PBHs and SGWB as a probe of nonstandard reheating history

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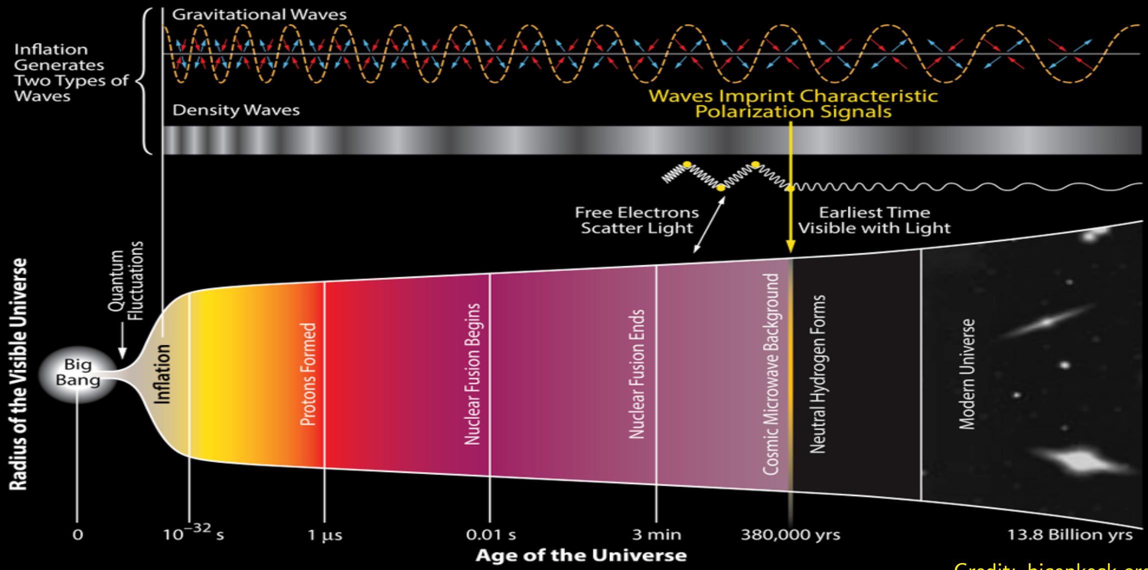
Hearing beyond the standard model with cosmic sources of Gravitational Waves,
ICTS Bengaluru, January 2, 2025

History of the Universe

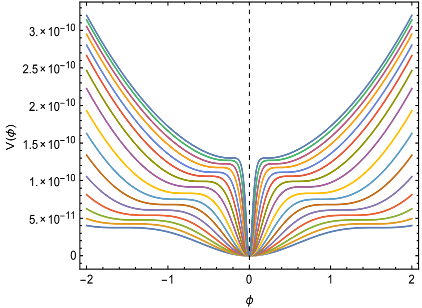


Credit: bicepkeck.org

Overview

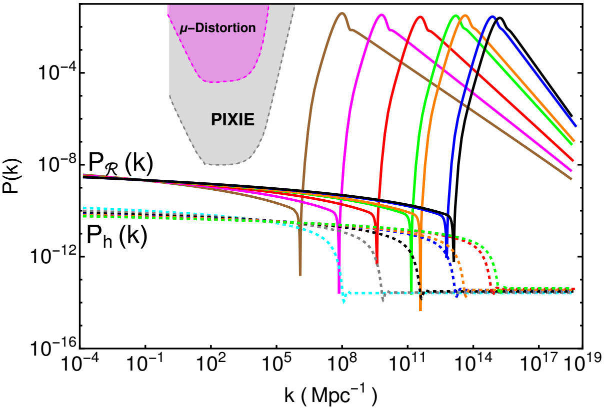
1. Efects of reheating on PBHs and SGWB
   - If PBHs do not form during reheating
   - If PBHs form during reheating
2. SGWB signatures from ultra low mass PBHs dominated universe
   - Ultra low mass PBHs form during radiaion domination
     - Explaining the NANOGrav signal
3. Ultra low mass PBHs form during reheating then dominates
   - Degeneracy with memory burden effect

Inflationary model and scalar power spectra



$$V(\phi) = V_0 \, \frac{a\phi^2 + b\phi^4 + c\phi^6}{(1+d\phi^2)^2}$$

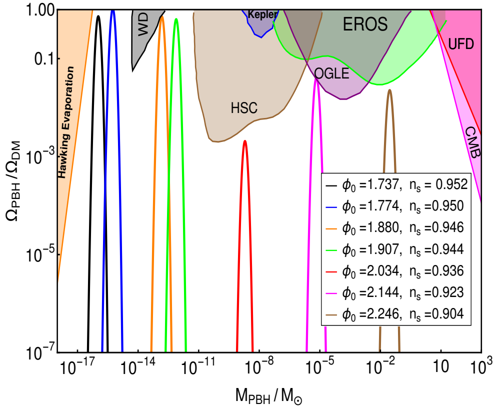
$$V'(\phi_0) = 0 \qquad V''(\phi_0) = 0$$

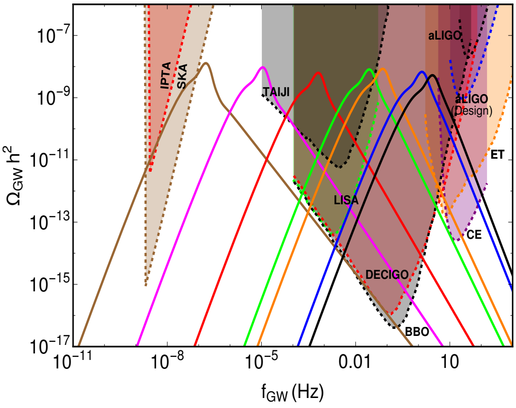


Mukhanov-Sasaki equations

$$\mathcal{R}_k'' + 2\left(\frac{z'}{z}\right)\mathcal{R}_k' + k^2\mathcal{R}_k = 0; \quad z = a\dot{\phi}/H = a\phi_N \qquad P_{\mathcal{R}} = \frac{k^3}{2\pi^2}|\mathcal{R}_k|^2$$

Primordial black holes and stochastic GW background





$$h''_{\mathbf{k}}(\tau) + 2\mathcal{H} h'_{\mathbf{k}}(\tau) + k^2 h_{\mathbf{k}}(\tau) = 4S_{\mathbf{k}}(\tau)$$

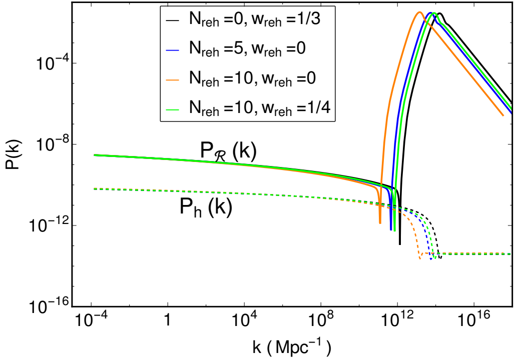
$$\Omega_{\rm GW}(\tau, k) \equiv \frac{1}{\rho_c}\frac{{\rm d}\,\rho_{\rm GW}}{{\rm d}\ln k} = \frac{1}{6}\left(\frac{k}{\mathcal{H}}\right)^2 \int_0^\infty {\rm d}v \int_{|1-v|}^{1+v} {\rm d}u \left(\frac{4v^2-(1+v^2-u^2)^2}{4uv}\right)^2 \times \overline{I^2_{\rm RD}(v,u,x)}\,\mathcal{P}_{\mathcal{R}}(kv)\mathcal{P}_{\mathcal{R}}(ku)$$

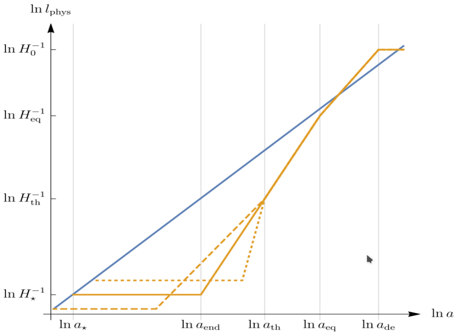
[Mollerach et al. 2003; Baumann et al. 2007 ...] [**NB**, Jain; JCAP 01(2020), 037]

Effects of reheating

We assume reheating phase with a constant equation of state w_{reh} and duration N_{reh}.

Remapping of scales : $k_e = k_{no-reheating} \times e^{-\frac{1}{4}N_{reh}(1-3w_{reh})}$





[**NB**, Jain; JCAP 01(2020), 037]

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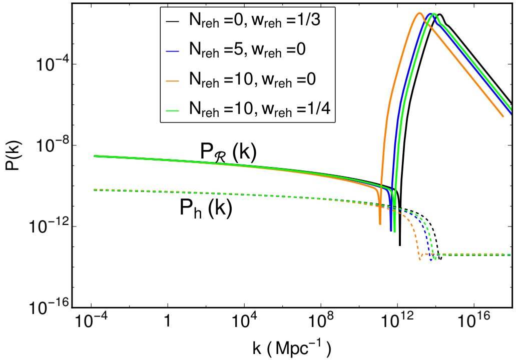
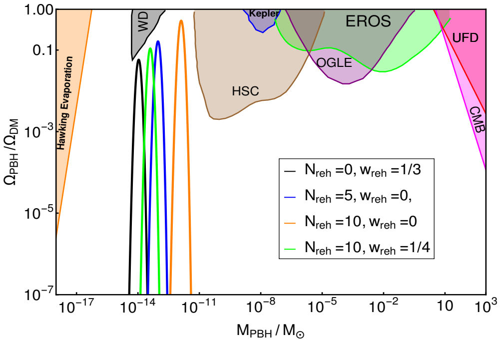




[**NB**, Jain; JCAP 01(2020), 037]

Effects of an early Matter dominated (eMD) reheating on ISGWB

Dependence on kernel I(u,v,x)

$$\Omega_{\rm GW}(\tau,k) = \frac{1}{6}\left(\frac{k}{\mathcal{H}}\right)^2 \int_0^\infty \mathrm{d}v \int_{|1-v|}^{1+v} \mathrm{d}u \left(\frac{4v^2-(1+v^2-u^2)^2}{4uv}\right)^2 \overline{I^2_{\rm eMD+RD}(v,u,x)}\mathcal{P}_\mathcal{R}(kv)\mathcal{P}_\mathcal{R}(ku)$$

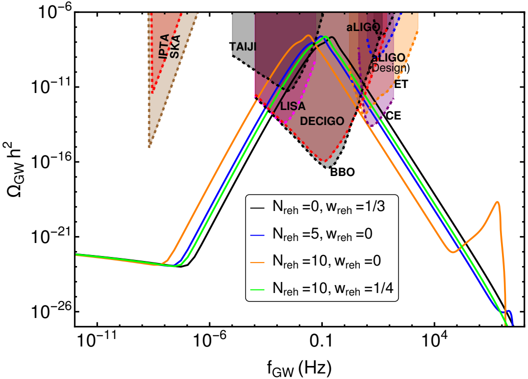
$$I(u,v,x,x_r) \simeq I_{RD}(u,v,x,x_r) = \int_{x_r}^{x} d\bar{x}\frac{a(\bar{x})}{a(x)} f(u,v,\bar{x},x_r) kG(\bar{x},x)$$

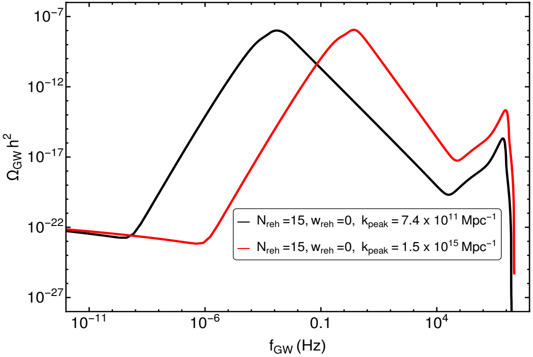
[Inomata, Kohri, Nakama, Terada; Phys. Rev. D 100, 043532 (2019)]

$$f(u,v,\bar{x},x_r) = \frac{4}{9}\Big[(\bar{x}-x_r/2)\partial_{\bar{x}}\mathcal{T}(u\bar{x},ux_r)((\bar{x}-x_r/2)\partial_{\bar{x}}\mathcal{T}(v\bar{x},vx_r)+\mathcal{T}(v\bar{x},vx_r))$$
$$+\mathcal{T}(u\bar{x},ux_r)((\bar{x}-x_r/2)\partial_{\bar{x}}\mathcal{T}(v\bar{x},vx_r)+3\mathcal{T}(v\bar{x},vx_r))\Big]$$

- $x=\tau k$ $\quad x_r=\tau_r k$ $\quad \frac{a(\tau)}{a(\tau_r)}=2\frac{\tau}{\tau_r}-1$ $\quad \mathcal{H}=aH=\frac{1}{\tau-\tau_r/2}$

ISGWB for different reheating histories





[**NB**, Jain; Phys. Rev. D 104, 023531 (2021)]

Formation of PBHs during reheating

1. PBH formation during reheating has not been studied very extensiely.
2. We follow Harada, Yoo and Kohri formalism [Harada et al, arXiv:1309.4201]
3. Mass : $M = \gamma\, M_{\mathrm{H}}$, $\gamma = w_{\mathrm{re}}^{3/2}$
4. Abundance :

$$\beta(M) \simeq \frac{1}{2}\,\mathrm{erfc}\left\{\frac{\delta_{\mathrm{c}}}{\sqrt{2}\,\sigma_\delta(M)}\right\},$$

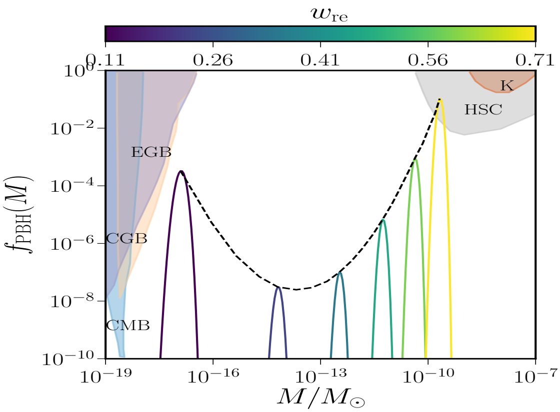
$$\mathcal{P}_\delta(k) = \left[\frac{2\,(1+w_{\mathrm{re}})}{5+3\,w_{\mathrm{re}}}\right]^2 \left(\frac{k}{a\,H}\right)^4 \mathcal{P}_{\mathcal{R}}(k).$$

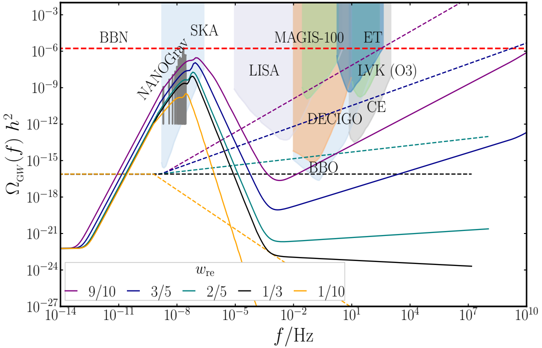
$$\sigma_\delta^2(R) = \int \mathrm{d}\ln k\, \mathcal{P}_\delta(k)\, W^2(k\,R).$$

$$\delta_{\mathrm{c}} = \frac{3\,(1+w_{\mathrm{re}})}{5+3\,w_{\mathrm{re}}}\,\sin^2\left(\frac{\pi\,\sqrt{w_{\mathrm{re}}}}{1+3\,w_{\mathrm{re}}}\right).$$

Please also check Prof. L Sriramkumar's talk for details.

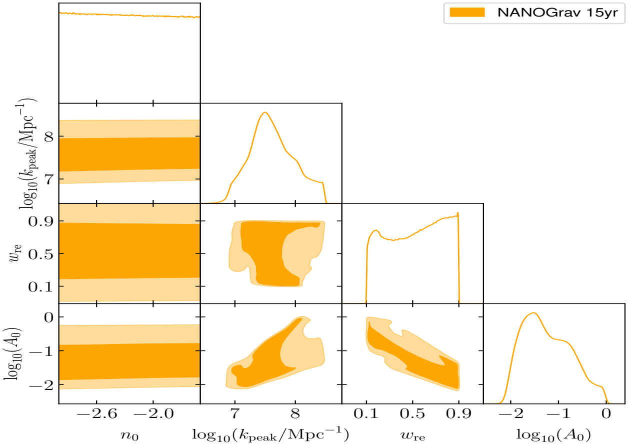
PBH formation and SGWB during w-domination

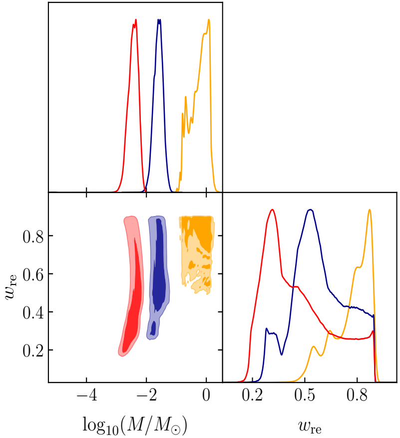




$$\mathcal{P}_{\mathcal{R}}(k) = A_{\rm S}\left(\frac{k}{k_*}\right)^{n_{\rm S}-1} + A_0 \begin{cases} \left(\frac{k}{k_{\rm peak}}\right)^4 & k \le k_{\rm peak} \\ \left(\frac{k}{k_{\rm peak}}\right)^{n_0} & k \ge k_{\rm peak} \end{cases},$$

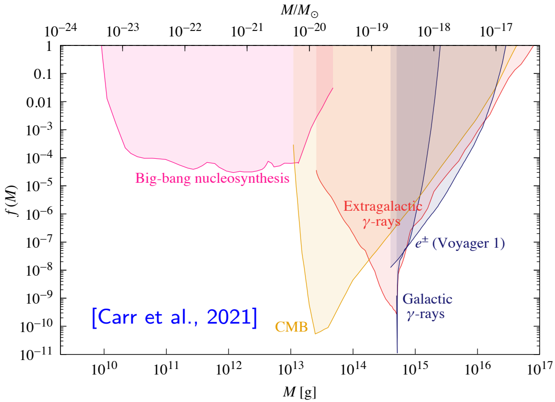
PBH formation during reheating can explain NANOGrav data

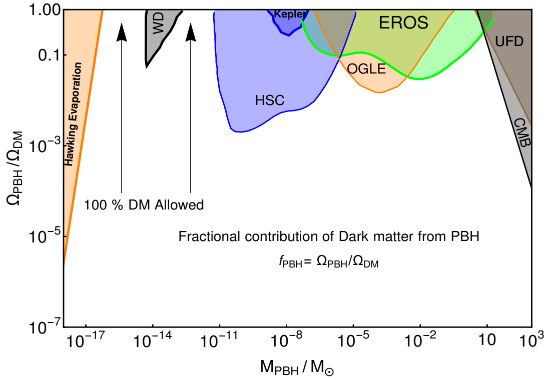




[S. Maity, **NB**, M. R. Haque, D. Maity, L. Sriramkumar, arXiv:2403.16963] (Accpted for publication in JCAP)

Ultra low mass PBHs form during radiaion domination





$$\text{Inflation} \xrightarrow{\text{eRD}} \text{PBH formation}(\tau_f) \xrightarrow{\text{eRD}} \text{PBH domination}(\tau_m) \xrightarrow{\text{eMD}} \text{PBH evaporation}(\tau_r) \xrightarrow{\text{RD}}$$

Adiabatic perturbations from two contributions

1. Poisson Distribution of PBHs
2. Cutoff for scales bellow PBH mean distance
3. Finite duration of PBH domination (Non-linearity bound)

$$\mathcal{P}_{\rm PBH}(k,\tau_r) = \frac{2}{3\pi}\left(\frac{k}{k_{\rm UV}}\right)^3\left(5+\frac{4}{9}\frac{k^2}{k_m^2}\right)^{-2}$$

$$k_{\rm UV} = \gamma^{-1/3}\beta^{1/3}k_f$$

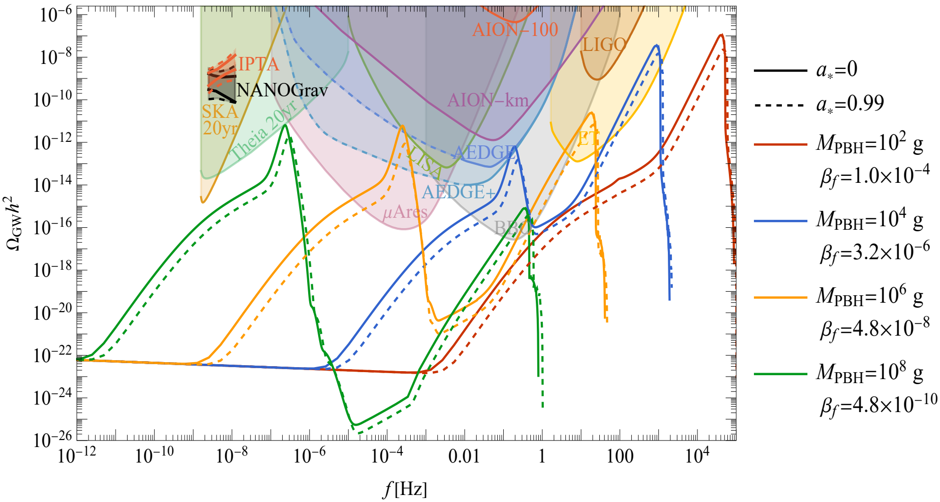
[Papanikolaou et al (2021), Domenech et al (2021)]

$$\mathcal{P}_{\rm infl}(k,\tau_r) = A_s\left(\frac{k}{k_p}\right)^{n_s-1}\theta_H(k_m-k)$$

$$\Phi(k,\tau_r) = \Phi_{\rm infl}(k,\tau_r) + \Phi_{\rm PBH}(k,\tau_r)$$

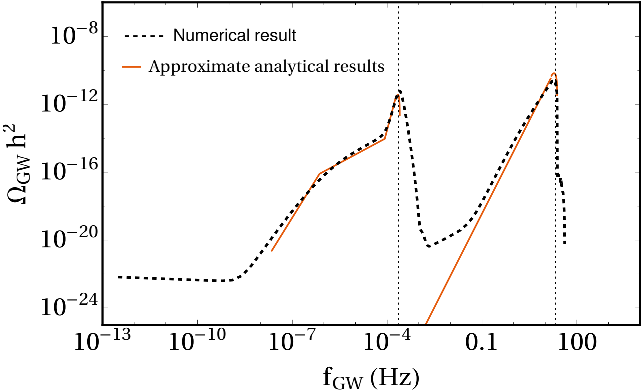
$$\mathcal{P}(k,\tau_r) = \mathcal{P}_{\rm infl}(k,\tau_r) + \mathcal{P}_{\rm PBH}(k,\tau_r)$$

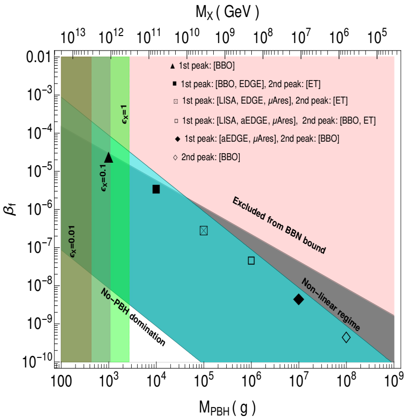
Two poltergeistic peaks of ISGWB



[**NB**, Ghoshal and Lewicki, JHEP 07, 130, 2022] [**NB**, Ghoshal, Jain and Lewicki, JHEP 05, 169, 2023]

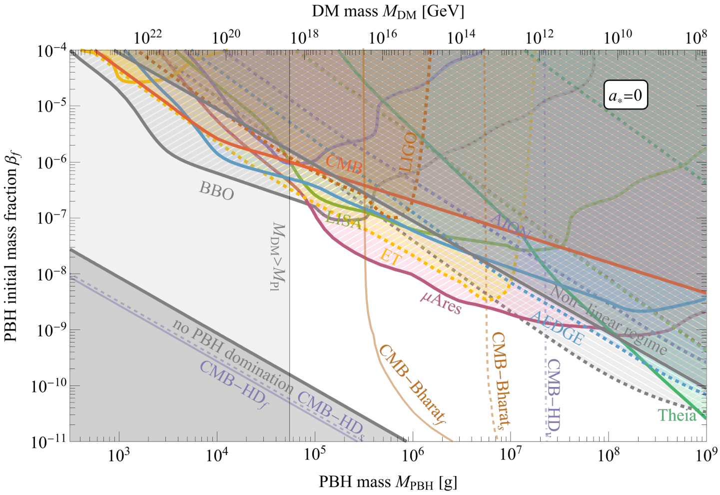
Two Peaks of ISGWB





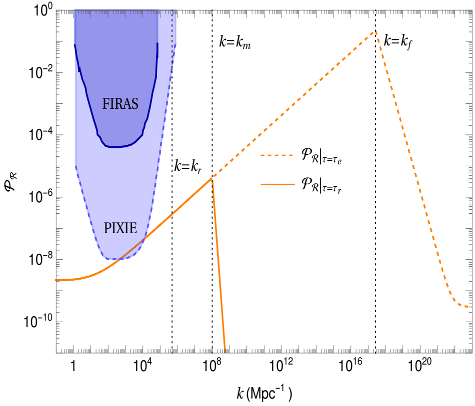
[**NB**, Ghoshal and Lewicki, JHEP 07, 130, 2022] [**NB**, Ghoshal, Jain and Lewicki, JHEP 05, 169, 2023]

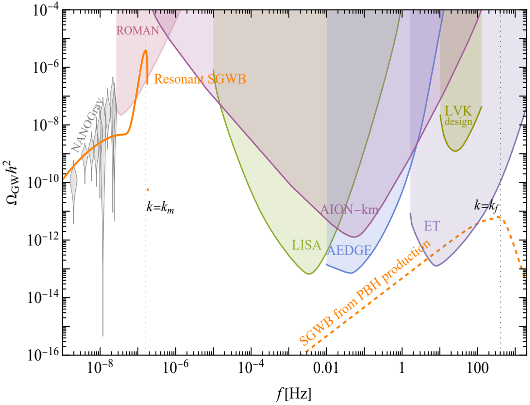
Detection prospects for ISGWB (SNR $\geq$ 10) and CMB Complementarity



[**NB**, Ghoshal, Jain and Lewicki, JHEP 05, 169, 2023]

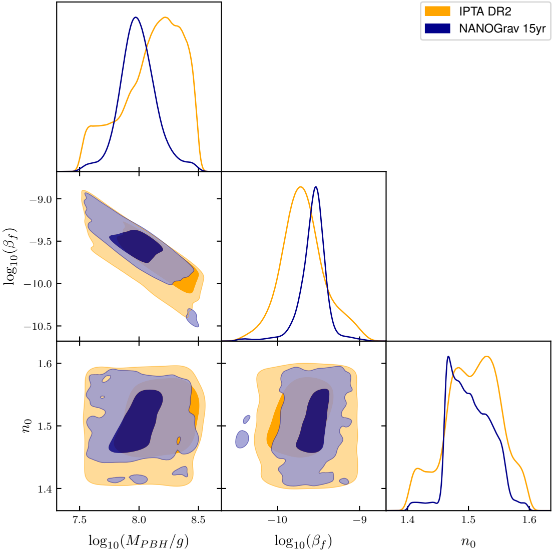
Signatures in NANOGrav signal





[**NB**, Jain and Lewicki, Phys.Rev.D 108 (2023) 12, arXiv:2308.07912]

Bayesian analysis



	Posterior mean	
Parameters	NG15	IPTA2
$\log_{10}(\frac{M_{\rm PBH}}{1\rm g})$	$7.99^{+0.13}_{-0.15}$	$8.11^{+0.35}_{-0.12}$
$\log_{10}(\beta_f)$	$-9.57^{+0.15}_{-0.11}$	$-9.69^{+0.22}_{-0.28}$
n_0	$1.503^{+0.025}_{-0.042}$	$1.507^{+0.047}_{-0.040}$

Model X	**Model Y**	$\mathrm{BF}_{Y,X}$	
		NG15	IPTA2
SMBHB	Ultra-low mass PBHs	18.00 ± 1.75	3.31 ± 0.09

[**NB**, Jain and Lewicki, Phys.Rev.D 108 (2023) 12]

Ultra low mass PBHs form during reheating

$$\text{Inflation} \xrightarrow{\text{reheating}} \text{PBH formation}(\tau_f) \xrightarrow{\text{reheating}} \text{PBH domination}(\tau_m) \xrightarrow{\text{eMD}} \text{PBH evaporation}(\tau_r) \xrightarrow{\text{RD}} \text{BBN}$$

$$s^2(1+s^{-3w})\frac{d^2\delta_m(k,s)}{ds^2} + \frac{3}{2}s(1+(1-w)s^{-3w})\frac{d\delta_m(k,s)}{ds} - \frac{3}{2}\delta_m(k,s) = 0,$$

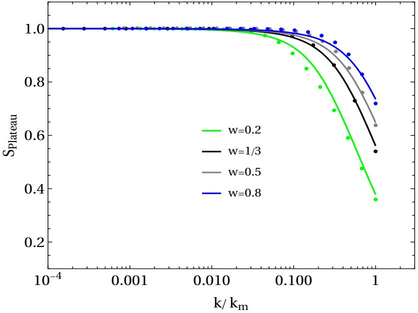
$$\Phi(k,\tau_m)|_{\rm PBH} = \frac{3}{2}\left(\frac{k_m}{k}\right)^2 {}_2F_1\left[-\frac{1}{3w},\frac{1}{2w},\frac{1}{2}+\frac{1}{6w},-1\right]\delta_i(k)$$

$$k_r \approx \left(\frac{2.1\times10^{11}}{\rm Mpc}\right)\left(\frac{M_{\rm PBH}}{10^4\,{\rm g}}\right)^{-\frac{3}{2}},$$

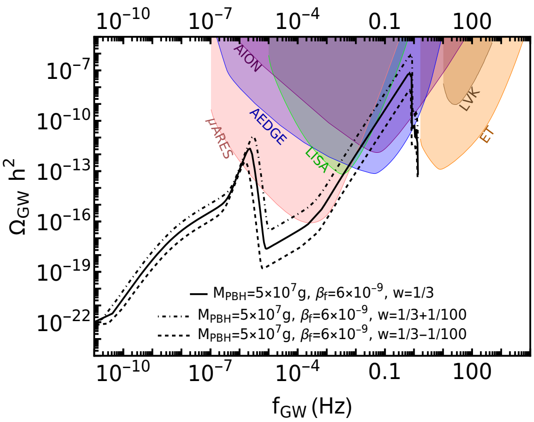
$$k_m \approx \left(\frac{8.7\times10^{17}}{\rm Mpc}\right)\left(\frac{M_{\rm PBH}}{10^4\,{\rm g}}\right)^{-\frac{5}{6}} \beta_f^{\frac{1+w}{6w}} w^{\frac{1}{2}}\left(w+\frac{1}{3}\right),$$

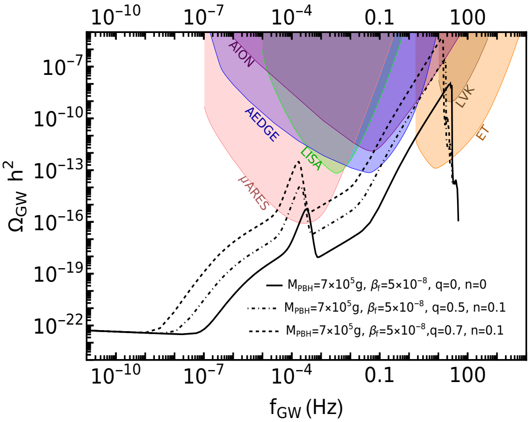
$$k_f \approx \left(\frac{8.7\times10^{17}}{\rm Mpc}\right)\left(\frac{M_{\rm PBH}}{10^4\,{\rm g}}\right)^{-\frac{5}{6}} \beta_f^{-\frac{1}{3}} w^{\frac{1}{2}}\left(w+\frac{1}{3}\right)$$

$$k_{\rm UV} \approx \left(\frac{8.7\times10^{17}}{\rm Mpc}\right)\left(\frac{M_{\rm PBH}}{10^4{\rm g}}\right)^{-\frac{5}{6}}\left(w+\frac{1}{3}\right)$$



The resultant SGWB spectra from reheating and memory burden effect





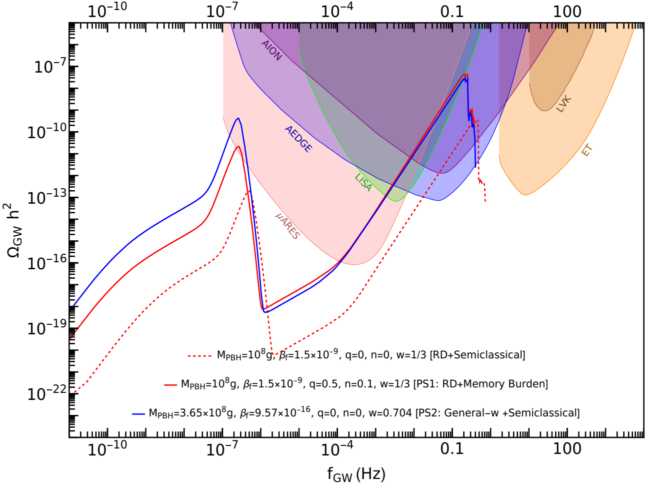
$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\frac{\pi\,\mathcal{G}\,g_{*,H}M_{\rm Pl}^4}{480M^2}\begin{cases}1 & \text{for } M > qM_{\rm PBH}\\ S^{-n}(M) & \text{for } M < qM_{\rm PBH}\end{cases}.$$

[Dvali et al, Phys. Rev. D 102, 103523 (2020)]

[Balaji et al, JCAP 11 026 (2024)]

[**NB**, Haque, Jain and Lewicki, JHEP 10 (2024) 142]

To break the degeneracy



[**NB**, Haque, Jain and Lewicki, JHEP 10 (2024) 142, arXiv:2409.04436]

Please attend Rajeev's talk for more details

Summary

- When there is a non-standard reheating phase, for a given inflationary model, PBH formation and SGWB generation is altered even if PBHs form during RD.
- PBH formation during reheating can explain NANOGrav 15 year data.
- Early ultra-low mass PBH-dominated universe leads to a uniquely shaped doubly peaked ISGWB spectrum; one peak from the inflationary adiabatic and another from isocurvature-induced adiabatic scalar modes.
- The amplitude of the ISGWB peaks enables us to constrain the initial abundance of PBHs.
- If a broadly peaked inflationary scalar spectra leads to ultra-low mass PBH formation and domination, it can also lead to resonant amplification of ISGWB just after PBH evaporation and explain the NANOGrav signal.
- The formation of ultra-low mass PBHs during non-stanadard reheating is degenerate with the memory burden effect for PBH density fluctuation induced SGWB peak, but this degeneracy is broken if we consider also the inflationary adiabatic peak.

Thank You

Effects of eMD on the transfer function

The scalar transfer function in RD

$$\mathcal{T}_k''(\tau) + 4\mathcal{H}\mathcal{T}_k'(\tau) + \frac{k^2}{3}\mathcal{T}_k(\tau) = 0 \quad (1)$$

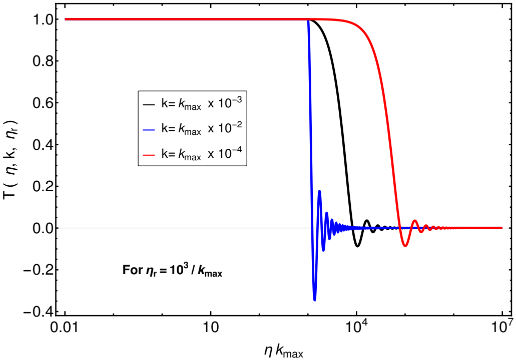
Background with eMD

$$\frac{a(\tau)}{a(\tau_r)} = 2\frac{\tau}{\tau_r} - 1$$

$$\mathcal{H} = \frac{1}{\tau - \tau_r/2}$$

Initial condition with eMD ($x = \tau k$)

- $\mathcal{T}(x, x_r)|_{x=x_r} = 1$
- $\partial_x \mathcal{T}(x, x_r)|_{x=x_r} = 0$



Scalar transfer function $\mathcal{T}_k(\tau)$ and Kernel I(u,v,x,x_r)

Oscillating terms

$$\mathcal{I} = I(u, v, x, x_r) \times (x - x_r/2)$$
$$\mathcal{I} = \mathcal{I}_s \sin(x) + \mathcal{I}_c \cos(x) + \text{4 other terms}$$

Oscillation avarage

$$\overline{\mathcal{I}^2} = \frac{1}{2}\left(\mathcal{I}_s^2 + \mathcal{I}_c^2\right)$$

Different k regiemes

$$\mathcal{I}_s \simeq \mathcal{I}_{ss} + \boxed{\mathcal{I}_{sl} x_r^4} + \text{other terms}$$
$$\mathcal{I}_c \simeq \mathcal{I}_{cs} + \boxed{\mathcal{I}_{cl} x_r^4} + \text{other terms}$$

$$I = I_{eRD} + I_{eMD} + I_{RD}$$
$$\Phi(k, \tau_r) = \Phi_{\rm infl}(k, \tau_r) + \Phi_{\rm PBH}(k, \tau_r)$$

Isocurvature Perturbation

$$d_f \equiv \left(\frac{3M_{\mathrm{PBH},f}}{4\pi\rho_{\mathrm{PBH},f}}\right)^{1/3} = \gamma^{1/3}\beta^{-1/3}H_f{}^{-1}$$

$$\langle\delta\rho_{\mathrm{PBH}}(k)\delta\rho_{\mathrm{PBH}}(k')\rangle = \frac{4\pi}{3}\left(\frac{d}{a}\right)^3 \rho_{\mathrm{PBH}}^2\,\delta(k+k')$$

$$S = \frac{\delta\rho_{\mathrm{PBH}}}{\rho_{\mathrm{PBH}}} - \frac{3}{4}\frac{\delta\rho_r}{\rho_r} = \frac{\delta\rho_{\mathrm{PBH}}}{\rho_{\mathrm{PBH}}} + \frac{3}{4}\frac{\delta\rho_{\mathrm{PBH}}}{\rho_r} \approx \frac{\delta\rho_{\mathrm{PBH}}}{\rho_{\mathrm{PBH}}} \quad \text{for } \rho_r \gg \rho_{\mathrm{PBH}}$$

$$\mathcal{P}_S(k) = \frac{2}{3\pi}\left(\frac{k}{k_{UV}}\right)^3$$

$$\Phi_{\mathrm{eMD}}(k; a \gg a_{\mathrm{eq}}) = S\begin{cases} \dfrac{1}{5} & k \ll k_{\mathrm{eq}} \\ \dfrac{3}{4}\left(\dfrac{k_{\mathrm{eq}}}{k}\right)^2 & k \gg k_{\mathrm{eq}} \end{cases}$$

Analytical form of ISGWB for inflationary adiabatic perturbation

$$\frac{\Omega_{GW}(\tau_0,k)}{A_s^2 c_g\, \Omega_{r,0}} \simeq \begin{cases} 3\times 10^{-7} x_r^3 x_{\rm max}^5 & 150 x_{\rm max}^{-5/3} \lesssim x_r \ll 1 \\ 6.6\times 10^{-7} x_r x_{\rm max}^5 & 1 \ll x_r \lesssim x_{\rm max}^{5/6} \\ 3\times 10^{-7} x_r^7 & x_{\rm max}^{5/6} \lesssim x_r \lesssim \frac{2}{1+\sqrt{3}} x_{\rm max} \\ \mathcal{C}(k) & \frac{2}{1+\sqrt{3}} \leq \frac{x_r}{x_{\rm max}} \leq \frac{2}{\sqrt{3}} \end{cases}, \tag{2}$$

where

$$\mathcal{C}(k) = 0.00638 \times 2^{-2n_s-13}\, 3^{n_s}\, x_r^7\, s_0 \left(\frac{x_r}{x_{max}}\right)^{2n_s-2} \times$$
$$\left(-s_0^2\, {}_2F_1(\frac{3}{2}, -n_s; \frac{5}{2}; \frac{s_0^2}{3}) + 4\, {}_2F_1(\frac{1}{2}, 1-n_s; \frac{3}{2}; \frac{s_0^2}{3}) - 3\, {}_2F_1(\frac{1}{2}, -n_s; \frac{3}{2}; \frac{s_0^2}{3})\right). \tag{3}$$

Analytical form for isocurvature induced adiabatic contribution to ISGWB

$$\Omega_{GW}(\tau_0, k) = c_g\, \Omega_{r,0}\, \mathcal{J} \int_{-s_0}^{s_0} \frac{27\sqrt[3]{3}\left(s^2-1\right)^2}{\left(9-3s^2\right)^{5/3}} \mathrm{d}s \tag{4}$$

$$= c_g\, \Omega_{r,0}\, \mathcal{J} \frac{2}{5} s_0 \left(\frac{3\left(14-3s_0^2\right)}{\left(1-\frac{s_0^2}{3}\right)^{2/3}} - 37\, {}_2F_1\left(\frac{1}{2}, \frac{2}{3}; \frac{3}{2}; \frac{s_0^2}{3}\right) \right) , \tag{5}$$

where,

$$\mathcal{J} = \frac{k^3 k_m^8 \left(\frac{k}{k_r}\right)^{2/3}}{1327104\sqrt[3]{2}\sqrt{3}\pi k_r^5 k_{\mathrm{UV}}^6} .$$

and

$$s_0 = \begin{cases} 1 & \frac{k}{k_{\mathrm{UV}}} \leq \frac{2}{1+\sqrt{3}} \\ 2\frac{k_{\mathrm{UV}}}{k} - \sqrt{3} & \frac{2}{1+\sqrt{3}} \leq \frac{k}{k_{\mathrm{UV}}} \leq \frac{2}{\sqrt{3}} \end{cases} . \tag{6}$$

Inflationary adiabatic and isocurvature-induced adiabatic perturbation

$$\Phi^{\rm eISO}_{\rm eMD}(k; a \gg a_{\rm eq}) \approx \begin{cases} \dfrac{1}{5} & k \ll k_{\rm eq} \\ \dfrac{3}{4}\left(\dfrac{k_{\rm eq}}{k}\right)^2 & k \gg k_{\rm eq} \end{cases},$$

$$\Phi^{e\rm CVT}_{e\rm MD}(k \gg k_{\rm eq}; a \gg a_{\rm eq}) \approx \frac{135}{16}\left(\frac{k_{\rm eq}}{k}\right)^4 \left(\ln 4 - \frac{7}{2} + \gamma_E + \ln\left(\sqrt{\frac{2}{3}}\frac{k}{k_{\rm eq}}\right)\right)$$

[KODAMA, SASAKI (1987). International Journal of Modern Physics A, 02(02), 491–560.]

Press Schechter Formalism

(Radiation dominated Epoch $\quad w = 1/3 \quad R = 1/k = (aH)^{-1}$)

$$\sigma_\delta^2(R) = \frac{16}{81}\int \frac{dk}{k}(kR)^4 P_{\mathcal{R}}(k) W^2(k,R) \tag{7}$$

$$\beta_f(M) = \frac{1}{2}\mathrm{erfc}\left(\frac{\delta_c}{\sqrt{2}\,\sigma_\delta(M(R))}\right) \tag{8}$$

$$M(R_f) = 4\pi\gamma M_{\mathrm{Pl}}^2\left(\frac{a_{eq}}{R_{eq}}\right)R_f^2$$

$$\beta_{eq}(M) = \beta_f(M)\left(\frac{a_{eq}}{a_f}\right) = \beta_f(M)\left(\frac{R_{eq}}{R_f}\right)$$

$$f_{PBH} = \frac{\beta_{eq}(M)}{\Omega_{DM}(M)}$$

Induced Stochastic GW Background (ISGWB)

$$h''_{\mathbf{k}}(\tau) + 2\mathcal{H}h'_{\mathbf{k}}(\tau) + k^2 h_{\mathbf{k}}(\tau) = 4S_{\mathbf{k}}(\tau)$$

$$S_k^s = \int \frac{\mathrm{d}^3 q}{(2\pi)^{3/2}} e_{ij}^s(k) q_i q_j \left[2\Phi_q \Phi_{k-q} + \frac{4}{3(1+w)}(\mathcal{H}^{-1}\Phi'_q + \Phi_q)(\mathcal{H}^{-1}\Phi'_{k-q} + \Phi_{k-q})\right]$$

$$\Omega_{\mathrm{GW}}(\tau, k) = \frac{1}{6}\left(\frac{k}{\mathcal{H}}\right)^2 \int_0^\infty \mathrm{d}v \int_{|1-v|}^{1+v} \mathrm{d}u \left(\frac{4v^2 - (1+v^2-u^2)^2}{4uv}\right)^2$$

$$\times \overline{I_{\mathrm{RD}}^2(v, u, x)} \mathcal{P}_{\mathcal{R}}(kv) \mathcal{P}_{\mathcal{R}}(ku)$$

$$I_{RD}(u, v, x) = \int_0^x d\bar{x} \frac{a(\bar{x})}{a(x)} f(u, v, \bar{x}) k G(\bar{x}, x)$$

Transfer function in RD in presence of eMD

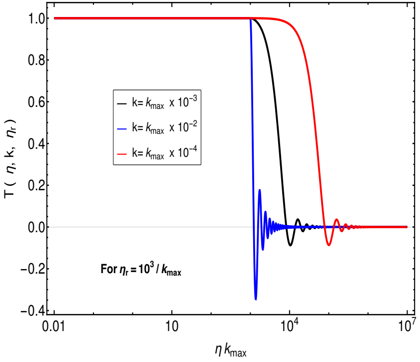
$$\mathcal{T}_k''(\tau) + 4\mathcal{H}\mathcal{T}_k'(\tau) + \frac{k^2}{3}\mathcal{T}_k(\tau) = 0$$

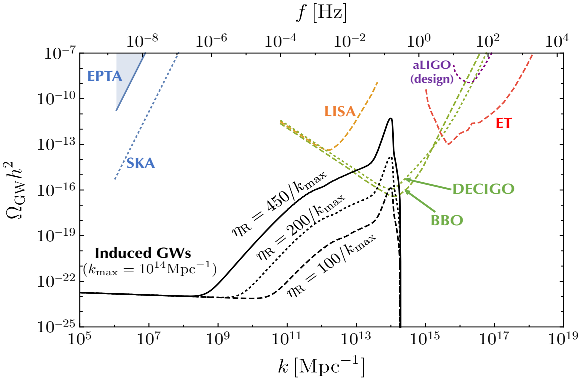
$$\mathcal{T}(x, x_r) = \frac{3\sqrt{3}\left[A(x_r)\, j_1\left(\frac{x-x_r/2}{\sqrt{3}}\right) + B(x_r)\, y_1\left(\frac{x-x_r/2}{\sqrt{3}}\right)\right]}{x - x_r/2}$$

$$\begin{aligned} A(x_r) &= \tfrac{x_r}{2\sqrt{3}} \sin\left(\tfrac{x_r}{2\sqrt{3}}\right) - \tfrac{1}{36}(x_r^2 - 36)\cos\left(\tfrac{x_r}{2\sqrt{3}}\right) \\ B(x_r) &= -\tfrac{1}{36}(x_r^2 - 36)\sin\left(\tfrac{x_r}{2\sqrt{3}}\right) - \tfrac{x_r}{2\sqrt{3}}\cos\left(\tfrac{x_r}{2\sqrt{3}}\right) \end{aligned}$$

$x = \tau k \qquad x_r = \tau_r k \qquad \frac{a(\tau)}{a(\tau_r)} = 2\frac{\tau}{\tau_r} - 1 \qquad \mathcal{H} = aH = \frac{1}{\tau - \tau_r/2} \qquad \mathcal{T}_k(\tau_r) = 1 \qquad \mathcal{T}_k'(\tau_r) = 0$

Resonant amplification of ISGWB





[Inomata et al (2019)]

Total efolds

In radiation dominated universe, $a \propto t^{1/2}$, $H \propto t^{-1}$ and $k = aH \propto t^{-1/2} \propto H^{1/2}$ .As $\rho = 3H^2 M_{pl}^2$, $k = aH \propto \rho^{1/4}$.

$$\frac{k}{k_{eq}} = \left(\frac{\rho}{\rho_{r,eq}}\right)^{1/4}$$
$$\rho_{r,eq} = \Omega_r (z_{eq}+1)^4 \rho_{critical} = \Omega_r (z_{eq}+1)^4 (3H_{present}^2 M_{pl}^2)$$

$$H_e = H_{ri} = \left(P_\zeta(k) 8\pi^2 M_{Pl}^2 \epsilon\right)^{1/2}$$
$$\rho_e = 3H_e^2 M_{pl}^2 = 8\pi^2 \left(P_\zeta(k) M_{Pl}^2 \epsilon\right)$$
$$k_{ri} = \frac{k_{eq}}{(z_{eq}+1)} \left(\frac{P_\zeta(k) 8\pi^2 \epsilon}{\Omega_r H_{present}^2}\right)^{1/4}$$

$$k_{ri} = \mathbf{1.94 \times 10^{24}} (\epsilon)^{\mathbf{1/4}} \mathbf{Mpc^{-1}}$$
$$N_e - N_p = ln\left(\frac{k_{ri}}{k_p}\right) = \mathbf{58.92} + \frac{\mathbf{1}}{\mathbf{4}} \mathbf{ln}(\epsilon)$$