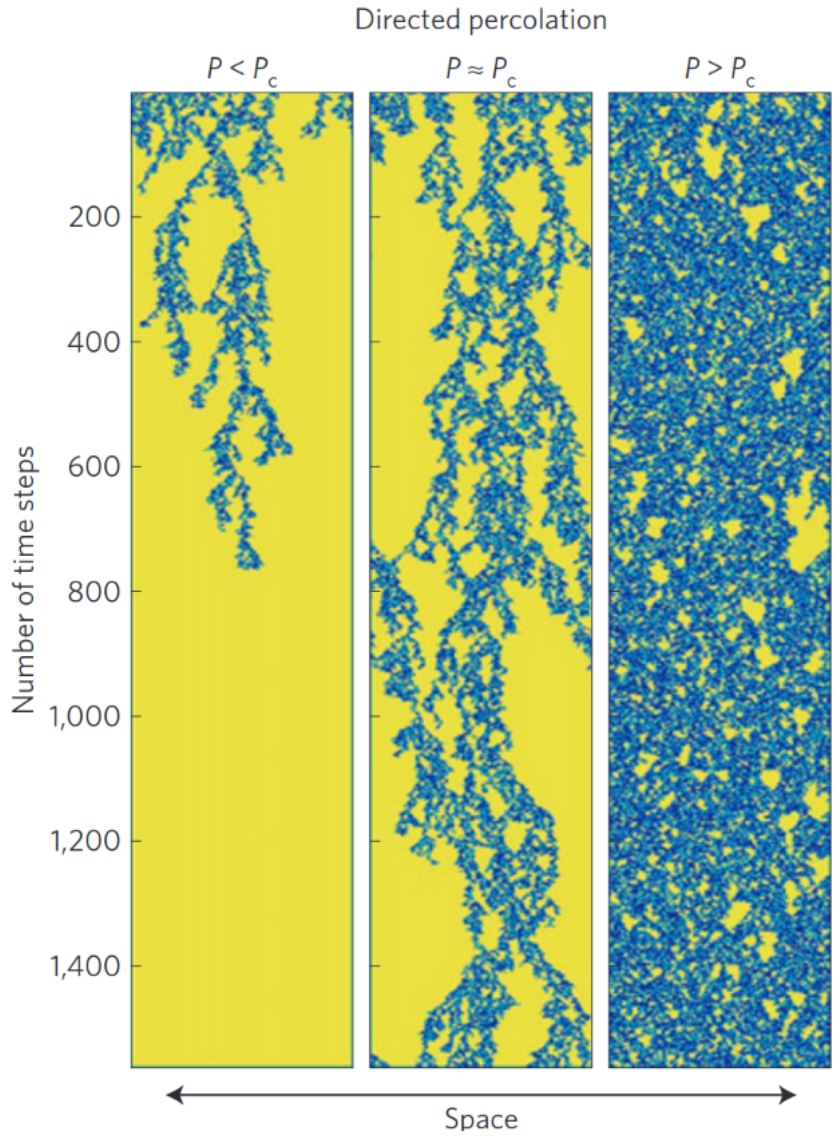
The model has remained unsolved even in $1+1$ dimensions, despite many efforts.

One can think of an infection in an orchard evolving in time, where the person recovers after one unit of time.

At low rates of infection, the infection eventually dies out, and the model shows an 'Active -absorbing state transition'

More hegerally, there are reaction diffusion systems, where the activity can grow, diffuse or die, that often show such absorbing state transitions.

There is a lot of work on absorbing state transitions. The Janssen -Grassberger conjecture states that if there is a unique absorbing state, and no special symmetries, then, the transition is in the Directed Percolation universality class.




A picture of some directed percolation clusters, taken from G. Lemoult et al, Nature Phys., 12,(2016) 254.

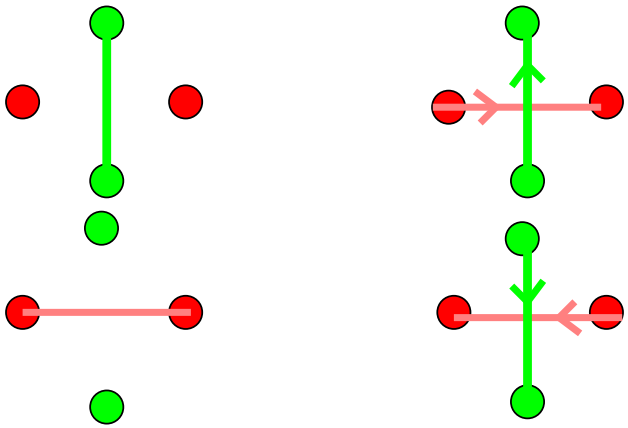
There are two correlation lengths: $\xi_{\|}$ , $\xi_{\perp}$ that diverge with different exponents $\epsilon^{-\nu_{\|}}$ and $\epsilon^{-\nu_{\perp}}$.

Equivalently, the relaxation time diverges as a nontrivial power of correlation length $\tau \sim \xi^{z}$, with $z > 1$.

# Duality transformations for Directed percolation

It is possible to extend the duality transformation for percolation in two dimensions to the case where each bond can be of any of four types: R, I, $D_+$ and $D_-$, which are resistor, insulator, forward and backward diodes.

The assignment of dual bonds is as follows:



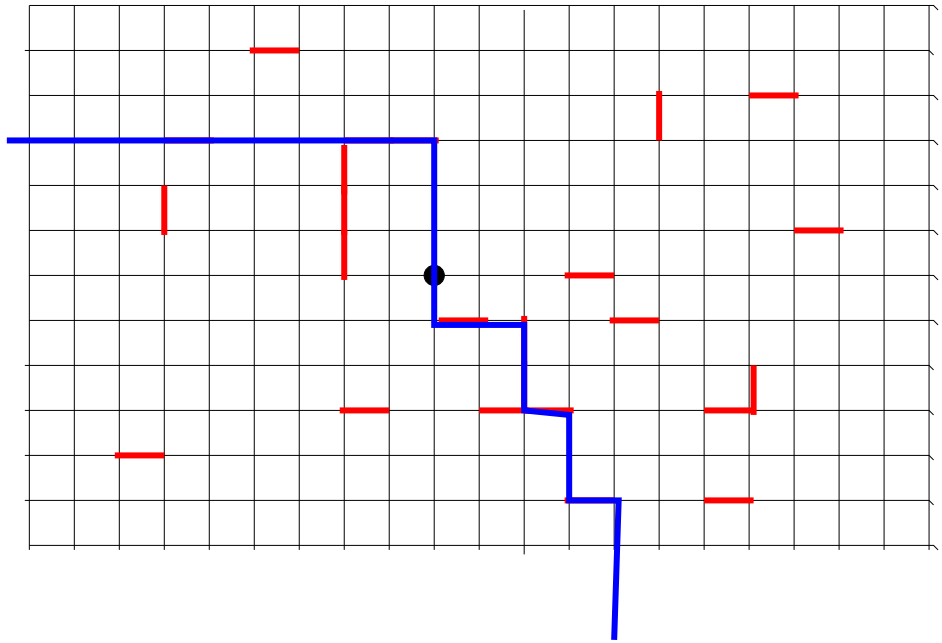
The red sites are original lattice sites, and the green sites are dual lattice sites. The I is denoted by absent bond, R is a undirected bond allowing two way flow, and the directed bonds denote allowed direction of flow. The assignment for vertical bonds on the original lattice is obtained by rotating figure anticlockwise by 90°.

If only R and I states are allowed, this is the old undirected percolation case. If only I and D+ are allowed, that is the usual directed percolation ( here we shall it DI-percolation).

The dual of the DI -percolation is the DR-percolation. This has an interesting structure. Assume the bonds allow only flow in the negative $x$ or negative y-direction with probability $p$, and two way flow with probability $(1-p)$.

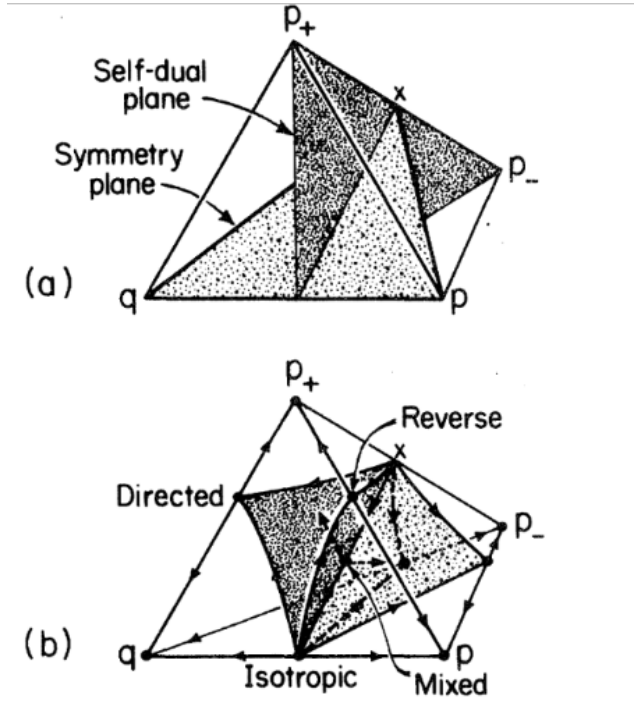
We consider a source at the origin, and look at the region than can be reached by the fluid. It is easy to see that flow in the negative direction is always allowed, then if site(x,y) is wet, so are all sites $(x', y')$ with $x' \leq x, y' \leq y$.

Hence the structure of the wetted cluster is like. single staircase' On large scales it forms an approximate wedge. The wedge angle $= 90^\circ$, and slowly increases as $p$ is increased. At a value $p^* = 1 - p_{c,DP,sq}$, the angle reaches $180^\circ$. Then there is an instability, and the angle jumps to $360^\circ$, for all $p > p^*$.



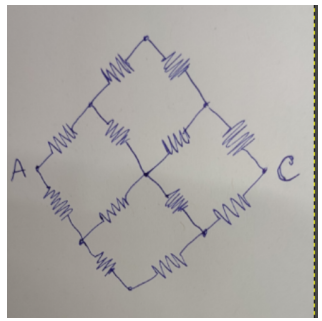
A picuture of Diode-resistor percolation. All the unmarked bonds allow flow left or down. The red bonds are resistors allowing two-way flow of the fluid. The blue line marks the boundary of wetted sites, wetted by a point source at the origin(black circle).

For the general case, where I,$D_+$, D-, and R cases occur with probabilities $p_0, p_+, p_-$ and $p_2$, the system has four possible phases : No flow to long distances, flow to long distances in the positive directions, only in negative directions, or both directions.
The phase diagram may be drawn in a tetrahedon ( with $p_0 + p_2 + p_+ + p_- = 1$.) There is a symmetry under the interchanging $p_0$ and $p_2$ and $p_+$ and $p_-$. A qualitative phae diagram is given below [ taken from S. Redner, Phys. Rev. B 25(1982)3242]



# Exercises

1. In the resistor network shown below, each resistance is $1\Omega$. Determine the effective resistance between the nodes $A$ and $C$.



# THANK YOU

# References

1. Duality transformation for two dimensional directed percolation and resistance problems, DD, M. Barma and M. K. Phani, Phys. Rev. Lett. **47** (1981) 1238.