Maximal surfaces in Lorentz-Minkowski 3-space

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Hypersurfaces in $\mathbb{L}^{n+1}$

$\mathbb{L}^{n+1}$: $(n+1)$-dim. Lorentz-Minkowski space.
$\langle\ ,\ \rangle := (dx^1)^2 + \cdots + (dx^n)^2 - (dx^{n+1})^2$.
U a compact set in $\mathbb{R}^n$, $u = (u^1, \ldots, u^n) \in U$.

Definition

An immersion $f : U \to \mathbb{L}^{n+1}$ is spacelike
$:\Longleftrightarrow g := f^* \langle\ ,\ \rangle$ is positive definite.

$$H^n_{\pm} := \{x = (x^1, \ldots, x^{n+1}) \in \mathbb{L}^{n+1} ;\ \langle x, x\rangle = -1,\ \ \pm x^{n+1} > 0\}.$$

$f : U \to \mathbb{L}^{n+1}$ a spacelike immersion.
ν the unit normal v.f. along f. i.e.

$\nu : U \to H^n_+$ or $\nu : U \to H^n_-$ such that

$$\langle f_* X, \nu\rangle = 0 \quad (\forall X \in T_pU), \quad \text{and} \quad \langle \nu, \nu\rangle = -1.$$

The first and second variation of the volume

$f : U \to \mathbb{L}^{n+1}$ a spacelike imm., f_t a variation of f,
ν the unit normal v.f. on U along f,
H the mean curvature of f, dV the volume element of f.

The first variation of the volume of f

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Vol}(f_t) = -n \int_U \beta H dV, \qquad \beta := -\left\langle (f_t)_* \left.\frac{\partial}{\partial t}\right|_{t=0}, \nu \right\rangle$$

Theorem

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Vol}(f_t) = 0 \text{ for } \forall f_t \text{ variation of } f \iff H \equiv 0.$$

The second variation of the volume of f

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathrm{Vol}(f_t) = -\int_U \beta \left(\triangle_g \beta - \beta |A|^2\right) dV$$
$$= -\int_U \left(|\nabla_g \beta|^2 + \beta^2 |A|^2\right) dV$$

Stability

Definition

Suppose that the first variation of $f : U \to \mathbb{L}^{n+1}$ vanishes for any variation (i.e. $H \equiv 0$). f is stable if the socond variation of f is always positive or always negative for any non-trivial variation.

Since $\langle \nu, \nu\rangle < 0$ and the $(n+1)$ th component of ν does not change, we have the following.

Proposition

$f : U \to \mathbb{L}^{n+1}$ an spacelike imm. with $H \equiv 0$. Then the second variation of the volume of f is always negative. That is, f is stable and has maximal volume.

A spacelike immersion with $H \equiv 0$ is called the maximal hypersurfaces.

Graph hypersurface

U a domain in $\mathbb{R}^n$, $u = (u^1, \ldots, u^n) \in U$, $\varphi : U \to \mathbb{R}$.
The mean curvature H of the spacelike graph hypersurface φ satisfies.

$$nH = \mathrm{div}\left(\frac{\nabla\varphi}{\sqrt{1 - |\nabla\varphi|^2}}\right), \quad |\nabla\varphi|^2 < 1.$$

Definition

$$\mathrm{div}\left(\frac{\nabla\varphi}{\sqrt{1 - |\nabla\varphi|^2}}\right) = 0 \qquad (|\nabla\varphi|^2 < 1)$$

is called the maximal hypersurface equation.

Remark

When $n = 2$, set $(u^1, u^2) = (x, y)$. Then the above eqn. is equivalent to

$$\left(1 - \varphi_y^2\right)\varphi_{xx} + 2\varphi_x\varphi_y\varphi_{xy} + \left(1 - \varphi_x^2\right)\varphi_{yy} = 0, \quad \varphi_x^2 + \varphi_y^2 < 1$$

This eqn is called the maximal surface equation.

Bernstein-type problem

Theorem (E. Calabi, 1968)

The graph maximal surface φ defined on the entire $\mathbb{R}^2$ must be a plane.

Remark

Without the assumption $\varphi_x^2 + \varphi_y^2 < 1$, $\exists\varphi$ nonlinear. For example,

$$\varphi(x, y) = \log\cosh x - \log\cosh y.$$

Theorem (Calabi 1968, Cheng-Yau 1976)

The graph maximal hypersurface φ defined on the entire $\mathbb{R}^n$ must be a hyperplane.

Other global properties:

- Complete maximal hypersurface must be a hyperplane (Calabi, Cheng-Yau).
- $\nexists$ nonorientable spacelike hypersurfaces.

Weierstrass-type representation

In the following, we assume $n = 2$.
$f : U \ni (x, y) \mapsto (f_1(x, y), f_2(x, y), f_3(x, y)) \in \mathbb{L}^3$ a maximal surface, (x, y) the isothermal coordinate of f.

Weierstrass-type representation (O. Kobayashi, 1983)

$$f = \mathrm{Re}\int \left(\left(1 + g^2\right), i\left(1 - g^2\right), 2g\right)\eta.$$

The first f.f. ds^2 of f, and the second f.f. A of f are given

$$ds^2 = \left(1 - |g|^2\right)^2 |\eta|^2, \qquad A = 2\,\mathrm{Re}\,Q, \qquad Q := \eta dg$$

respectively.
$\nu : U \to H^2_+$ the unit normal v.f. on U along f,
$\sigma : H^2_+ \to \mathbb{D} = \{z \in \mathbb{C}\,;\,|z| < 1\}$ the stereographic projection. Then,

$$g = \sigma \circ \nu$$

Hence we call g the Gauss map of f.

Maximal surfaces with singularities (before 2010)

- O. Kobayashi “conelike singularities” (1984).
- F. J. M. Estudillo and A. Romero “generalized maximal surfaces” (1992).
- F. J. López, R. López and R. Souam “Riemann type maximal surface” (2000).
- L. J. Alías, R. M. B. Chaves and P. Mira “Björling problem” (2003).
- I. Fernández, F. J. López and R. Souam “moduli space” (2005).
- I. Fernández and F. J. López “periodic maximal surfaces” (2007).
- T. Imaizumi and S. Kato “flux” (2008).
- F. Martín, M. Umehara and K. Yamada “bounded maximal surfaces” (2009).

Maxfaces

Definition (Umehara-Yamada, 2006)

$f : M \to \mathbb{L}^3$is a maxface $:\Longleftrightarrow$

- $\exists W \subset M$ (open dense) s.t. $f|_W$ a conformal maximal immersion,
- $df_p \neq 0$ $(\forall p \in M)$.

For a maxface, $\left(1 + |g|^2\right)^2 |\eta|^2$ is always positive definite.
This the set of singular points of f is $\{p \in M \mid |g(p)| = 1\}$.

Definition (Umehara-Yamada, 2006)

A maxface $f : M \to \mathbb{L}^3$ is complete if $\exists C \underset{\text{cpt}}{\subset} M$, $\exists$symmetric $(0, 2)$ tensor $T \in \Gamma(T^*M^2 \otimes T^*M^2)$ such that $T \equiv 0$ on $M \setminus C$ and $ds^2 + T$ is an complete Riemannian metric.

Ossermn-type inequality

Theorem (Umehara-Yamada, 2006)

$f : M \to \mathbb{L}^3$ a complete maxface, (g, η) the Weierstrass data of f.
Then $\exists$a cpt Riem. surf. $\overline{M}$, $\exists p_1, \dots, p_n \in \overline{M}$ such that

- $M = \overline{M} \setminus \{p_1, \dots, p_n\}$ (biholomorphic).
- g, η extend meromorphically to $\overline{M}$.

$p_1, \dots p_n$ are the ends of f ($\nexists$compact maxface).

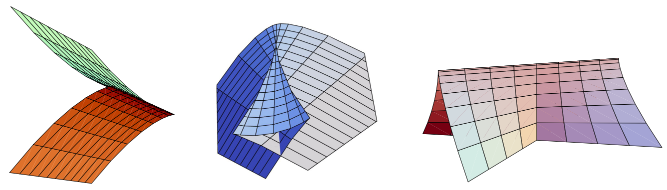
Theorem (Umehara-Yamada, 2006)

$f : M = \overline{M} \setminus \{p_1, \dots, p_n\} \to \mathbb{L}^3$ a complete maxface,
(g, η) the Weierstrass data of f. Then

- $2 \deg g \geq -\chi(\overline{M}) + 2n$.
- "=" $\iff$ each end is properly embedded.

Singularities

$$f_C(u, v) = (u^2, u^3, v) \quad \text{(Cuspidal edge)}$$
$$f_S(u, v) = (3u^4 + u^2 v, 2u^3 + uv, v) \quad \text{(Swallowtail)}$$
$$f_{CCR}(u, v) = (u, v^2, uv^3) \quad \text{(Cuspidal cross cap)}$$



f_C f_S f_{CCR}

Definition

Two C^∞-maps $f_j : (\mathbb{R}^2, p_j) \to \mathbb{R}^3$ $(j = 1, 2)$ are right-left equivalent if $\exists$local diffeo's $\varphi : (\mathbb{R}^2, p_1) \to (\mathbb{R}^2, p_2)$ and $\Phi : (\mathbb{R}^3, f_1(p_1)) \to (\mathbb{R}^3, f_2(p_2))$ such that $\Phi \circ f_1 = f_2 \circ \varphi$.

Criteria for singularities

Theorem (Umehara-Yamada, 2006)

$f : U \to \mathbb{L}^3$ a maxface with Weierstrass data (g, η). We set
$\alpha = \dfrac{dg}{g^2 \eta}$ and $\beta = g \dfrac{d\alpha}{dg}$. Then

- $p \in U$ is a singular point of f iff $|g(p)| = 1$.
- f is right-left equivalent to a cuspidal edge at p iff $\operatorname{Re} \alpha \neq 0$ and $\operatorname{Im} \alpha \neq 0$.
- f is right-left equivalent to a swallowtail at p iff $\alpha \in \mathbb{R} \setminus \{0\}$ and $\operatorname{Re} \beta \neq 0$.
- (Fujimori-Saji-Umehara-Yamada, 2008)
  f is right-left equivalent to a cuspidal cross cap at p iff $\alpha \in i\mathbb{R} \setminus \{0\}$ and $\operatorname{Im} \beta \neq 0$.

Generic singularities of maxfaces

$U \subset \mathbb{C}$ a simply connected domain,
$h \in \mathcal{O}(U) :=$ {holomorphic function on U}
endowed with the compact open C^∞-topology.
$f_h : U \to \mathbb{L}^3$ a maxface with Weierstrass data $(g, \eta) = (e^h, dz)$.

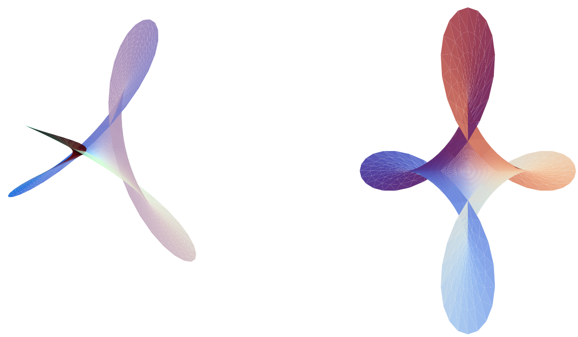
Theorem (Fujimori-Saji-Umehara-Yamada, 2008)

$K \subset U$ an arbitrary compact set,

$$S(K) := \left\{ h \in \mathcal{O}(U) \;\middle|\; \begin{array}{l} \text{singular points of } f_h \text{ are cuspidal edges,} \\ \text{swallowtails or cuspidal cross caps on } K. \end{array} \right\}$$

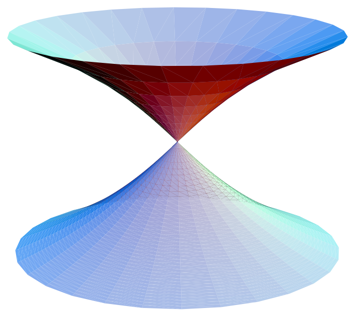
Then $S(K)$ is an open and dense subset of $\mathcal{O}(U)$.

Examples

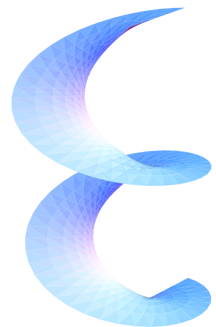


Maximal Enneper

Examples



Maximal catenoid
(cone-like singular points)



Maximal helicoid
(fold singular points)

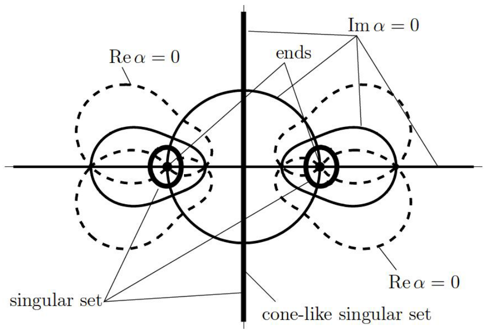
Ex. (Fujimori-Rossman-Umehara-Yamada-Yang, 2009)

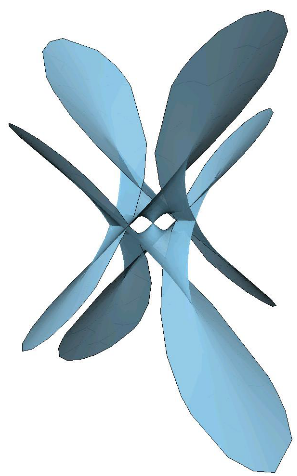
$M = (\mathbb{C} \cup \{\infty\}) \setminus \{1, -1\}$,

$g = \dfrac{(z-1)(z^2+az+1)}{(z+1)(z^2-az+1)}$,

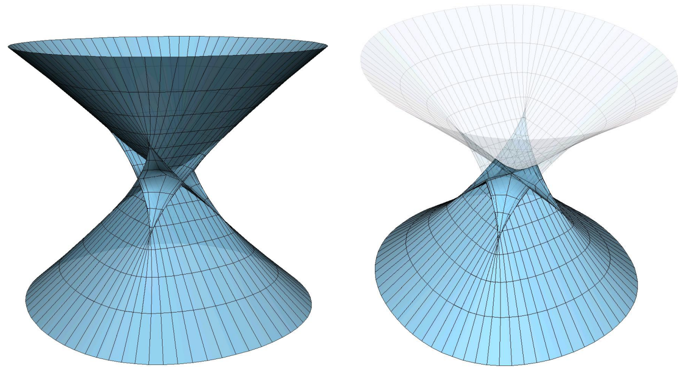
$\eta = \dfrac{(z^2-az+1)(z^2+az-1)}{(z+1)^2(z-1)^4}dz$,

where $a \in (1,4) - \{2\}$.





Kim-Yang catenoid

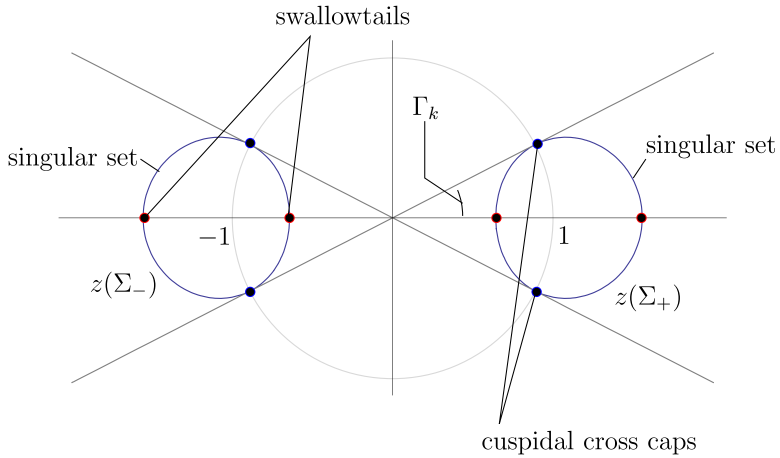


Y. W. Kim and S.-D. Yang (2006) found a complete maxface of genus 1 with two embedded ends, whose Weierstrass data are:

$M = \{(z, w) \in (\mathbb{C} \cup \{\infty\})^2 \mid w^2 = z(z^2-1)\} \setminus \{(0,0), (\infty, \infty)\}$,

$g = c\dfrac{w}{z}$ (for some $c > 0$), $\eta = \dfrac{dz}{w}$.

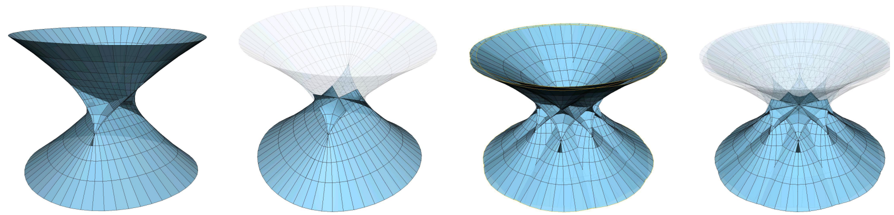
The singular set



Higher genus version

Example (F.-Rossman-Umehara-Yamada-Yang, 2009)

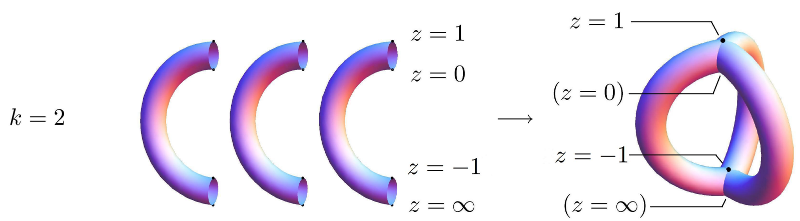
$M_k = \{(z,w) \in (\mathbb{C} \cup \{\infty\})^2 \mid w^{k+1} = z(z^2-1)^k\} \setminus \{(0,0),(\infty,\infty)\}$ $(\forall k \in \mathbb{Z}_+)$, $g = c\dfrac{w}{z}$ (for some $c > 0$), $\eta = \dfrac{dz}{w}$.



$k = 2$ $k = 3$

The Riemann surface $\overline{M}_k$

$\overline{M}_k = \{(z,w) \in (\mathbb{C} \cup \{\infty\})^2 \mid w^{k+1} = z(z^2-1)^k\}$.



Reduction for $k = 2m$ case

$\overline{M}_{2m} = \{(z,w) \in (\mathbb{C} \cup \{\infty\})^2 \mid w^{2m+1} = z(z^2-1)^{2m}\}$,
$\overline{M}'_m = \{(Z,W) \in (\mathbb{C} \cup \{\infty\})^2 \mid W^{2m+1} = Z^{m+1}(Z-1)^{2m}\}$.

$$M_{2m} = \overline{M}_{2m} \setminus \{(0,0),(\infty,\infty)\}, \quad M'_m = \overline{M}'_m \setminus \{(0,0),(\infty,\infty)\}.$$

Then

$$\varpi : M_{2m} \ni (z,w) \mapsto (Z,W) = (z^2, zw) \in M'_m$$

is a double cover. Let

$$g_1 = c\frac{W}{Z}, \qquad \eta_1 = \frac{dZ}{2Z}.$$

Then $g = g_1 \circ \varpi$ and $\eta = \varpi^* \eta_1$ hold, and hence (g_1, η_1) are the Weierstrass data for the maxface $f_1 : M'_m \to \mathbb{L}^3$.

(1)

$\overline{M}_k : w^{k+1} = z(z^2-1)^k$ $\qquad \lambda = \frac{\pi}{k+1}$

$\mu_j : \overline{M}_k \to \overline{M}_k$ $\quad (j=1,2,3,4)$

$\mu_1(z,w) = (\bar{z}, \bar{w})$, $\mu_2(z,w) = (\bar{z}, e^{2k\lambda i}\bar{w})$

$\mu_3(z,w) = (-\bar{z}, e^{-i\lambda}\bar{w})$, $\mu_4(z,w) = \left(\frac{1}{\bar{z}}, e^{ik\lambda}\frac{\bar{w}}{\bar{z}^2}\right)$

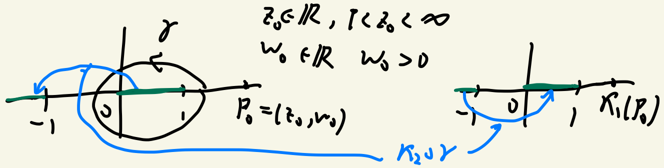
$\kappa_1 = \mu_2\circ\mu_1$, $\kappa_2 = \mu_3\circ\mu_1$

Lem $\pi_1(\overline{M}_k)$ is generated by

$[(\kappa_1)^j\circ\gamma]$, and $[(\kappa_1)^j\circ(\kappa_2\circ\gamma)]$ $(j=0,\dots,k)$

$z_0\in\mathbb{R}$, $1<z_0<\infty$

$w_0\in\mathbb{R}$ $\quad w_0>0$



(2)

$\Phi := \begin{pmatrix} 1+g^2 \\ i(1-g^2) \\ 2g \end{pmatrix}\eta$ $\qquad f = \mathrm{Re}\int\Phi$

$g = c\frac{w}{z}$, $\eta = \frac{dz}{w}$

$R(\theta) := \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Lem $\kappa_1^*\Phi = R(2k\lambda)\bar{\Phi}$, $\kappa_2^*\Phi = R(-k\lambda)\bar{\Phi}$

$\mathrm{Re}\oint_{\kappa_j\circ\gamma}\Phi = \mathrm{Re}\oint_\gamma \kappa_j^*\Phi = \mathrm{Re}\int_\gamma R(\theta)\bar{\Phi} = R(\theta)\,\mathrm{Re}\oint_\gamma\Phi$

Prop

f is well-defined iff $\mathrm{Re}\oint_\gamma\Phi = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$\mathrm{Re}\oint_\gamma 2g\eta = \mathrm{Re}\oint_\gamma 2c\frac{dz}{z} = 2c\oint_\gamma d(\log z) = 2c\,\mathrm{Re}(2\pi i) = 0$

(3)

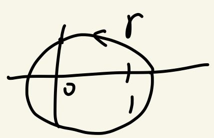
$\mathrm{Re}\oint_\gamma (1+g^2)\eta = \frac{1}{2}\left(\oint\eta + \oint g^2\eta + \overline{\oint\eta} + \overline{\oint g^2\eta}\right)$

$-i\,\mathrm{Re}\oint_\gamma i(1-g^2)\eta = \frac{1}{2}\left(\oint\eta - \oint g^2\eta - \overline{\oint\eta} + \overline{\oint g^2\eta}\right)$

$\oint_\gamma\eta + \overline{\oint_\gamma g^2\eta} = 0$

$\mathrm{Ord}_{z=0}\,\eta = k-1$ $\qquad \mathrm{Ord}_{z=1}\,\eta = 0$

$\mathrm{Ord}_{z=0}\,g^2\eta = -k-1$ $\qquad \mathrm{Ord}_{z=1}\,g^2\eta = 2k$



$\oint_\gamma\eta = 2i\sin k\lambda \boxed{\int_0^1 \frac{dt}{t^{\frac{1}{k+1}}(1-t^2)^{\frac{k}{k+1}}}}$ > 0

$\overset{\text{"}}{B_k}$

(4)

Lem $g^2\eta + c^2\frac{k+1}{k}d\left(\frac{w}{z}\right) = -2c^2\frac{w}{1-z^2}dz$

$\oint_\gamma c^2\frac{k+1}{k}d\left(\frac{w}{z}\right) = 0$

$\oint_\gamma g^2\eta = \oint_\gamma g^2\eta + \oint_\gamma c^2\frac{k+1}{k}d\left(\frac{w}{z}\right) = \oint_\gamma -2c^2\frac{w}{1-z^2}dz$

$= 4ic^2\sin k\lambda \boxed{\int_0^1\left(\frac{t}{1-t^2}\right)^{\frac{1}{k+1}}dt}$ > 0

$\overset{\text{"}}{A_k}$

Period problem is solved

$\Leftrightarrow$ c satisfies ~~$2i\sin k\lambda$~~ B_k ~~$-2i$~~ c^2 ~~$\sin k\lambda$~~ $A_k = 0$

$\Leftrightarrow$ c satisfies $\quad B_k - c^2 2A_k = 0 \Leftrightarrow c = \sqrt{\frac{B_k}{2A_k}} > 0$