Continued Fractions: Exploration Sheet

Biman B Nath, RRI

- Based on the method of continued fraction, show that

$$3.1416 = 3 + \cfrac{1}{7 + \frac{1}{16 + \frac{1}{11 + ...}}} \,. \tag{1}$$

From this, derive the approximation for $\pi \approx 22/7$. Also, show that the next best approximation is 355/113. This is known as the Metius number, after Adrian Adrianszoon (Metius) who found it around 1585.

- Show the following:

$$\begin{aligned} \sqrt{3} &= [1; 1, 2, 1, 2, 1, 2, ...] \\ \sqrt{5} &= [2; 4, 4, 4, 4, ...] \\ \sqrt{6} &= [2; 2, 4, 2, 4, 2, 4...] \\ \sqrt{7} &= [2; 1, 1, 1, 4, 1, 1, 1, 4...] \,. \end{aligned} \tag{2}$$

Notice the periodicity in the numbers in the brackets. Lagrange proved that this is not unexpected and that a real number has a periodic continued fraction expansion if and only if the real number is a real quadratic irrational. There are other interesting characteristics here. If you take the last number in the period (say, in $\sqrt{6}$, the numbers 2,4,2,4. The last number 4 is the double of $a_0 = 2$). Some quadratic irrationals have longer periods, and the numbers in between form a palindrome:

$$\sqrt{23} = [4; 1, 3, 1, 8, 1, 3, 1, 8, ...] \,. \tag{3}$$

Here, the last number in the periodic sequence 1,3,1,8 is double of $a_0 = 4$. Also, the numbers in the middle (1,3,1) form a palindromic sequence!

When it comes to the partial quotients in the continued fraction for π, there are many interesting questions that number theorists ponder on... But we will go on to a more immediate problem, that of leap years.

- The time taken by the Sun to return to the same position with respect to the fixed stars is what we call the Solar year or Tropical year. And it is roughly 365 days 5 hours 48 minutes 46 seconds. Let us find the equivalent continued fraction. We can ignore the integral component (365) and only deal with the decimal fraction. 5 h 48 m 46 s equals 20926 seconds. We need to divide this by the number of seconds in a day, which is 86400. Dividing both by 2, we have the number 10463/43200. Let us invert it to get 43200/10463 for which we need to find the best rational approximation.
  Show that as a first approximation, 1 leap year in a 4 year cycle works. The next best approximation is to have 7 leap years in 29 years, and that the next one suggests 8 leap years in 33 years. This is what Omar Khayyam suggested in 1079.

- Next, consider the periodicity of Solar eclipses. The Babylonians noticed Solar eclipses recur after 18 years and 6 months. We can try to understand the Saros cycle in the following manner in retrospect. One notices that there are three distinct periods regarding the motion of the Moon around the Earth. (1) There is a period from full moon to full moon, called the Synodic month, which lasts 29.530589 days, or 29 days 12 hours 44 minutes and 3 seconds. This is somewhat longer than the actual period of revolution of the Moon around the Earth (roughly 27.5 days) , since by the time the Moon goes around once, the Earth would have moved further along in its orbit, and it takes roughly 2 more days to come to the same alignment with the Sun. (2) Then there is another time period regarding Moon's orbit. Note that the Moon's orbit is inclined to the plane of Earth's orbit around the Sun, and they intersect at two points, which are called 'nodes'. The time period between going from one node to the other is called Draconic month, and it is 27.212221 days, or 27 days 05 hours 5 minutes and 36 seconds.
  Any two successive solar eclipses would come after a period of time that is an integral multiple of all these two months. If one can find a common time period which is an integral multiple of Synodic and Draconian months, then the eclipses would share the same geometry. And this is what the Chaldeans determined. Their Saros cycle, within a few hours, consists of 223 Synodic months, 239 anomalistic months, and 242 draconic months.
  Let us call the first period (Synodic month) 'SM', and the second period is half of Draconic month, and we can write it as 'DM/2'. We need

to find the smallest common multiple. we have (DM/2)/(SM)=2.170391682. The best way to find a smallest common multiple (within a given accuracy) is to look for the continued fraction expansion of it. Show that the continued fraction for this number is $[2; 5, 1, 6, 1, 1, 1, 1...]$, and the approximations are,

$$\frac{2}{1}, \frac{11}{5}, \frac{13}{6}, \frac{89}{41}, \frac{102}{47}, \frac{191}{88}, \frac{293}{135}, \frac{484}{223}, ... \tag{4}$$

It is the 8th ratio that the Chaldeans used, corresponding to 223 Synodic months.

- We know that initially the time keeping for the year was done win the help of lunar months– because the lunar phases are more conspicuous than, say, equinoxes and solstices. But then it was realised that a lunar calendar, based on lunar months, are not good for keeping track of seasons, which are ultimately important for farming and so on. So, some method was found that can make a lunar calendar consistent with a solar calendar. In ancient India, for example, there was this custom of adding an 'extra month' (adhika-masa) every three solar years.
  The basic problem is that the tropical year (that tracks seasons) is not a multiple of the synodic lunar month, which we have encountered earlier. The tropical year is 365.2422 days, while a Synodic lunar month is 29.5306 days (the interval between two full moons). So, a tropical year is longer than 12 lunar month, but less than 13 lunar months. It is said that Meton of Athens, in the 5th century BC, realised that a particular lunar phase falls on the same date in the solar calendar after 19 years. He found that 235 synodic lunar months corresponds to 19 solar years. This is therefore called the Metonic cycle. It also gave rise to the Metonic calendar, in which there were 12 years of 12 lunar months, followed by 7 years with 13 lunar months. The famous Antykherra mechanism is based on the Metonic cycle of 19 years.
  Based on the method of continued fraction, show that the ratio of a tropical year and a synodic lunar month can be expressed as a continued fraction, as follows:

$$\frac{\text{Tropicalyear}}{\text{Synodic Lunar Month}} = 12 + \frac{1}{2 + \frac{1}{1+\frac{1}{2+\frac{1}{1+\frac{1}{1+\frac{1}{16+\frac{1}{1+\frac{1}{5+...}}}}}}}} \,. \tag{5}$$

Show that the approximation 37/3 corresponds to the ancient India's custom of adding one extra month every 3 years. The approximation of 235/19 is even better, and is the familiar Metonic cycle. Note that the next accurate approximation is given by 315 solar years (corresponding to 3896 lunar months). Clearly, this was deemed to be too long to be of any practical use. Therefore Metonic cycle has been extensively used.