An Introduction to Percolation Theory -V

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Invasion percolation

This problem originates in the important real-world problem of oil-extraction from porous oil-bearing rock (called shale).

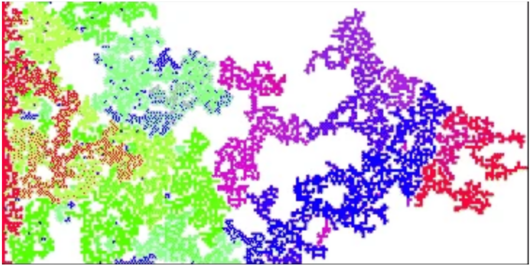
The method of extraction is to push hot steam from one side of rock at high pressure, which displaces the oil, and it comes out at the other end.

It was found that some fraction of the oil remains in the rock, and is hard to extract.
Wilkinson and Willemsen proposed a model for this.

The porous rock is modelled as a percolation structure (the unoccupied sites are the pores).
A fraction of the pores are full of oil. Adjacent pores are connected by necks of variable sizes. It takes a higher pressure head to push oil through a narrower neck.

At any time, as the steam is pushed, there is the advancing front of the invading fluid. The fluid invades at the bond corresponding to the smallest pressure head.

Take the pressure head to a a radom variable uniformly distributed between 0 and 1.



A schematic representaion of the advancing front. Different colors represent different time intervals of invasion. Picture taken from The Encyclopedia of Complexity and Systems Science.

Interestingly, it was observed that the invaded cluster looks like the percolation cluster at its critical point.

The dynamics of system seems to reach the critical point without any external fine tuning of parameters.

This was actually the first example of Self-organized Criticality.

Consider a bond occupied if its pressure -head is less than a value p, with $p = p_c^+$.
Then, we have an infinite spanning cluster.

Once the invaded cluster hits this spanning cluster, all further growth will be on this cluster. Oil trapped in finite clusters cannot be recovered by fluid displacement.

All subsequently selected bonds will have a value of pressure head uniformly distributed between 0 and p_c.

Self -organized directed percolation

In the previous model, self-organization required "extremal dynamics". Can one get self-organized criticality without this, involving only local evolution rules?

The answer was provided in a model of SOC-DP by Grassberger and Zhang.

Consider sites on a line. At each site the is a random number in $[0, 1]$. Start with $f(x, t = 0) = 0$, at all sites x.
The sites are updated in parallel with the following simple rule:

$$f(x, t + 1) = Max\left[Min[f(x - 1, t), f(x + 1, t)], \eta(x, t + 1)\right]. \quad (1)$$

where $\eta(x, t)$ are i.i.d. random variables uniformly distributed in $[0, 1]$.

Then, one finds that at large times, the values of f lie only in the interval $f > p_{c,DP}$, and in fact are proportional $P_\infty(p = f)$. This looks fairy mysterious at first. But may be seen as follows:

Think of all directed paths from (x, t) to the bottom line $t = 0$. For each path there is a maximum value of the noise variable $\eta(x', t')$, with (x', t') on the path. Then find the least value of this value amongst all possible directed paths. Call that $f(x, t)$.

Then, it is easily that $f(x, t)$ satisfies the evolution equation.

Then the result follows that all directed long paths to infinity must have at least some site where $\eta(x, t)$ is above $p_{c,DP}$.

The treatment is clearly valid for other lattices, and other dimensions.

Other properties of these optimal paths may be studied. And follow from / are same as the expected behavior of directed percolation clusters.

Self-organized Undirected Percolation

The directed case has some special features (you cannot influence your past). It is interesting to ask if one can make an undirected variation of the Grassberger-Zhang model.

This is possible, but not obvious. But one can check that the following works.

• We consider a finite $L \times L$ square lattice, with open boundaries.
• There is a real variable $\eta(x, y)$ at each site (x, y). There are i.i.d. variables distributed uniformly within $[0, 1]$.

• Define a real variable $f(x, y)$ at each site. the starting values are $f(x, y) = 1$, for all x, y.
Define $f(x, y) = 0$ at all external perimeter sites of the square.
• These variables f are updated in parallel using the following rule:

$$f(x, y, t+1) = Max\left[Min\ of\ f\ of\ all\ neighbors\ at\ time\ t, \eta(x, y)\right].$$

It is easy to see that time $t = 1$, at all internal point $f(x, y)$ remains 1. But it would change at the boundary sites.

At $t = 2$, the sites at the next layer would also be affected. And with time, the disturbance propagates inwards.

At large times, the f values stablize, and do not change anymore.

On this stable attractor, let the value of the function $f(x, y) = f^*(x, y)$.

The variable $f^*(x, y)$ is a random variable distributed in the interval $[p_{c,undirected}, 1]$ with density proportional to $P_\infty(p)$.

As in the SODP case, we consider a path P from site (x, y) to the boundary, and find the maximum value of the quenched variable $\eta(x', y')$ along sites (x', y') on the path. Call this $v(P)$ Then determine the minimum value of $v(P)$ over all paths P is $f^*(x, y)$.

Then on a large lattice, with (x, y) away from the boundary, the allowed values of $f^*(x, y)$ are $\geq p_{c,undirected}$.

However, calculating $f^*(x, y)$ is not trivial. We imagine that there is an agent at each site, who at each time step declares that, based on available information, he concludes that $f^*(x, y) \leq f(x, y, t)$, and shares this information with his neighbors.

Iterating this process of information sharing, we are able to compute $f^*(x, y)$.

In the beginning, the agent has no information, and $f(x, y, t = 0)$ is equal to the default value 1.

At $t = 1$, the sites at the boundary can use a lower value equal to their own random η.

Then at $t = 2$, the some of sites in the next layer are able to set their thresholds to a value lower than 1, and so on.

Eventually, when the f-values do not change, we have calculated the function $f^*(x, y)$.

Chase -Escape Percolation

As a final example of variations of the percolation problems, I will discuss the following variation of the first passage percolation problem:

We consider a d-dimensional lattice. Each site can be in one of three states: empty, occupied by a prey , or occupied by predator. For simplicity we will call these white, red and blue sites.

The initial state will a single blue site, with all its neighbors red, and all other sites empty.

The time evolution rule is this : A white site having a red neighbor becomes red with rate λ (preys reproduce). A red site having a blue neighbor becomes blue with rate 1 (predators need prey to eat and reproduce).

If there are no predators, the behavior of the model is quite simple: (this is a continuous time formulation of a model called the Eden model): The cluster of red sites grows with time, and the diameter of the cluster grows linearly with time.

If $\lambda > 1$, we get a red cluster that grows with time, and inside it a blue cluster grows with time, at a smaller rate.

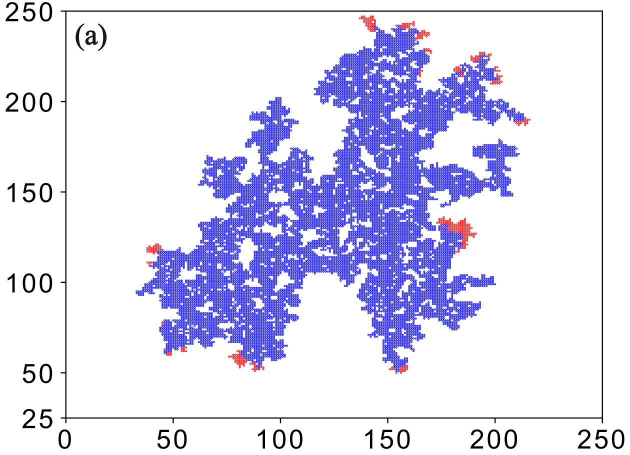
If $\lambda \ll 1$, the predators are in a hurry, and eat up the prey, before they have a chance to grow. Then the prey die out, and the predators cannot grow, as they need prey to reproduce. Then the system grows into an absorbing state with a non-growing cluster of blue sites, in a background of white, with no red sites.

So, there will be critical value λ^*, such that for $\lambda < \lambda^*$, all prey die, but for $\lambda > \lambda^*$, the prey can survive to large times with a non-zero probability.

The interesting, and non-obvious point is λ^* is much less than 1. In fact, it is closer to $1/2$.

How can the prey survive if predator can move much faster than them ?

The red can survive , even for some range of $\lambda < 1$, as red sites on the average have more available white neighbors, than the typical number of red neighbors of a blue site (even in the growth region).



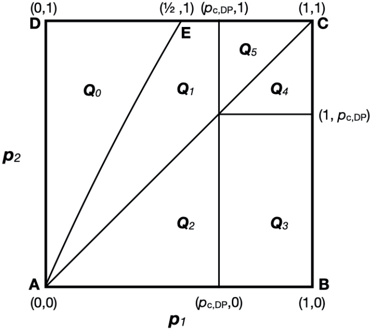
A configuration at $\lambda = 0.495$, near the critical value at $t = 500$. Note the irregular shape of frozen parts of the blue cluster. Some red sites are presents near the boundary of the growing cluster.

It is interesting that in this problem, one can can find both the directed and undirected critical thresholds.

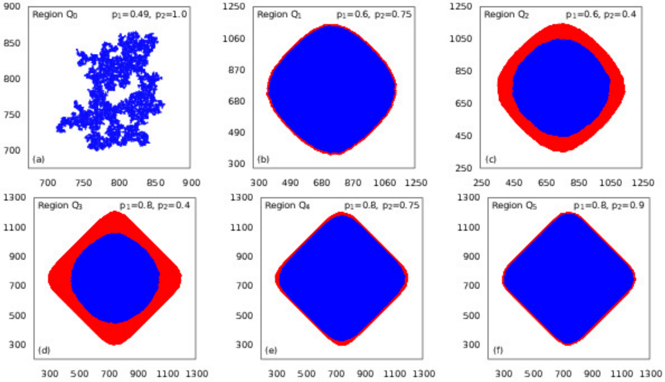
We define a discrete time version of the model as follows:

• Evoltion occurs with parallel update of all sites. A white site with a red neighbor at time t becomes red at time $(t+1)$ with probability p_1. A red site with a blue neighbor at time t becomes blue at time $(t+1)$ with probability p_2.

This problem has a more complicated phase diagram in the (p_1, p_2) plane.



These phases are distinguished by the behavior of red and blue boundaries at large times.



Region Q_0
p_1=0.49, p_2=1.0
(a)
Region Q_1
p_1=0.6, p_2=0.75
(b)
Region Q_2
p_1=0.6, p_2=0.4
(c)
Region Q_3
p_1=0.8, p_2=0.4
(d)
Region Q_4
p_1=0.8, p_2=0.75
(e)
Region Q_5
p_1=0.8, p_2=0.9
(f)

We note that the point E in this phase diagram must have the p_1-coordinate = the Undirected percolation threshold.

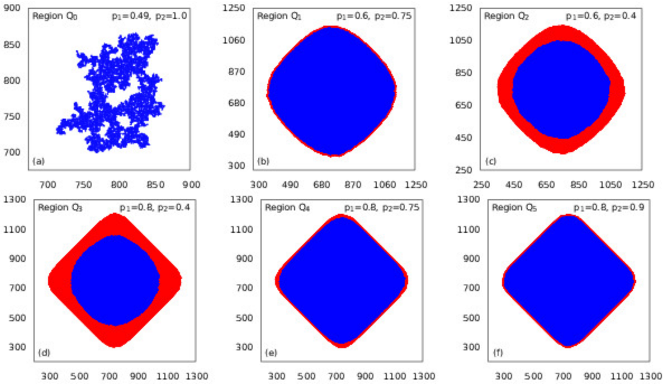
To prove this, we designate all the bonds of the lattice as 'active' or 'inactive', with probabilities p_1 and $(1 - p_1)$ at the beginning of the simulation itself.

A red site at time t will give rise to a neighboring site becoming red, IFF it is connected by an 'active' bond.

Thus, the threshold for survival of prey for long times is the critical concentration for undirected bond percolation on the lattice.

A key observation is that for the $p > p_{c,directed}$, in the discrete time evolution model, the asymptotic shape of the cluster has flat parts.

No red sites, or red boundary detached or not detached from blue boundary, and red or blue boundaries having flat segements or not.



Problems:

1. Can you construct a dynamics for a system of Ising spins that self-organizes to the critical point with only local rules?

2. Consider the chase-escape percolation with $\lambda > \lambda_c$, but $\lambda < 1$. Prove that the asymptotic velocity of the red front is strictly less than it would be if no predators were present.

THANK YOU

References

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