Nonorientable maximal surfaces with one end in the Lorentz-Minkowski 3-space

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Minimal surfaces in $\mathbb{R}^3$

Theorem (Weierstrass representation)

M a Riemann surface, $f : M \to \mathbb{R}^3$ a minimal surface (i.e. $H \equiv 0$). Then
$\exists$a merom. function g and a holom. 1-form η on M such that

$$f = \mathrm{Re} \int \left(1 - g^2, i\left(1 + g^2\right), 2g\right) \eta, \quad ds^2 = \left(1 + |g|^2\right)^2 |\eta|^2.$$

(g, η) the Weierstrass data of f, g is called the Gauss map of f.

Theorem (Huber (1957) / Osserman (1963))

$f : M \to \mathbb{R}^3$ a complete minimal surface of f.t.c. with the W-data (g, η). Then $\exists \overline{M}$ a cpt Riem. surf., $\exists p_1, \ldots, p_n \in \overline{M}$ such that

- $M = \overline{M} - \{p_1, \ldots, p_n\}$ (biholom.).
- g, η extend meromorphically to $\overline{M}$.
- $2 \deg g \geq -\chi(\overline{M}) + 2n$.

Period problem

M a Riemann surface. g a merom. fct on M, η a holom. 1-form on M such that $(1 + |g|^2)^2|\eta|^2$ gives a complete Riemannian metric of finite total curvature on M. If M is not simply connected, then

$$f = \mathrm{Re} \int \left(1 - g^2, i\left(1 + g^2\right), 2g\right) \eta.$$

might not be well-defined on M.

Period problem

$f : M \to \mathbb{R}^3$ is well-defined on $M \iff$

$$\mathrm{Re} \oint_\gamma \left(1 - g^2, i\left(1 + g^2\right), 2g\right) \eta = (0, 0, 0) \qquad \forall \gamma \in H_1(M, \mathbb{Z})$$

Nonorientable minimal surfaces

M' a nonorientable surface.

$f' : M' \to \mathbb{R}^3$ a nonorientable minimal surfaces $:\iff$ the mean curvature w.r.t. a unit normal vanishes identically.

$f' : M' \to \mathbb{R}^3$ a complete nonorientable minimal surface of f.t.c.
Take a double cover $\pi : M \to M'$ (M a orientable surface), then
$f := f' \circ \pi : M \to \mathbb{R}^3$ is an orientable minimal surface.
$\longrightarrow$ one can apply the Weierstrass rep.
(g, η): the Weierstrass data of f.
$I : M \to M$ the anti-holomorphic deck transf w.r.t. π. Then,

$$f \circ I(p) = f(p) \qquad (\forall p \in M).$$

Lemma

$$f \circ I = f \quad \iff \quad g \circ I = -\frac{1}{\bar{g}} \quad \text{and} \quad I^*\eta = \overline{g^2 \eta}.$$

The Gauss map

$f' : M' \to \mathbb{R}^3$ a complete nonorientable minimal surface of f.t.c.
$g : M \to \mathbb{C} \cup \{\infty\}$ the Gauss map of $f = f' \circ \pi$.
$I : M \to M$ the anti-holomorphic deck transf w.r.t. π.
Then, $\exists 1\ \hat{g} : M' \to \mathbb{RP}^2$ s.t. the following diagram is commutative.

$$\begin{array}{ccc} M & \xrightarrow{\ g\ } & \mathbb{C} \cup \{\infty\} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p_0} \\ M' & \xrightarrow{\ \hat{g}\ } & \mathbb{RP}^2 \end{array}$$

where $p_0 : \mathbb{C} \cup \{\infty\} \to \mathbb{RP}^2 = (\mathbb{C} \cup \{\infty\})/\langle I_0 \rangle$ is the natural projection, $I_0(z) := -1/\bar{z}$.

Definition

The above $\hat{g}$ is called the Gauss map of a nonorientable minimal surface $f' : M' \to \mathbb{R}^3$.

Remark. Since $\deg(\pi) = \deg(p_0) = 2$, can define $\deg \hat{g}$: $\deg \hat{g} = \deg g$.

$\deg \hat{g}$

Theorem (Meeks, 1981)

$f' : M' \to \mathbb{R}^3$ a complete nonorientable minimal surface of f.t.c.
$\hat{g}$ the Gauss map of f'. Then,

$$\deg \hat{g} \equiv \chi(\overline{M'}) \pmod 2.$$

Lemma (Meeks, 1981)

M_j a compact 2-mfd such that $\partial M_j = \varnothing$ $(j = 1, 2)$.
$p : M_1 \to M_2$ a branched covering.

- $\chi(M_2)$ is even $\Longrightarrow \chi(M_1)$ is even.
- $\chi(M_2)$ is odd $\Longrightarrow \chi(M_1) \equiv \deg p \pmod 2$.

$\deg \hat{g}$

Corollary (Meeks, 1981)

$f' : M' \to \mathbb{R}^3$ a complete nonorientable minimal surface of f.t.c.
$\hat{g}$ the Gauss map of f'. Then,

$$\deg \hat{g} \geq 3.$$

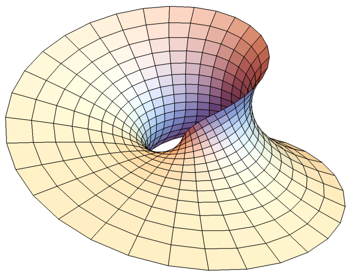
(Proof) Let $\pi : M \to M'$ the double cover.

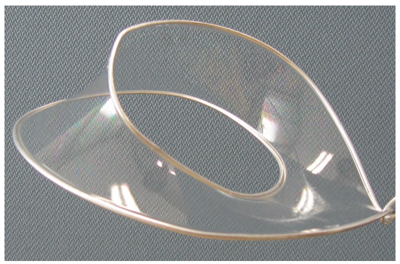
- $\deg \hat{g} = 1 \Rightarrow M = S^2 - \{p, q\}$ (embedded ends) $\Rightarrow M$: catenoid.
- $\deg \hat{g} = 2 \Rightarrow M = T^2 - \{p, q\}$ (embedded ends) $\Rightarrow \nexists$. □

Example: Möbius strip ($\deg \hat{g} = 3$)

$M = \mathbb{C} - \{0\}$, $I(z) = -1/\bar{z}$, $M' = M/\langle I \rangle = \mathbb{RP}^2 - \{\pi(0)\}$,

$$g = z^2 \frac{z+1}{z-1}, \qquad \eta = i \frac{(z-1)^2}{z^4} dz.$$





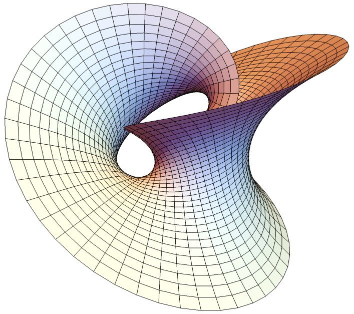
Theorem (Meeks, 1981)

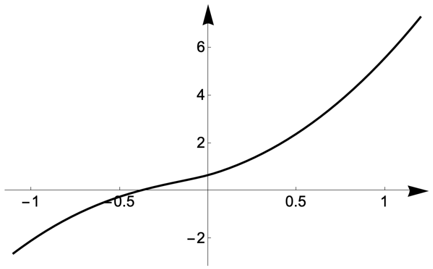
This is the unique example with $\deg \hat{g} = 3$.

Remark. There exists a Möbius strip with $\deg \hat{g}$ is odd (≥ 5).

Example: Klein bottle−{1 pt} (López, deg $\hat{g}=4$)

$$M=\left\{(z,w)\in(\mathbb{C}\cup\{\infty\})^2\ ;\ w^2=z\frac{rz-1}{z+r}\right\}-\{(0,0),(\infty,\infty)\},$$
$$(r\in\mathbb{R}-\{0\}),\ I(z,w)=\left(-\frac{1}{\bar{z}},\frac{1}{\bar{w}}\right),\ g=w\frac{z+1}{z-1},\ \eta=i\frac{(z-1)^2}{z^2w}dz.$$



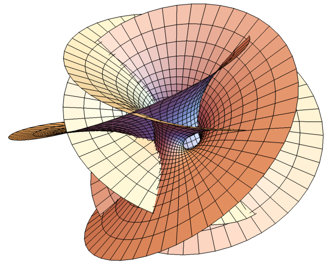


Theorem (López, 1996)

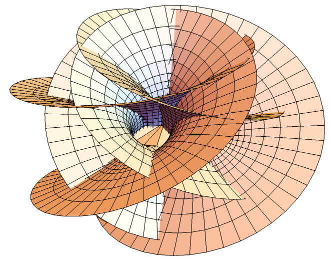
This is the unique example with deg $\hat{g}=4$.

Example: Higher genus (López-Martín, deg $\hat{g}=3k+1$)

$$M=\left\{(z,w)\in(\mathbb{C}\cup\{\infty\})^2\ ;\ w^{k+1}=\frac{z(z-r)}{rz+1}\right\}-\{(0,0),(\infty,\infty)\},$$
$$k\in\mathbb{Z}_{>0},\ I(z,w)=\left(-\frac{1}{\bar{z}},\frac{1}{\bar{w}}\right),\ g=w^k\frac{z-1}{z+1},\ \eta=i\frac{(z+1)^2}{z^2w^k}dz.$$



$k=2$



$k=3$

$$\exists 1r\ :\ \int_0^1\Big(\big(k+(k+1)r\big)r^2+\big(k+(2k+1)r\big)t\Big)\left(\frac{1-t}{t(t+r^2)}\right)^{\frac{1}{k+1}}dt=0$$

Maximal surfaces in $\mathbb{L}^3$

$\mathbb{L}^3$ the Lorentz-Minkowski 3-space. $\langle\ ,\ \rangle:=dx_1^2+dx_2^2-dx_3^2$.
M a 2-dim. mfd.

- $f:M\to\mathbb{L}^3$ is a spacelike if $\langle df,df\rangle$ is positive definite.
- A maximal surface is a spacelike surface with $H\equiv 0$.

Theorem (O. Kobayashi, 1983 / L. McNertney, 1980)

M a Riemann surface, $f:M\to\mathbb{L}^3$ a maximal surface. Then $\exists$a merom. function g and a holom. 1-form η on M such that

$$f=\mathrm{Re}\int\left(1+g^2,\,i\left(1-g^2\right),\,2g\right)\eta.\quad ds^2=\left(1-|g|^2\right)^2|\eta|^2.$$

(g,η) the Weierstrass data of f, g is called the Gauss map of f.

- Complete maximal surface is a plane (Calabi, 1970).
- $\nexists$ nonorientable spacelike surface.

Maxfaces

Definition (Umehara-Yamada, 2006)

$f:M\to\mathbb{L}^3$ is a maxface $:\Longleftrightarrow$

- $\exists W\subset M$ (open dense) s.t. $f|_W$ a conformal maximal immersion,
- $df_p\neq 0$ ($\forall p\in M$).

For a maxface, $\left(1+|g|^2\right)^2|\eta|^2$ is always positive definite.
The set of singular points of f is $\{p\in M\mid |g(p)|=1\}$.

Definition (Umehara-Yamada, 2006)

A maxface $f:M\to\mathbb{L}^3$ is complete if $\exists C\underset{\text{cpt}}{\subset}M$, $\exists$symmetric $(0,2)$ tensor $T\in\Gamma(T^*M^2\otimes T^*M^2)$ such that $T\equiv 0$ on $M-C$ and ds^2+T is a complete Riemannian metric.

Ossermn-type inequality

Theorem (Umehara-Yamada, 2006)

$f : M \to \mathbb{L}^3$ a complete maxface, (g, η) the Weierstrass data of f.
Then $\exists$a cpt Riem. surf. $\overline{M}$, $\exists p_1, \ldots, p_n \in \overline{M}$ such that

- $M = \overline{M} - \{p_1, \ldots, p_n\}$ (biholomorphic).
- g, η extend meromorphically to $\overline{M}$.

$p_1, \ldots, p_n$ are the ends of f ($\nexists$compact maxface).

Theorem (Umehara-Yamada, 2006)

$f : M = \overline{M} - \{p_1, \ldots, p_n\} \to \mathbb{L}^3$ a complete maxface,
(g, η) the Weierstrass data of f. Then

- $2 \deg g \geq -\chi(\overline{M}) + 2n$.
- "=" $\iff$ each end is properly embedded.

Nonorientable maxface

Definition

1. M' a nonorientable surface. $f' : M' \to \mathbb{L}^3$ is a nonorientable maxface if $\exists$a Riemann surface M, $\exists$the double cover $\pi : M \to M'$ such that $f = f' \circ \pi : M \to \mathbb{L}^3$ is a maxface.
2. $f' : M' \to \mathbb{L}^3$ is complete if $f = f' \circ \pi : M \to \mathbb{L}^3$ is complete.

(g, η) the Weierstrass data of f. $I : M \to M$ the anti-holom. order 2 deck transf. associated to π. Then,

$$f \circ I(p) = f(p) \qquad (\forall p \in M).$$

Lemma

$$f \circ I = f \quad \text{iff} \quad g \circ I = \frac{1}{\bar{g}} \quad \text{and} \quad I^*\eta = \overline{g^2\eta}.$$

Gauss map

$f' : M' \to \mathbb{L}^3$ a nonorientable maxface,
$\pi : M \to M'$ the double cover.
$g : M \to \mathbb{C} \cup \{\infty\}$ the Gauss map of $f = f' \circ \pi$,
$A : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$, $A(z) := 1/\bar{z}$.
$p_0 : \mathbb{C} \cup \{\infty\} \to (\mathbb{C} \cup \{\infty\})/\langle A \rangle$ the projection. Then, $\exists 1$the conformal map $\hat{g} : M' \to (\mathbb{C} \cup \{\infty\})/\langle A \rangle$ such that $\hat{g} \circ \pi = p_0 \circ g$.

$$\begin{array}{ccc} M & \xrightarrow{g} & \mathbb{C} \cup \{\infty\} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p_0} \\ M' & \xrightarrow{\hat{g}} & (\mathbb{C} \cup \{\infty\})/\langle A \rangle \end{array}$$

Definition

The above $\hat{g}$ is called the Gauss map of $f' : M' \to \mathbb{L}^3$.

Remark. If f' is complete, we can define $\deg \hat{g}$. $\deg \hat{g} = \deg g$.

Degree of the Gauss map

Theorem (Fujimori-López, 2010)

$f' : M' \to \mathbb{L}^3$ a complete nonorientable maxface,
$\hat{g} : M' \to (\mathbb{C} \cup \{\infty\})/\langle A \rangle$ the Gauss map of f'.
$\Longrightarrow$ $\deg \hat{g}$ is even and greater than 2.

Lemma (Ross, 1992)

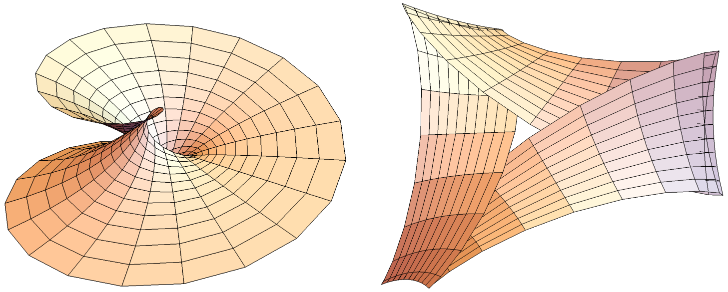
$\overline{M}$ a cpt Riem. surf., $I : \overline{M} \to \overline{M}$ anti-holom. invol. without fixed pt.
$\Longrightarrow \quad \exists h : \overline{M} \to \mathbb{C} \cup \{\infty\}$ such that $h \circ I = -1/\bar{h}$.

(Proof of Thm) Define $G : \overline{M} \to \mathbb{C} \cup \{\infty\}$ by $G(p) = g(p)h(p)$ $(p \in \overline{M})$.
Since $G \circ I = (gh) \circ I = (g \circ I)(h \circ I) = (1/\bar{g})(-1/\bar{h}) = -1/\bar{G}$, Meeks' lemma yields $\chi(\overline{M}') \equiv \deg G \pmod 2$. Also, $\chi(\overline{M}') \equiv \deg h \pmod 2$.
Since $\deg G = \deg(gh) = \deg h + \deg g$,

$$\deg h \equiv \deg h + \deg g \pmod 2. \qquad \text{Hence} \qquad \deg g = \text{even}.$$

Moreover it is easy to verify that $\deg g$ cannot be 2. $\square$

Möbius strip ($\deg \hat{g} = 4$)



Left: g is branched at the ends. Right: g is not branched at the ends.

Theorem (Fujimori-López, 2010)

Möbius strip with $\deg \hat{g} = 4$ are the LHS one or one of the 2-parameter family of the RHS one.

Remark. For minimal Möbius strip ($\deg \hat{g} = 3$), g must be branched at the ends (Meeks, 1981).

Weierstrass data of Möbius strip ($\deg \hat{g} = 4$)

$M = \mathbb{C} - \{0\}$, $I(z) = -1/\bar{z}$, $M' = M/\langle I \rangle = \mathbb{RP}^2 - \{\pi(0)\}$,

- (Left) $g = z^3 \dfrac{z+1}{z-1}$, $\eta = i\dfrac{(z-1)^2}{z^5}dz$.
- (right) $g = z\dfrac{(rz-1)(sz-1)(tz-1)}{(z+r)(z+\bar{s})(z+\bar{t})}$,

  $\eta = i\dfrac{(z+r)^2(z+\bar{s})^2(z+\bar{t})^2}{z^5}dz$,

  where $r > 0$, $s, t \in \mathbb{C} - \{0\}$.

Example: Two-ended projective plane ($\deg \hat{g} = 4$)

$M = \mathbb{C} - \{0, \pm 1\}$, $I(z) = -1/\bar{z}$, $M' = M/\langle I \rangle = \mathbb{RP}^2 - \{\pi(0), \pi(1)\}$,

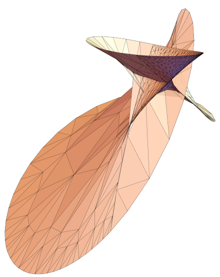
$$g = \frac{z(z-\alpha)(z-\beta)(z-\gamma)}{(\bar{\alpha}z+1)(\bar{\beta}z+1)(\bar{\gamma}z+1)}, \qquad \eta = i\frac{(\bar{\alpha}z+1)^2(\bar{\beta}z+1)^2(\bar{\gamma}z+1)^2}{z^2(z^2-1)^3}dz,$$

where $\alpha, \beta, \gamma \in \mathbb{C}$.

Lemma (Kaneda, 2023)

$\exists 1\{\alpha, \beta, \gamma\}$ such that $f : M \to \mathbb{L}^3$ is well-defined on M.

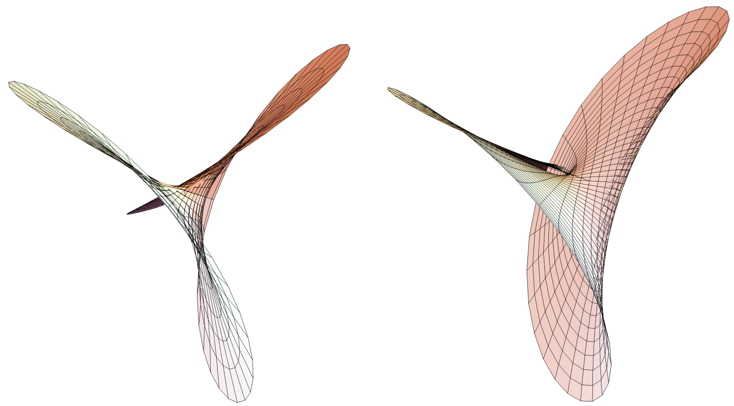
$\alpha \approx 0.929495 - 2.31357i$,
$\beta \approx -1.48442 + 1.9773i$,
$\gamma \approx 0.554922 + 0.336273i$.



Theorem (Kaneda, 2023)

This is the unique example with this topology and $\deg \hat{g} = 4$.

One-ended Klein bottle ($\deg \hat{g} = 4$)



Theorem (Fujimori-López 2010)

One-ended Klein bottle with $\deg \hat{g} = 4$ and a certain symmetry must be one of them.

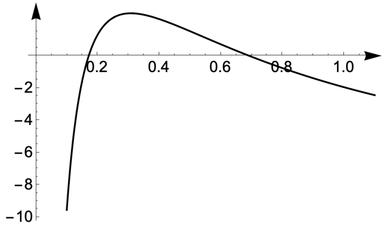
W-data of one-ended Klein bottle ($\deg \hat{g} = 4$)

$$M = \left\{(z,w) \in (\mathbb{C} \cup \{\infty\})^2 \,;\, w^2 = \frac{z(z-r)}{rz+1}\right\} - \{(0,0),(\infty,\infty)\},$$

$(r \in \mathbb{R} - \{0,1\}),$

$$I(z,w) = \left(-\frac{1}{\bar{z}}, \frac{1}{\bar{w}}\right), \qquad g = \frac{w(z+1)}{z(z-1)}, \qquad \eta = i\frac{(z-1)^2}{zw}dz.$$

(Left) $r \approx 0.17137$, (Right) $r \approx 0.691724$.



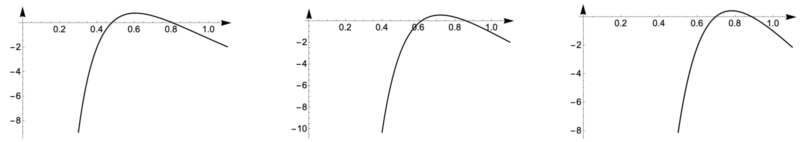
Higher genus version ($\deg \hat{g} = 2(k+1)$, $k \in \mathbb{Z}_{>0}$)

$$M = \left\{(z,w) \in (\mathbb{C} \cup \{\infty\})^2 \,;\, w^{k+1} = \frac{z(z-r)}{rz+1}\right\} - \{(0,0),(\infty,\infty)\},$$

$$I(z,w) = \left(-\frac{1}{\bar{z}}, \frac{1}{\bar{w}}\right),\ g = \frac{w^k(z+1)}{z(z-1)},\ \eta = i\frac{(z-1)^2}{zw^k}dz \ (r \in \mathbb{R} - \{0,1\}).$$

Main Theorem (Fujimori-Kaneda, 2023)

For each $k \in \mathbb{Z}_{>0}$, there exist exactly two r for which the maxface $f : M \to \mathbb{L}^3$ is well-defined and induces a one-ended complete nonorientable maxface $f' : M' = M/\langle I \rangle \to \mathbb{L}^3$ of genus $k+1$.



$k = 2$ $\qquad$ $k = 3$ $\qquad$ $k = 4$

Remark. $k = 1 \Longrightarrow$ One-ended Klein bottle by Fujimori-López.

Outline of Proof: Divisors

$$M = \left\{(z,w) \in (\mathbb{C} \cup \{\infty\})^2 \,;\, w^{k+1} = \frac{z(z-r)}{rz+1}\right\} - \{(0,0),(\infty,\infty)\},$$

$$I(z,w) = \left(-\frac{1}{\bar{z}}, \frac{1}{\bar{w}}\right),\ g = \frac{w^k(z+1)}{z(z-1)},\ \eta = i\frac{(z-1)^2}{zw^k}dz \ (r \in \mathbb{R} - \{0,1\}).$$

(z,w)	$(-1,*)$	$(-1/r,\infty)$	$(0,0)$	$(1,*)$	$(r,0)$	(∞,∞)
g	0^1	∞^k	∞^1	∞^1	0^k	0^1
η	–	0^{2k}	∞^{k+1}	0^2	–	∞^{k+3}
$g\eta$	0^1	0^k	∞^{k+2}	0^1	0^k	∞^{k+2}
$g^2\eta$	0^2	–	∞^{k+3}	–	0^{2k}	∞^{k+1}

$$\deg g = 2k + 2.$$

Outline of Proof: Symmetry

$$f = \mathrm{Re}\int \Phi, \qquad \Phi = \begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} = \begin{pmatrix}(1+g^2)\eta\\ i(1-g^2)\eta\\ 2g\eta\end{pmatrix}.$$

Define conformal maps $\kappa_j : \overline{M} \to \overline{M}$ $(j = 1,2)$ as follows:

$$\kappa_1(z,w) = \left(z, e^{\frac{2\pi i}{k+1}}w\right), \qquad \kappa_2(z,w) = (\bar{z},\bar{w}).$$

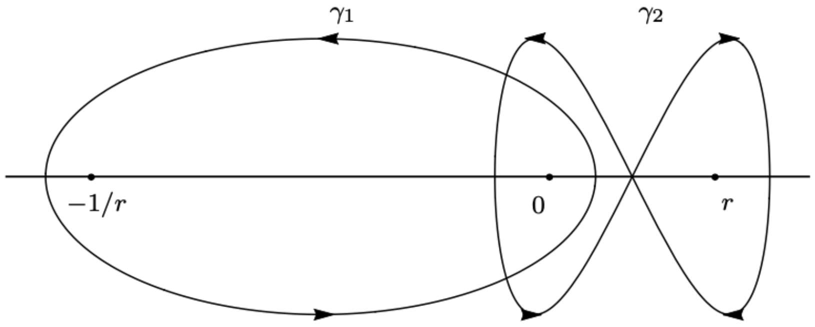
Then we have the following:

$$\kappa_1^*\Phi = K_1\Phi, \qquad \kappa_2^*\Phi = K_2\overline{\Phi},$$

where

$$K_1 = \begin{pmatrix}\cos\frac{2\pi}{k+1} & \sin\frac{2\pi}{k+1} & 0\\ -\sin\frac{2\pi}{k+1} & \cos\frac{2\pi}{k+1} & 0\\ 0 & 0 & 1\end{pmatrix}, \qquad K_2 = \begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}.$$

Outline of Proof: Homology basis of $\overline{M}$



Let γ_1 and γ_2 be two loops in $\overline{M}$ whose projections to the z-plane are as above. Then

$$\{\kappa_j^m \circ \gamma_l \;;\; j, l \in \{1,2\},\;\; m \in \{1, \ldots, k+1\}\}$$

contains a homology basis of $\overline{M}$.

Period Problem

$$\mathrm{Re} \oint_{\gamma_l} \phi_j = 0, \qquad j = 1,2,3, \;\; l = 1,2.$$

Since $\phi_3 = 2g\eta = d\left(\frac{2i(z^2+1)}{z}\right)$ is exact, $\oint_\gamma \phi_3 = 0$ for any γ.
Since $\oint_\gamma \phi_j = \oint_{I_*(\gamma)} I^*(\phi_j) = \oint_{I_*(\gamma)} \overline{\phi_j}$, the period problem reduces to

$$\oint_{\gamma_l + I_*(\gamma_l)} \phi_j = 0, \qquad j = 1,2, \;\; l = 1,2.$$

Lemma

$$I_*(\gamma_1) = \gamma_1, \qquad I_*(\gamma_2) = \gamma_1 - \gamma_2 + (\kappa_1)_*^k(\gamma_1).$$

Thus the period problem reduces to

$$\oint_{\gamma_1} \phi_j = 0, \qquad j = 1,2.$$

Since $\phi_1 = (1+g^2)\eta$, $\phi_2 = i(1-g^2)\eta$, the period problem is

$$\oint_{\gamma_1} \eta = \oint_{\gamma_1} g^2\eta = 0.$$

Since $\oint_{\gamma_1} \eta = \oint_{I_*(\gamma_1)} I^*\eta = \oint_{\gamma_1} \overline{g^2\eta}$, the period problem is

$$\oint_{\gamma_1} g^2\eta = \oint_{\gamma_1} \frac{w^k(z+1)^2}{z^3} dz = 0.$$

We set $F = \frac{(k+1)(z-r)(2rz^2-((k+1)r^2-2(k+2)r+k)z+r)}{(k+2)rwz}$, then we have

$$\frac{w^k(z+1)^2}{z^3} dz + dF = \frac{a(r) + 2(2k+1)rz}{(k+2)rw} dz,$$

where $a(r) = -(k+1)(k+2)r^2 + 2k(k+2)r - k(k-1)$. Thus

$$f \text{ is well-defined on } M \iff \psi(r) := \int_{-1/r}^{0} \frac{a(r) + 2(2k+1)rz}{r|w|} dz = 0.$$

Roots of $\psi(r)$

We set $t = -rz$, then

$$\psi(r) = |r|^{\frac{-2k}{k+1}} \int_0^1 (a(r) - 2(2k+1)t) \left(\frac{1-t}{t(t+r^2)}\right)^{\frac{1}{k+1}} dt.$$

$$\lim_{r \to -\infty} \psi(r) < 0, \quad \lim_{r \to 0} \psi(r) = -\infty, \quad \lim_{r \to +\infty} \psi(r) < 0.$$

Lemma

$$\psi\big(k/(k+1)\big) > 0, \qquad \psi(1) < 0.$$

Therefore, $\psi(r)$ has at least two roots in $(0,1)$.
Moreover, by considering the signs of $\psi'(r)$ and $\psi''(r)$ near the roots of $\psi(r)$, we see that $\psi(r)$ has exactly two real roots on $\mathbb{R} - \{0,1\}$.

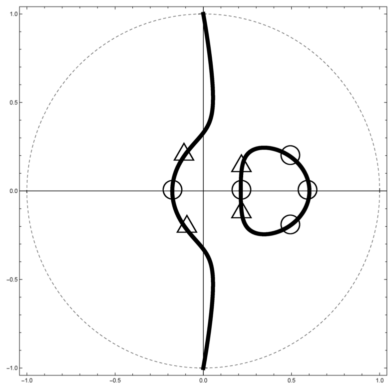
Example: Higher genus ($\deg \hat{g} = 2(k+1)$, $k \in \mathbb{Z}_{>0}$)

$$M = \left\{(z,w) \in (\mathbb{C} \cup \{\infty\})^2 \; ; \; w^{k+1} = \frac{z(z-r)}{rz+1}\right\} - \{(0,0),(\infty,\infty)\},$$

$$I(z,w) = \left(-\frac{1}{\bar{z}}, \frac{1}{\bar{w}}\right),\; g = \frac{w^k(z+1)}{z(z-1)},\; \eta = i\frac{(z-1)^2}{zw^k}dz.$$

$k=2$	$k=2$	$k=3$	$k=3$
$r \approx 0.478169$	$r \approx 0.807158$	$r \approx 0.615965$	$r \approx 0.859345$

Singularities



The singular set of f' ($k=2, r=r_1 \approx 0.478169$).
The thick curves indicate the singular points.
◯ indicates a cuspidal cross cap and △ indicates a swallowtail.
The other singularities are cuspidal edges.

References

- Shoichi Fujimori and Francisco J. López,
  Nonorientable maximal surfaces in the Lorentz-Minkowski 3-space,
  Tohoku Mathematical Journal, **62** (2010), 311–328.
- Shoichi Fujimori and Shin Kaneda,
  Higher genus nonorientable maximal surfaces in the Lorentz-Minkowski 3-space,
  Tohoku Mathematical Journal, **75** (2023), 1–14.
- Shin Kaneda,
  Some new examples of nonorientable maximal surfaces in the Lorentz-Minkowski 3-space,
  Hiroshima Mathematical Journal, **53** (2023), 311–334.

Lem. If $r_0 \in \mathbb{R}$ satisfies $\Psi(r_0) = 0$

$\Longrightarrow 0 \le r_0^- < r_0$ $\qquad r_0^- := \dfrac{k(k+2) - \sqrt{k(k+2)(2k+1)}}{(k+1)(k+2)}$

thus $r_0 \in (\infty, \infty)$

Lem. If $r_0 \in (0, \infty)$ satisfies $\Psi(r_0) = 0$

$$\Longrightarrow \begin{cases} \Psi'(r_0) = \dfrac{k(k+2) r_0^{-\frac{3k+1}{k+1}} \rho_1(r_0)}{(k+1)(r_0^2+1)} A_0(r_0) \\[2ex] \Psi''(r_0) = \dfrac{k(k+2) r_0^{-\frac{4k+2}{k+1}} \rho_2(r_0)}{(k+1)(r_0^2+)} A_0(r_0), \end{cases}$$

$A_0(r_0) > 0$, $\rho_1(r)$, $\rho_2(r)$ are poly. of degree 4

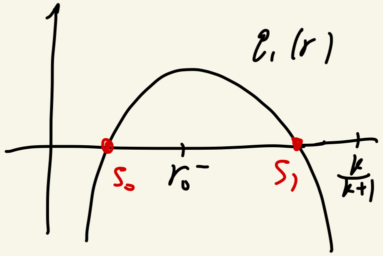
Lem. $\rho''(r) < 0 \quad \forall r \in \mathbb{R}$

$\rho_1(0) < 0$, $\rho_1(r_0^-) > 0$, $\rho\left(\frac{k}{k+1}\right) < 0$

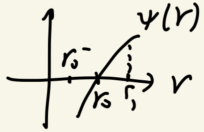
$\exists! s_0 \in (0, r_0^-)$

$\exists! s_1 \in (s_0^-, \frac{k}{k+1})$

s.t. $\rho_1(s_0) = \rho_1(s_1) = 0$

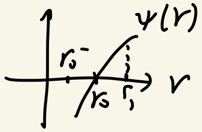


- For $r_0 \in (r_0^-, s_1)$ $\Psi'(r_0) > 0$

→ 

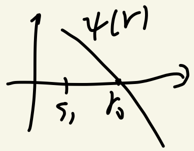
Thus $\Psi(r)$ has at most one root in (r_0^-, s_1)

- For $r_0 \in (r_0^-, s_1)$ $\Psi'(r_0) > 0$

→ 

Thus $\Psi(r)$ has at most one root in (r_0^-, s_1)

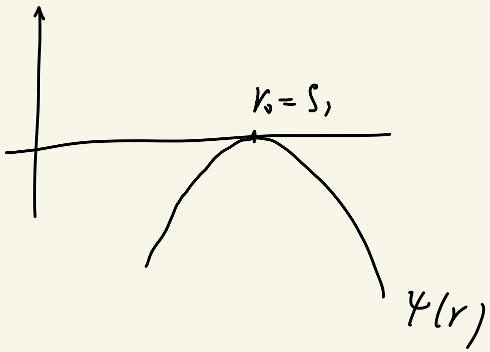
- For $r_0 \in (s_1, \infty)$ $\Psi'(r_0) < 0$

→ 

Thus $\Psi(r)$ has at most one root in (s_1, ∞)

Lem $\rho_2(s_1) < 0 \iff \Psi''(s_1) < 0$.

If $r_0 = s_1 \Longrightarrow \Psi(r_0) = \Psi'(r_0) = 0$ and $\Psi''(r_0) < 0$



impossible