Stable model categories

(Edinburgh 3/20, 3×1hr)

model category: category with "homotopy" between morphisms

Def: A model cat. is a category $\mathcal{C}$ with three distinguished classes of morphisms

- weak equivalences $\xrightarrow{\sim}$
- cofibrations $\hookrightarrow$
- fibrations $\twoheadrightarrow$

(think: w.eq. = homotopy equiv.)

s.th.

(M1) $\mathcal{C}$ has finite limits and colimits

(M2) 2-out-of-3: If two out of f, g and $g \circ f$ are w.eq., then so is the third

(M3) Retracts A retract of a w.eq. /cof. / fib is again a w.eq. /cof. /fib.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & X \arrow[r] \arrow[d, "f"] & A \arrow[d, "g"] \\ B \arrow[r] & Y \arrow[r] & B \end{tikzcd}$$

(retract of f)

(M4) Lifts

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dotted, "h"] & Y \end{tikzcd}$$

A lift exist if

- $i \hookrightarrow$ and $f \overset{\sim}{\twoheadrightarrow}$ (acyclic fib)
- $i \overset{\sim}{\hookrightarrow}$ (acyclic cof) and $f \twoheadrightarrow$

(Rmk: this is $\Leftrightarrow$!)

(M5) Factorisation Every $f: X \to Y$ can be factored as

$$X \overset{\sim}{\hookrightarrow} A \twoheadrightarrow Y$$

$$X \hookrightarrow B \overset{\sim}{\twoheadrightarrow} Y.$$

(think: top. spaces)

Examples:

$\underline{Top}$: w.eq. = π_*-isos

cof = retracts of relative CW-complexes

fib:

$$\begin{tikzcd} A\times\{0\} \arrow[r] \arrow[d] & X \arrow[d] \\ A\times[0,1] \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

, A CW-complex

⤳ Serre model str. (there's others!)

$\underline{Ch(\mathcal{A})_{\geq 0}}$, $\mathcal{A}$ abelian cat. with enough projectives

w.eq. = H_*-isos

cof: $f_n: C_n \to D_n$ injective with proj. cokernel, $n\geq 0$

fib: f_n surjective, $n \geq 1$

⤳ projective model str. (also: injective)

also: $Ch(\mathcal{A})$ unbounded

$\underline{sSet}$: functors $\Delta^{op} \to Set$

Δ = finite ordered sets

f w.eq. $\Leftrightarrow$ $|f|$ π_*-iso

f cofibration $\Leftrightarrow$ $f([n]): X([n]) \to Y([n])$ inclusion

fibrations = "Kan fibrations"

$\underline{Sp}$ sequential spectra:

objects $X = (X_0, X_1, X_2, \ldots)$, $X_i \in Top_*$ + $\Sigma X_i \xrightarrow{\sigma_i} X_{i+1}$

morphisms ....

$$\begin{tikzcd} \Sigma X_i \arrow[r, "\Sigma f_i"] \arrow[d, "\sigma_i"'] & \Sigma Y_i \arrow[d, "\sigma_i"] \\ X_{i+1} \arrow[r, "f_{i+1}"] & Y_{i+1} \end{tikzcd}$$

$$\pi_n(X) = \operatorname{colim}_k \pi_{n+k}(X_k)$$

$\left(\to \pi_{n+k}(X_k) \xrightarrow{\Sigma} \pi_{n+k+1}(\Sigma X_k) \xrightarrow{\sigma_*} \pi_{n+k+1}(X_{k+1}) \to \ldots \right)$

ex: $\pi_n(\mathbb{S}^0) = \pi_n^{st} S^0$

$\mathbb{S}^0 = (S^0, S^1, S^2, \ldots)$

Def: X is cofibrant if $\emptyset \hookrightarrow X$ cofibration
X is fibrant if $X \twoheadrightarrow *$ fibration

Example: • in Top: CW-complexes are cofibrant
• in $Ch(A)_{\geq 0}$: cofibrant $\Leftrightarrow$ deg-wise projective
everything is fibrant
• all simplicial sets are cofibrant

mention spectra

cofibrant replacement: $\emptyset \hookrightarrow X^{cof} \overset{\sim}{\twoheadrightarrow} X$
fibrant replacement: $X \overset{\sim}{\hookrightarrow} X^{fib} \twoheadrightarrow *$

think: CW-approx. / proj. resolution

Homotopy

cylinder object: $X \sqcup X \overset{\sim}{\hookrightarrow} Cyl(X) \twoheadrightarrow X$

left homotopy:

$$\begin{tikzcd} X \sqcup X \arrow[r, "f \sqcup g"] \arrow[d] & Y \\ Cyl(X) \arrow[ur, "H"'] & \end{tikzcd} \qquad f \simeq_l g$$

homotopy without unit interval!

path object: $X \hookrightarrow P(X) \overset{\sim}{\twoheadrightarrow} X \times X$

right homotopy:

$$\begin{tikzcd} X \arrow[r, "(f,g)"] \arrow[dr, "H"'] & Y \times Y \\ & P(Y) \arrow[u] \end{tikzcd} \qquad f \simeq_r g$$

Properties: Let X, Y be fibrant & cofibrant. Then:
- left homotopic $\Leftrightarrow$ right homotopic ("homotopic") $\simeq$
- $\simeq$ equivalence relation on $\mathcal{C}(X,Y)$
- weak equivalence $\Leftrightarrow$ homotopy equivalence

<u>Def</u>: homotopy category of $\mathcal{C}$:

$ob(Ho(\mathcal{C})) = ob(\mathcal{C})$

$Ho(\mathcal{C})(X, Y) = \mathcal{C}(X^{cof}, Y^{fib}) / \underline{\sim} =: [X, Y]$

<u>Ex</u>: derived category $\mathcal{D}(\mathcal{A}) = Ho(Ch(\mathcal{A}))$

(stable homotopy category $SHC = Ho(Sp)$)

$[A[n], B[m]] = Ext_{\mathcal{A}}^{m-n}(A, B)$

<u>Def</u>: Quillen adjunction $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$:

- F preserves $\hookrightarrow$ and $\overset{\sim}{\hookrightarrow}$

$\Leftrightarrow$ • G preserves $\twoheadrightarrow$ and $\overset{\sim}{\twoheadrightarrow}$.

$\rightsquigarrow$ induces adjunction $Ho(\mathcal{C}) \underset{RG}{\overset{LF}{\rightleftarrows}} Ho(\mathcal{D})$

Quillen equivalence: LF, RG are equivalences of cat's.

"Quillen equivalent model categories have the same homotopy theory" e.g. $sSet \underset{Q}{\simeq} Top$

<u>Stability</u> (pointed model cats from now on)

The suspension and loop functors:

$$\begin{tikzcd} X \amalg X \arrow[r] \arrow[d] & Cyl(X) \arrow[d] \\ * \arrow[r] & \Sigma X \end{tikzcd} \qquad \begin{tikzcd} \Omega X \arrow[r] \arrow[d] & P(X) \arrow[d] \\ * \arrow[r] & X \times X \end{tikzcd}$$

$\rightsquigarrow$ functors $\Sigma, \Omega : Ho(\mathcal{C}) \longrightarrow Ho(\mathcal{C})$

$f: \Sigma X \to Y$ is the same as a left homotopy from $0: X \to Y$ to $0: X \to Y$

$g: X \to \Omega Y$ is the same as a right homotopy from 0 to 0

Thm: $\Sigma : Ho(\mathcal{C}) \rightleftarrows Ho(\mathcal{C}) : \Omega$ is an adjunction

Proof via correspondence left homotopy $\leftrightarrow$ right homotopy

Def: $\mathcal{C}$ is stable if Σ is an equivalence.

Ex: Top_* is not stable: $\pi_2(S^1) = 0$, $\pi_3(S^2) = \mathbb{Z}$, $\pi_4(S^3) = \mathbb{Z}/2$

$Ch(\mathcal{A})$ is stable: $\Sigma C_* = C_{*-1}$, $\Omega C_* = C_{*+1}$

Sp is stable: $X \to \Omega \Sigma X$ is a π_*-isomorphism

Σ, Ω applied levelwise

(proof:
$$\begin{tikzcd}
\dots \arrow[r] & \pi_{n+k}(X_k) \arrow[r] \arrow[d] & \pi_{n+k+1}(\Sigma X_k) \arrow[r] \arrow[d] & \pi_{n+k+1}(X_{k+1}) \arrow[r] \arrow[d] & \dots \\
& \pi_{n+k}(\Omega\Sigma X_k) \arrow[r] & \pi_{n+k+2}(\Sigma^2 X_k) \arrow[r] & \pi_{n+k+2}(\Sigma X_{k+1}) \arrow[r] & \dots
\end{tikzcd}$$
$= \pi_{n+k+1}(\Sigma X_k)$

$\rightsquigarrow$ same colim)

Thm If $\mathcal{C}$ is stable, then $Ho(\mathcal{C})$ is triangulated.

exact triangles: $\underbrace{A \to B \to C}_{\text{cofibre seq.}} \to \Sigma A$

Rmk: cofibre sequences $\rightsquigarrow$ triangulated str. $\rightsquigarrow$ cofibre seq. $=$ fibre seq.

I: Chromatic basics

contains everything: cws, H_*, algebra..! $\exists X$----

goal: study stable homotopy category $SHC = Ho(Sp)$ ← Sp model category with w.eq. = π_*-isos, cofibs + fibs

$\rightsquigarrow$ invert "W-equivalences" instead of π_*-isos

e.g. E_*-isos, rational htpy theory

break up into more manageable pieces: CHT

machinery for inverting:

W class of maps in Sp:

- X is <u>W-local</u> if $\forall f: A \to B$ in W, $f^*: [B, X] \xrightarrow{\cong} [A, X]$
- $f: C \to D$ <u>W-equivalence</u> if $f^*: [D, X] \xrightarrow{\cong} [C, X]$ $\forall X$ W-local.

<u>N.B:</u> $W \subseteq$ W-equivalences $\underset{=}{\cancel{\subsetneq}}$ W-(W-equivalences)

W-acyclic: $[Z, X] = 0$ $\forall$ local X

also, π_*-isos $\subseteq$ W-equivalences

<u>Def:</u> W-local model structure on Sp (if exists): $\boxed{L_W Sp}$

weak equiv. = W-equivalences

cofibs = old cofibs

fibrations = what they have to be $\rightsquigarrow$ form $\underset{\text{W-local SHC}}{Ho(L_W Sp)}$

<u>Properties</u> if exists

- $id: Sp \rightleftarrows L_W Sp : id$ Quillen adjunction
- fibrant replacement gives $X \xrightarrow[\text{W-equivalence}]{} \underset{\text{local object}}{L_W X}$
- universal property:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \\ L_W X \arrow[ur, dashed] & \end{tikzcd}$$

$\leftarrow$ W-local

need W to be a set, find suitable set J_E

$\boxed{\text{Good news:}}$ For E a homology theory, $L_E Sp$ with w.eq. $= E_*$-isos $= \pi_*(E \wedge \_)$-isos exists!

<u>Examples:</u> $E = H\mathbb{Q}$: $S^0 \longrightarrow L_{H\mathbb{Q}} S^0 = H\mathbb{Q}$ ← something with $\pi_*(X)$ already rational

π_*-iso after smashing with $H\mathbb{Q}$ and $H\mathbb{Q}$ detects $H\mathbb{Q}$-isos

(PTO)

X $H\mathbb{Q}$-local $\Leftrightarrow [B, X] \xrightarrow{\cong} [A, X]$ for $A \to B$ $H\mathbb{Q}$-equiv.

$\Leftrightarrow [B, X \wedge H\mathbb{Q}] \cong [A, X \wedge H\mathbb{Q}]$ as $X = X \wedge H\mathbb{Q}$

$\Leftrightarrow [B \wedge H\mathbb{Q}, X] \cong [A \wedge H\mathbb{Q}, X]$ ← true as $A \wedge H\mathbb{Q} \to B \wedge H\mathbb{Q}$ w.eq.

e p-localisation: $E = H\mathbb{Z}_{(p)}$

(similar to $H\mathbb{Q}$) $L_{H\mathbb{Z}_{(p)}} X =: X_{(p)} = X \wedge H\mathbb{Z}_{(p)}$

$X \to X \wedge H\mathbb{Z}_{(p)}$ is $H\mathbb{Z}_{(p)}$-iso + $X_{(p)} \to X_{(p)} \wedge H\mathbb{Z}_{(p)}$ π_*-iso

$\pi_*(X_{(p)}) = \pi_*(X) \otimes \mathbb{Z}_{(p)}$ (universal coefficient s. ex. seq.)

in particular, $S^0_{(p)} = H\mathbb{Z}_{(p)}$, $X_{(p)} = X \wedge S^0_{(p)}$ ⇝ useful to study SHC and $\pi_* S^0$

<u>Def:</u> A localisation is <u>smashing</u> if $L_E X = X \wedge L_E S^0$.

- smashing localisations preserve compact objects
- $L_E S^0$ is a compact generator of $Ho(L_E Sp)$
- L_E commutes with coproducts

⋮ and much more

<u>Example</u> p-completion $E = H\mathbb{Z}/p$

$L_{H\mathbb{Z}/p} X =: X^\wedge_p$

If $\pi_n(X)$ is finitely generated, then $\pi_n(X^\wedge_p) = \pi_n(X) \otimes \mathbb{Z}^\wedge_p$

$(S^0)^\wedge_p = Map(\Sigma^{-1} H(\mathbb{Z}/p^\infty), S^0)$ ⇝ <u>not</u> smashing.

<u>Bousfield squares</u>

pullbacks

$$\begin{array}{ccc} X & \longrightarrow & \prod_p X^\wedge_p \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & L_{\mathbb{Q}}(\prod_p X^\wedge_p) \end{array} \qquad \begin{array}{ccc} X & \longrightarrow & X_{(P_1)} \\ \downarrow & & \downarrow \\ X_{(P_2)} & \longrightarrow & L_{\mathbb{Q}} X \end{array}$$

In algebra, this is "enough" – in homotopy theory, one can do more!

⇝ chromatic homotopy theory

MU complex cobordism

$MU_* = \mathbb{Z}[x_1, x_2, \ldots\ldots]$ $|x_i| = 2i$ "universal"

$MU_{(p)} = \bigvee_{\ldots} \Sigma^{\ldots} BP$, $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots\ldots]$ $|v_i| = 2p^i - 2$

⇝ from now on, everything is p-local

...nson-Wilson spectra $E(n)$ with $E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_n, v_n^{-1}]$
Morava-K-theories $K(n)$ with $K(n)_* = \mathbb{Z}/p[v_n, v_n^{-1}]$

$\boxed{E(n):}$ start with $BP \xrightarrow{\text{kill hpy}} BP\langle n\rangle_* = \mathbb{Z}_{(p)}[v_1, \dots, v_n]$

$\xrightarrow{\text{invert } v_n^{-1}} E(n) = \mathrm{colim}(BP\langle n\rangle \xrightarrow{v_n} \Sigma^{-(2p^n-2)} BP\langle n\rangle \xrightarrow{v_n} \dots)$

$\boxed{K(n):}$ $BP \xrightarrow{\text{kill hpy}} k(n)_* = \mathbb{Z}/p[v_n] \xrightarrow{\text{invert}} K(n) = \mathrm{colim}(k(n) \xrightarrow{v_n} \dots)$

Convention: $E(0) = K(0) = H\mathbb{Q}$

$\boxed{n=1:}$ K-theory $\xrightarrow{?} E(1)$ $\quad E(1)_* = \mathbb{Z}_{(p)}[v_1^{\pm 1}]$

$\phantom{K\text{-theory}} \xrightarrow{?} K(1)$ $\quad K(1)_* = \mathbb{Z}/p[v_1^{\pm 1}]$

X is KU-local $\iff$ X is KO-local

exact triangle $\Sigma KO \xrightarrow{\eta \wedge 1} KO \longrightarrow KU \longrightarrow \Sigma^2 KO$

$\Rightarrow$ $KO \wedge X = 0 \Rightarrow KU \wedge X = 0$

Assume $KU \wedge X = 0 \Rightarrow \Sigma KO \wedge X \xrightarrow{\eta \wedge 1 \wedge X} KO \wedge X$ iso

but $\eta^4 = 0$, so $KO \wedge X = 0$.

$\Rightarrow L_{K_{(p)}} = L_{KO_{(p)}} = L_{KU_{(p)}} = L_{E(1)} =: L_1$

[Adams]: $K^*_{(p)}(X) = \bigoplus_{0 \le i \le p-2} \Sigma^{2i} G^*(X)$

"Adams summand" $G = E(1)$

$KU_{(p)*} = \mathbb{Z}_{(p)}[\beta^{\pm 1}] \underset{\text{at } p=2}{\overset{=}{\pi}} E(1)_* \Rightarrow KU_{(2)} = E(1)$

$|\beta| = 2$

$L_{K(1)} X = L_{E(1) \wedge M\mathbb{Z}/p} X = (L_{E(1)} X)^\wedge_p$

Recall: need set J_E s.th. J_E-equivalences = E_*-isos

For K-theory, just need one map.

$M = M\mathbb{Z}/p$ has a v_1-self map $v_1: \Sigma^{2p-2} M \longrightarrow M$ $(p>2)$

$v_1^4: \Sigma^8 M \longrightarrow M$ $(p=2)$

c] X $E(1)$-local $\Leftrightarrow$ $[M,X]_* \xrightarrow{(v_1)^4} [M,X]_{*+2p-2}$ iso

$\rightsquigarrow L_1 X = L_{\{v_1\}} X$

Iso: $\underset{E(1)}{K}$-localisation is smashing, i.e. $L_1 X \simeq X \wedge L_1 S^0$.

$\rightsquigarrow \pi_* L_1 S^0 = ?$

Relationship between $E(1)$ and $E(0)$:
If X is rational, then it is K-local. (obvs, not the other way.)

$$L_1 X = L_1 L_{H\mathbb{Q}} X = X \wedge L_{H\mathbb{Q}} S^0 \wedge L_1 S^0 = X \wedge L_{H\mathbb{Q}} S^0 = X$$ $\square$

<u>What about the higher n?</u>

$n=2 \rightsquigarrow$ elliptic cohomology theories

but in general, the interaction between the levels $/K(n)/E(n)$ is interesting in itself.

~~smashing~~ smashing? $L_n := L_{E(n)}$ is smashing [Ravenel]

$L_{K(n)}$ is not smashing: take a $K(n)$-local spectrum E s.th. $L_{H\mathbb{Q}} E \neq *$.

Assume $L_{K(n)}$ was smashing: $E = E \wedge L_{K(n)} S^0$
$L_{K(n)} H\mathbb{Q} = H\mathbb{Q} \wedge L_{K(n)} S^0 \simeq *$

$$\rightsquigarrow 0 \neq \pi_*(E \wedge H\mathbb{Q}) = \pi_*(E \wedge \underbrace{L_{K(n)} S^0 \wedge H\mathbb{Q}}_{=*}) = 0$$ ↯

(Does such an E exist? yes - $E = E_n$.)

Landweber exactness: M_* BP_*-module $\rightsquigarrow$ $\forall n$: $(p, v_1, v_2, \ldots, v_n)$ regular sequence for M, i.e. v_i not zero-divisor for $M/(p, \ldots, v_{i-1})M$

$\rightsquigarrow$ explicit algebraic conditions s.th.
$$M_*(X) := BP_*(X) \otimes_{BP_*} M_*$$ is a homology theory.

$E(n)_*$ is Landweber exact
$K(n)_*$ is not.

Künneth iso $K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y) \cong K(n)_*(X \wedge Y)$ ($K(n)$ graded field)

but nothing like that for $E(n)$.

nce [Hopkins-Smith]

$f: X \to Y$ smash nilpotent $\Leftrightarrow$ $K(n)_* f = 0$, $0 \le n \le \infty$

f nilpotent $\Leftrightarrow$ $K(n)_* f = 0$, $0 \le n < \infty$, ($K(\infty) = H\mathbb{F}_p$)

periodicity Are there any maps on a spectrum X that never die?

Let n be the largest integer s.th. $K(m)_*(X) = 0$, $m < n$

$\Rightarrow$ X has a v_n-self map $\alpha: \Sigma^d X \to X$, i.e.

- $K(n)_* \alpha$ is mult. by v_n^k for some k
- $K(m)_* \alpha = 0$ for $m > n$.

...tion between $E(n)$ and $K(n)$ ⑤

Theorem [Ravenel] $L_n = L_{(K(0) \vee K(1) \vee \dots \vee K(n))}$

Corollary $E(n+1)_*(X) = 0 \Rightarrow E(n)_*(X) = 0$

X is $E(n)$-local $\Rightarrow X$ is $E(n+1)$-local (remember: rational $\Rightarrow$ K-local)

$\rightsquigarrow$ nat. trf. $L_{n+1} \to L_n$

$E(n)_*(X) = 0 \Rightarrow K(n)_*(X) = 0$

X $K(n)$-local $\Rightarrow X$ $E(n)$-local $\rightsquigarrow$ nat. trf. $L_n \to L_{K(n)}$

$\rightsquigarrow$ homotopy pullback square

$$\begin{tikzcd}
L_n X \arrow[r] \arrow[d] & L_{K(n)} X \arrow[d] \\
L_{n-1} X \arrow[r] & L_{n-1} L_{K(n)} X
\end{tikzcd}$$

Chromatic convergence [Ravenel]

X p-localisation of finite CW-spectrum

$\Rightarrow \quad X \simeq \operatorname{holim}(L_0 X \leftarrow L_1 X \leftarrow L_2 X \leftarrow \dots\dots)$

X needs to be finite: $L_n HG = HG_{\mathbb{Q}}$

Thick Subcategory Theorem

$\mathcal{F}$ thick subcat. of triangulated category $\mathcal{T}$:
full subcat. closed under Δ and retracts.

The nontrivial thick subcat. of $Ho(\mathcal{S}p_{(p)})^{\omega}$ are the

$\mathcal{F}_n = \{X \text{ finite } p\text{-local}, K(n-1)_*(X)\} = 0.$

$\rightsquigarrow$ "atomic pieces"

more generally:

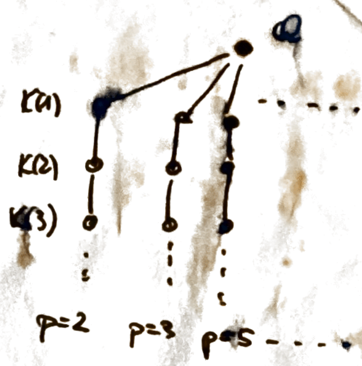
$\mathcal{T}$ $\otimes$-category (tensor-triangulated, e.g. $Ho(\mathcal{C})$ stable monoidal model cat)

$\mathcal{F}$ thick subcategory is an ideal if $X \in \mathcal{T}, Y \in \mathcal{F} \Rightarrow X \wedge Y \in \mathcal{F}$

prime ideal: $X \wedge Y \in \mathcal{F} \Rightarrow X \in \mathcal{F}$ or $Y \in \mathcal{F}$

the $K(n)$-acyclics, for each p, form the thick prime ideals of $Ho(Sp)^\omega$

⇝ "Balmer spectrum"



inclusion of acyclics

can study Balmer spectrum of other ⊗-categories, e.g. $Ho(G\text{-}Sp)^\omega$ — G compact Lie

$$\mathcal{P}(\underbrace{H}_{\text{subgp of } G}, \underbrace{p}_{\text{prime}}, n) = \{ X \in Ho(G\text{-}Sp)^\omega \mid K(n)_*(\underbrace{\Phi^H}_{\text{geometric fixed points}}(X)) = 0 \}$$

<u>return to chromatic square:</u>

$$\begin{tikzcd} L_n S^0 \arrow[r] \arrow[d] & L_{K(n)} S^0 \arrow[d] \\ L_{n-1} S^0 \arrow[r] & L_{n-1} L_{K(n)} S^0 \end{tikzcd}$$

⇝ $L_n S^0\text{-mod} = L_n Sp \underset{Q}{\cong} \left(L_{n-1} Sp \longrightarrow L_{n-1} L_{K(n)} S^0\text{-mod} \longleftarrow L_{K(n)} S^0\text{-mod} \right)$

with homotopy limit model structure [Balchin - Greenlees]

$C_0 \xrightarrow{F_0} C_{01} \xleftarrow{F_1} C_1$ ⇝ col. objects

s.th. in htpy category $F_0(X_0) \cong F_1(X_1)$

- rigidity + exotic objects
- adelic rigidity

In general,

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \Lambda_w \mathcal{C} \arrow[d] \\ L_w \mathcal{C} \arrow[r] & (\ldots) \end{tikzcd}$$

OR

<u>blue shift</u> G finite group, K, H p-groups, $s = \log_p(|H/K|)$

$\rightsquigarrow \mathcal{P}(K,p,n+s) \subseteq \mathcal{P}(H,p,n)$

What is the minimal i with $\mathcal{P}(K,p,n+i) \subseteq \mathcal{P}(H,p,n)$?

"nth blue shift"

<u>$L_{K(2)}S^0$</u>

$$\cdots \to KO_{(2)} \xrightarrow{\psi^3 - 1} KO_{(2)} \to L_1 S^0 \to \Sigma KO_{(2)} \to \cdots$$

Adams op (+ complete version)

$$\left(\cdots \to K_{(p)} \xrightarrow{\psi^\ell - 1} K_{(p)} \to L_1 S^0 \to \Sigma K_{(p)} \to \cdots \right)$$

$\rightsquigarrow$ can calculate $\pi_* L_1 S^0$ from l.ex. seq. and ASS

Morava-E-theories / Lubin-Tate spectra:

E_n $K(n)$-local spectrum with $(E_n)_* = \mathbb{W}(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]][u^{\pm 1}]$

$\circlearrowleft \mathbb{G}_n$ Morava stabilizer group $\qquad |u_i| = 0 \quad |u| = -2$

$\rightsquigarrow$ cofiber sequence $L_{K(1)}S^0 \simeq E_1^{h\mathbb{G}_1} \to \underbrace{E_1^{hC_2}}_{= KO_2^\wedge} \to E_1^{hC_2}$

[Goess et al]

$$L_{K(2)}S^0 \to E_2^{hG_{24}} \to \dots \to \dots \to \dots \to \Sigma^{48} E_2^{hG_{24}}$$

5 terms s.th. $L_{K(2)}S^0 = \lim(\dots)$ $\qquad (p=3)$

$\rightsquigarrow$ extending this range [Goess-Henn, Rezk, Stojanoska, Bobkova]

chromatic square

$$\begin{tikzcd} L_n S^0 \arrow[r] \arrow[d] & L_{K(n)} S^0 \arrow[d] \\ L_{n-1} S^0 \arrow[r] & L_{n-1} L_{K(n)} S^0 \end{tikzcd}$$

[Balchin-Greenlees] ~~Take $\mathcal{K} \subseteq \mathcal{C}$ set of compact objects~~

$$L_n \mathrm{Sp} = L_n S^0\text{-mod} \underset{\text{Quillen}}{\simeq} \left(\begin{tikzcd} & L_{K(n)} S^0\text{-mod} \arrow[d] \\ L_{n-1} S^0\text{-mod} \arrow[r] & L_{n-1} L_{K(n)} S^0\text{-mod} \end{tikzcd} \right)$$

with homotopy limit model structure

$$\mathcal{C}_0 \xrightarrow{F_0} \mathcal{C}_{01} \xleftarrow{F_1} \mathcal{C}_1$$

cofibrant objects: $F_0(X_0) \cong F_1(X_1)$ in htpy cat

$\mathcal{K}$ set of compact objects:

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \Lambda_{\mathcal{K}} \mathcal{C} \arrow[d] \\ L_{\mathcal{K}} \mathcal{C} \arrow[r] & (\dots) \end{tikzcd}$$

if $\mathcal{C} = \mathbb{1}\text{-mod}$:

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \Lambda_{\mathcal{K}} \mathbb{1}\text{-mod} \arrow[d] \\ L_{\mathcal{K}} \mathbb{1}\text{-mod} \arrow[r] & L_{\mathcal{K}} \Lambda_{\mathcal{K}} \mathbb{1}\text{-mod} \end{tikzcd}$$

__Rigidity questions__ $\mathcal{C} \simeq \mathcal{D} \Rightarrow \mathrm{Ho}(\mathcal{C}) \cong \mathrm{Ho}(\mathcal{D})$, but not necessarily "$\Leftarrow$". $\rightsquigarrow$ $\mathrm{Ho}(\mathcal{C})$ "rigid"

__Ex:__ $\mathrm{Ho}(K(n)\text{-mod}) \underset{\Delta}{\cong} \mathcal{D}(K(n)_*\text{-mod})$

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[Schwede] $\mathrm{Ho}(\mathcal{S}p) \underset{\Delta}{\cong} \mathrm{Ho}(\mathcal{D}) \Rightarrow \mathcal{S}p \underset{QE}{\simeq} \mathcal{D}$

[R] $(p=2)$ $\mathrm{Ho}(L_1 \mathcal{S}p) \cong \mathrm{Ho}(\mathcal{D}) \Rightarrow L_1 \mathcal{S}p \underset{QE}{\simeq} \mathcal{D}$

$(p>2)$: not true [Franke, Patchkoria – Pstrągowski]

Idea: $\mathrm{Ho}(\mathcal{S}p) = \mathrm{End}(S^0)\text{-mod}$

$\mathrm{Ho}(\mathcal{D}) = \mathrm{End}(X)\text{-mod}$ (X compact gen.)

$\pi_* \mathrm{End}(S^0) = \pi_* \mathrm{End}(X)$

$\Rightarrow (\dots)$ $\mathrm{End}(X) = S^0$

and K-locally $\mathrm{End}(X) = L_1 S^0$

$\rightsquigarrow$ difficulties for $\mathrm{Ho}(L_2 \mathcal{S}p)$

but: <u>Pre-Theorem</u> [Balchin – R – Williamson]

(tt-rigid: ask for tt-equivalence $\mathrm{Ho}(\mathcal{C}) \cong \mathrm{Ho}(\mathcal{D})$)

unitally tt-rigid: —— " ——, and the QE $F: \mathcal{C} \to \mathcal{D}$ sends unit to unit)

- $L_n \mathcal{S}p$ is unitally tt-rigid $\Leftrightarrow$ $L_{K(i)} \mathcal{S}p$ unitally tt-rigid for $1 \le i \le n$.
- If $L_I \mathcal{C}$ and $\Lambda_I \mathcal{C}$ are unitally tt-rigid, then so is $\mathcal{C}$.