Clebsch-Gordan coefficients for Macdonald polynomials

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- There are two types of Macdonald polynomials:
  (a) The symmetric Macdonald polynomials

$$P_m(X) \in \mathbb{C}[X^{\pm 1}] \qquad m \in \mathbb{Z}_{\geq 0}$$

  (b) The nonsymmetric Macdonald polynomials

$$E_m(X) \in \mathbb{C}[X^{\pm 1}] \qquad m \in \mathbb{Z}$$

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  $$E_m(X) \in \mathbb{C}[X^{\pm 1}] \qquad m \in \mathbb{Z}$$
- $\{P_m(X) : m \in \mathbb{Z}_{\geq 0}\}$ forms a basis for $\mathbb{C}[X + X^{-1}]$.
- $\{E_m(X) : m \in \mathbb{Z}\}$ forms a basis for $\mathbb{C}[X^{\pm 1}]$.

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- $\{P_m(X) : m \in \mathbb{Z}_{\geq 0}\}$ forms a basis for $\mathbb{C}[X + X^{-1}]$.
- $\{E_m(X) : m \in \mathbb{Z}\}$ forms a basis for $\mathbb{C}[X^{\pm 1}]$.
- This talk will be about the products $E_\ell P_m$ and $P_\ell P_m$.

Examples of Symmetric Macdonald Polynomials

The symmetric Macdonald polynomials

$$P_m(X) \in \mathbb{C}[X + X^{-1}], \qquad m \in \mathbb{Z}_{\geq 0}$$

$$
\begin{aligned}
P_0 &= 1 \\
P_1 &= X + X^{-1} \\
P_2 &= (X^2 + X^{-2}) + \frac{(1-q^2)(1-t)}{(1-q)(1-qt)} \\
P_3 &= (X^3 + X^{-3}) + \frac{(1-q^3)(1-t)}{(1-q^2t)(1-q)}(X + X^{-1}) \\
P_4 &= (X^4 + X^{-4}) + \frac{(1-q^4)(1-t)}{(1-q^3t)(1-q)}(X^2 + X^{-2}) \\
&\quad + \frac{(1-q^4)(1-q^3)(1-qt)(1-t)}{(1-q^3t)(1-q^2t)(1-q^2)(1-q)} \\
&\vdots
\end{aligned}
$$

Examples of NonSymmetric Macdonald Polynomials

The non-symmetric Macdonald polynomials

$$E_m(X) \in \mathbb{C}[X^{\pm 1}], \qquad m \in \mathbb{Z}$$

$$\begin{aligned}
&\vdots \\
E_{-2}(X) &= X^{-2} + \frac{(1-t)(1-q^2)}{(1-q)(1-q^2t)} + \frac{(1-t)}{(1-q^2t)}X^2 \\
E_{-1}(X) &= X^{-1} + \frac{1-t}{1-qt}X, \\
E_0(X) &= 1, \\
E_1(X) &= X, \\
E_2(X) &= X^2 + q\frac{(1-t)}{(1-qt)}, \\
&\vdots
\end{aligned}$$

Examples of Products

$$E_1 P_m = E_{m+1} + \frac{(1-q^m)}{(1-tq^m)} E_{-m+1}$$

$$E_2 P_m = E_{m+2} + \frac{(1-t)}{(1-tq)} \cdot \frac{(1-q^m)}{(1-tq^{m-1})} \cdot \frac{(1-t^2q^m)}{(1-tq^m)} E_m$$

$$+ \frac{(1-q^{m-1})}{(1-tq^{m-1})} \frac{(1-q^m)}{(1-tq^m)} \cdot \frac{(1-t^2q^{m-1})}{(1-tq^{m-1})} E_{-m+2}$$

$$+ q\frac{(1-t)}{(1-tq)} \cdot \frac{(1-q^m)}{(1-tq^{m+1})} E_{-m}$$

Examples of Products

$$E_3 P_m = E_{m+3} + \frac{(1-t)(1-q^2)}{(1-q)(1-tq^2)} \cdot \frac{(1-q^m)}{(1-tq^{m-1})} \cdot \frac{(1-t^2q^{m+1})}{(1-tq^{m+1})} \, E_{m+1}$$

$$+ \frac{(1-t)}{(1-tq^2)} \cdot \frac{(1-q^{m-1})(1-q^m)}{(1-tq^{m-2})(1-tq^{m-1})} \cdot \frac{(1-t^2q^{m-1})(1-t^2q^m)}{(1-tq^{m-1})(1-tq^m)} E_{m-1}$$

$$+ \frac{(1-q^{m-2})(1-q^{m-1})(1-q^m)}{(1-tq^{m-2})(1-tq^{m-1})(1-tq^m)} \cdot \frac{(1-t^2q^{m-2})(1-t^2q^{m-1})}{(1-tq^{m-2})(1-tq^{m-1})} E_{-m+3}$$

$$+ q\frac{(1-t)(1-q^2)}{(1-q)(1-tq^2)} \cdot \frac{(1-q^{m-1})(1-q^m)}{(1-tq^m)(1-tq^{m+1})} \cdot \frac{(1-t^2q^m)}{(1-tq^{m-1})} \, E_{-m+1}$$

$$+ q^2\frac{(1-t)}{(1-tq^2)} \cdot \frac{(1-q^m)}{(1-tq^{m+2})} \, E_{-m-1}$$

Examples of Products

$$E_3 P_m = E_{m+3} + \frac{(1-t)(1-q^2)}{(1-q)(1-tq^2)} \cdot \frac{(1-q^m)}{(1-tq^{m-1})} \cdot \frac{(1-t^2q^{m+1})}{(1-tq^{m+1})}\, E_{m+1}$$
$$+ \frac{(1-t)}{(1-tq^2)} \cdot \frac{(1-q^{m-1})(1-q^m)}{(1-tq^{m-2})(1-tq^{m-1})} \cdot \frac{(1-t^2q^{m-1})(1-t^2q^m)}{(1-tq^{m-1})(1-tq^m)} E_{m-1}$$
$$+ \frac{(1-q^{m-2})(1-q^{m-1})(1-q^m)}{(1-tq^{m-2})(1-tq^{m-1})(1-tq^m)} \cdot \frac{(1-t^2q^{m-2})(1-t^2q^{m-1})}{(1-tq^{m-2})(1-tq^{m-1})} E_{-m+3}$$
$$+ q\frac{(1-t)(1-q^2)}{(1-q)(1-tq^2)} \cdot \frac{(1-q^{m-1})(1-q^m)}{(1-tq^m)(1-tq^{m+1})} \cdot \frac{(1-t^2q^m)}{(1-tq^{m-1})}\, E_{-m+1}$$
$$+ q^2\frac{(1-t)}{(1-tq^2)} \cdot \frac{(1-q^m)}{(1-tq^{m+2})}\, E_{-m-1}$$

- All the coefficients are products,

Examples of Products

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\begin{aligned}
E_3P_m = E_{m+3} &+ \frac{(1-t)(1-q^2)}{(1-q)(1-tq^2)} \cdot \frac{(1-q^m)}{(1-tq^{m-1})} \cdot \frac{(1-t^2q^{m+1})}{(1-tq^{m+1})}\ E_{m+1} \\
&+ \frac{(1-t)}{(1-tq^2)} \cdot \frac{(1-q^{m-1})(1-q^m)}{(1-tq^{m-2})(1-tq^{m-1})} \cdot \frac{(1-t^2q^{m-1})(1-t^2q^m)}{(1-tq^{m-1})(1-tq^m)} E_{m-1} \\
&+ \frac{(1-q^{m-2})(1-q^{m-1})(1-q^m)}{(1-tq^{m-2})(1-tq^{m-1})(1-tq^m)} \cdot \frac{(1-t^2q^{m-2})(1-t^2q^{m-1})}{(1-tq^{m-2})(1-tq^{m-1})} E_{-m+3} \\
&+ q\frac{(1-t)(1-q^2)}{(1-q)(1-tq^2)} \cdot \frac{(1-q^{m-1})(1-q^m)}{(1-tq^m)(1-tq^{m+1})} \cdot \frac{(1-t^2q^m)}{(1-tq^{m-1})}\ E_{-m+1} \\
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\end{aligned}
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- All the coefficients are products,
- coefficients are functions of q^m,

Examples of Products

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\begin{aligned}
E_3 P_m = E_{m+3} &+ \frac{(1-t)(1-q^2)}{(1-q)(1-tq^2)} \cdot \frac{(1-q^m)}{(1-tq^{m-1})} \cdot \frac{(1-t^2q^{m+1})}{(1-tq^{m+1})} \, E_{m+1} \\
&+ \frac{(1-t)}{(1-tq^2)} \cdot \frac{(1-q^{m-1})(1-q^m)}{(1-tq^{m-2})(1-tq^{m-1})} \cdot \frac{(1-t^2q^{m-1})(1-t^2q^m)}{(1-tq^{m-1})(1-tq^m)} E_{m-1} \\
&+ \frac{(1-q^{m-2})(1-q^{m-1})(1-q^m)}{(1-tq^{m-2})(1-tq^{m-1})(1-tq^m)} \cdot \frac{(1-t^2q^{m-2})(1-t^2q^{m-1})}{(1-tq^{m-2})(1-tq^{m-1})} E_{-m+3} \\
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&+ q^2\frac{(1-t)}{(1-tq^2)} \cdot \frac{(1-q^m)}{(1-tq^{m+2})} \, E_{-m-1}
\end{aligned}
$$

- All the coefficients are products,
- coefficients are functions of q^m,
- in $E_\ell P_m$ the terms that appear are $E_{\pm m+\ell-2j}$ where $j \in \{0, \ldots, \ell-1\}$

Examples of Products

- All coefficients are products,
- coefficients are functions of q^m,
- in $E_{-\ell+1}P_m$ the terms are $E_{\pm m+(-\ell+1)+2j}$ where $j \in \{0, \dots, \ell-1\}$

$$
\begin{aligned}
E_{-1}P_m = E_{-m-1} &+ \frac{(1-t)}{(1-tq)} \cdot \frac{(1-q^m)}{(1-tq^{m-1})} \cdot \frac{(1-t^2q^m)}{(1-tq^m)} E_{-m+1} \\
&+ t\frac{(1-q^{m-1})}{(1-tq^{m-1})} \frac{(1-q^m)}{(1-tq^m)} \cdot \frac{(1-t^2q^{m-1})}{(1-tq^{m-1})} E_{m-1} \\
&+ tq\frac{(1-t)}{(1-tq)} \cdot \frac{(1-q^m)}{(1-tq^{m+1})} E_{m+1}
\end{aligned}
$$

$$P_m = E_0P_m = E_{-m} + t\frac{(1-q^m)}{(1-tq^m)} E_m$$

Examples of Products

$$P_1 P_m = P_{m+1} + \frac{(1-q^m)}{(1-tq^m)} \frac{(1-t^2q^{m-1})}{(1-tq^{m-1})} P_{m-1}$$

$$P_2 P_m = P_{m+2} + \frac{(1-q^2)(1-t)}{(1-tq)(1-q)} \cdot \frac{(1-q^m)}{(1-tq^{m+1})} \cdot \frac{(1-t^2q^m)}{(1-tq^{m-1})} P_m + \frac{(1-q^{m-1})(1-q^m)}{(1-tq^{m-1})(1-tq^m)} \cdot \frac{(1-t^2q^{m-2})(1-t^2q^{m-1})}{(1-tq^{m-2})(1-tq^{m-1})} P_{m-2}$$

Examples of Products

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$$\begin{aligned} P_2 P_m = P_{m+2} &+ \frac{(1-q^2)(1-t)}{(1-tq)(1-q)} \cdot \frac{(1-q^m)}{(1-tq^{m+1})} \cdot \frac{(1-t^2 q^m)}{(1-tq^{m-1})} P_m \\ &+ \frac{(1-q^{m-1})(1-q^m)}{(1-tq^{m-1})(1-tq^m)} \cdot \frac{(1-t^2 q^{m-2})(1-t^2 q^{m-1})}{(1-tq^{m-2})(1-tq^{m-1})} P_{m-2} \end{aligned}$$

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Examples of Products

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$$P_2 P_m = P_{m+2} + \frac{(1-q^2)(1-t)}{(1-tq)(1-q)} \cdot \frac{(1-q^m)}{(1-tq^{m+1})} \cdot \frac{(1-t^2 q^m)}{(1-tq^{m-1})} P_m + \frac{(1-q^{m-1})(1-q^m)}{(1-tq^{m-1})(1-tq^m)} \cdot \frac{(1-t^2 q^{m-2})(1-t^2 q^{m-1})}{(1-tq^{m-2})(1-tq^{m-1})} P_{m-2}$$

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Examples of Products

$$P_1 P_m = P_{m+1} + \frac{(1-q^m)}{(1-tq^m)} \frac{(1-t^2q^{m-1})}{(1-tq^{m-1})} P_{m-1}$$

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- All the coefficients are products,
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- in $P_\ell P_m$ the terms that appear are $P_{m+\ell-2j}$ where $j \in \{0, \dots, \ell\}$

A formula for the Macdonald Polynomials

- q-Pochammer symbols

$$(z;q)_j = (1-z)(1-zq)(1-zq^2)\cdots(1-zq^{j-1})$$

- q,t-binomial coefficients

$$\begin{bmatrix} m \\ j \end{bmatrix}_{q,t} = \frac{\frac{(q;q)_m}{(t;q)_m}}{\frac{(q;q)_j}{(t;q)_j}\frac{(q;q)_{m-j}}{(t;q)_{m-j}}}$$

(Macdonald):

$$P_m(X) = \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_{q,t} X^{m-2j} \in \mathbb{C}[X+X^{-1}],$$

$$E_{m+1}(X) = \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_{q,t} \frac{(1-tq^{m-j})}{(1-tq^m)} q^j X^{m+1-2j}, \qquad (m \in \mathbb{Z}_{\geq 0})$$

$$E_{-m}(X) = \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_{q,t} \frac{(1-tq^{j})}{(1-tq^m)} X^{m-2j} \in \mathbb{C}[X^{\pm 1}].$$

Product Rules (Main Result)

Theorem

For $\ell \in \mathbb{Z}_{>0}$ and $m \in \mathbb{Z}_{\geq 0}$,

$$P_\ell P_m = \sum_{j=0}^{\ell} c_j^{(\ell)}(q^m)\, P_{m+\ell-2j}\,,$$

$$E_\ell P_m = \sum_{j=0}^{\ell-1} a_j^{(\ell)}(q^m) E_{m+\ell-2j} + b_j^{(\ell)}(q^m) E_{-m+\ell-2j}\,,$$

$$E_{-\ell+1} P_m = \sum_{j=0}^{\ell-1} t \cdot b_j^{(\ell)}(q^m) E_{m+(-\ell+1)+2j} + a_j^{(\ell)}(q^m) E_{-m+(-\ell+1)+2j}$$

where $c_j^{(\ell)}(q^m)\,, a_j^{(\ell)}(q^m)\,, b_j^{(\ell)}(q^m)$ are "products".

Product Rules

$$P_\ell P_m = \sum_{j=0}^{\ell} c_j^{(\ell)}(q^m)\, P_{m+\ell-2j}\,,$$

$$E_\ell P_m = \sum_{j=0}^{\ell-1} a_j^{(\ell)}(q^m) E_{m+\ell-2j} + b_j^{(\ell)}(q^m) E_{-m+\ell-2j}\,,$$

$$E_{-\ell+1} P_m = \sum_{j=0}^{\ell-1} t \cdot b_j^{(\ell)}(q^m) E_{m-(\ell-1-2j)} + a_j^{(\ell)}(q^m) E_{-m-(\ell-1-2j)}\,.$$

$$c_j^{(\ell)}(q^m) = \begin{bmatrix} \ell \\ j \end{bmatrix}_{q,t} \frac{(q^m q^{-(j-1)};q)_j}{(tq^m q^{-j};q)_j} \frac{(t^2 q^m q^{\ell-2j};q)_j}{(tq^m q^{\ell-2j+1};q)_j},$$

$$a_j^{(\ell)}(q^m) = c_j^{(\ell)}(q^m) \cdot \frac{(1-q^{\ell-j})}{(1-q^\ell)} \cdot \frac{(1-tq^m q^{\ell-j})}{(1-tq^m q^{\ell-2j})}$$

$$b_j^{(\ell)}(q^m) = c_{\ell-j}^{(\ell)}(q^m) \cdot q^j \cdot \frac{(1-q^{\ell-j})}{(1-q^\ell)} \cdot \frac{(1-tq^m q^{-(\ell-j)})}{(1-t^2 q^m q^{-(\ell-2j)})}$$

Fix $q^{\frac{1}{2}}, t^{\frac{1}{2}} \in \mathbb{C}^\times$. The double affine Hecke algebra (DAHA) for SL_2 is

$$\tilde{H}_{\mathrm{int}} = \mathbb{C}\,\mathrm{algebra}\langle T_1^{\pm 1}, X^{\pm 1}, Y^{\pm 1}, T_\pi^{\pm 1} \,|\, \mathrm{relations}\rangle$$

$$T_\pi = YT_1^{-1} = T_1 Y^{-1}\,, \qquad T_\pi X T_\pi^{-1} = q^{\frac{1}{2}} X^{-1}\,, \qquad T_\pi^2 = 1\,,$$
$$T_1 X T_1 = X^{-1}, \qquad T_1 Y^{-1} T_1 = Y, \qquad (T_1 - t^{\frac{1}{2}})(T_1 + t^{-\frac{1}{2}}) = 0.$$

" Can move all the Xs to the left, all the Ys to the right etc. "
PBW Theorem (Cherednik):

$$\tilde{H}_{\mathrm{int}} = \bigoplus_{\substack{n,m \in \mathbb{Z} \\ \varepsilon \in \{0,1\}}} \mathbb{C}\{X^n T_1^\varepsilon Y^m\}$$

Polynomial Representation

Let $\mathcal{H}_Y = \langle T_1^{\pm 1}, Y^{\pm 1} \rangle \subset \tilde{H}_{\mathrm{int}}$.
$\mathcal{H}_Y$ has a 1-dimensional representation $\mathbb{C}\mathbf{1}_Y$:

$$T_1 \mathbf{1}_Y = t^{\frac{1}{2}} \mathbf{1}_Y, Y \mathbf{1}_Y = t^{\frac{1}{2}} \mathbf{1}_Y$$

The polynomial representation is $\mathrm{Ind}_{\mathcal{H}_Y}^{\tilde{H}_{\mathrm{int}}} \mathbb{C}\mathbf{1}_Y$.
By the PBW theorem, $\mathrm{Ind}_{\mathcal{H}_Y}^{\tilde{H}_{\mathrm{int}}} \mathbb{C}\mathbf{1}_Y \cong \mathbb{C}[X^{\pm 1}]\mathbf{1}_Y$ as $\mathbb{C}$-vector spaces.

Macdonald E-polynomials

The nonsymmetric Macdonald polynomials $E_m(X) \in \mathbb{C}[X^{\pm 1}]$ are eigenvectors for the action of Y on the polynomial representation:

$$YE_m(X)\mathbf{1}_Y = \mathrm{ev}_m(Y)E_m(X)\mathbf{1}_Y$$

where

$$\mathrm{ev}_m(Y) = \begin{cases} q^{-\frac{m}{2}}t^{-\frac{1}{2}}, & m > 0\,, \\ q^{-\frac{m}{2}}t^{\frac{1}{2}}, & m \leq 0\,. \end{cases}$$

with normalization $E_m(X) = X^m + \ldots$.

Examples of products

$$\mathrm{ev}_m(Y) = q^{-\frac{m}{2}} t^{-\frac{1}{2}} \implies \mathrm{ev}_m(Y^{-2}) = q^m t$$

$$\begin{aligned} E_1 P_m &= E_{m+1} + \frac{(1-q^m)}{(1-tq^m)} E_{-m+1} \\ &= E_{m+1} + \mathrm{ev}_m\left(\frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})}\right) E_{-m+1} \end{aligned}$$

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- coefficients are functions of $q^m \implies \mathrm{ev}_m(\text{functions of } Y^{-2})$

The symmetric Macdonald polynomials can be constructed from the nonsymmetric Macdonald polynomials:

$$\mathbf{1}_0 = T_1 + t^{-\frac{1}{2}} \in \tilde{H}_{\mathrm{int}}$$

$$\begin{aligned} P_m(X)\mathbf{1}_Y &= t^{\frac{1}{2}}\mathbf{1}_0 E_m(X)\mathbf{1}_Y \\ &= E_{-m}(X)\mathbf{1}_Y + t\frac{1-q^m}{1-q^m t}E_m(X)\mathbf{1}_Y \end{aligned}$$

Strategy of proof

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$$E_\ell(X)P_m(X)\mathbf{1}_Y = t^{\frac{1}{2}} E_\ell(X)\mathbf{1}_0 E_m(X)\mathbf{1}_Y$$

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Compute $E_\ell(X)\mathbf{1}_0 \in \tilde{H}$ separately. Then apply on $E_m(X)\mathbf{1}_Y$.

Strategy of proof

$$E_\ell(X)P_m(X)\mathbf{1}_Y = t^{\frac{1}{2}} E_\ell(X)\mathbf{1}_0 E_m(X)\mathbf{1}_Y$$

Compute $E_\ell(X)\mathbf{1}_0 \in \tilde{H}$ separately. Then apply on $E_m(X)\mathbf{1}_Y$.

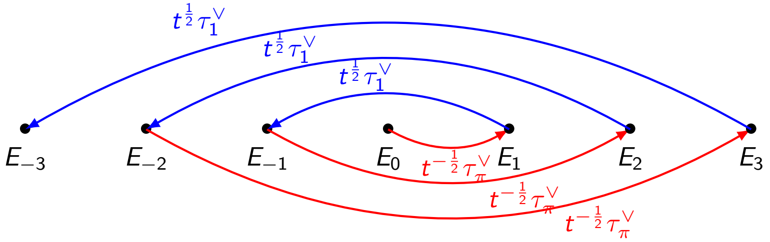
$$P_\ell(X)P_m(X)\mathbf{1}_Y = t^{\frac{1}{2}}\mathbf{1}_0 E_\ell(X) t^{\frac{1}{2}}\mathbf{1}_0 E_m(X)\mathbf{1}_Y$$

Compute $\mathbf{1}_0 E_\ell(X)\mathbf{1}_0 \in \tilde{H}$ separately. Then apply on $E_m(X)\mathbf{1}_Y$.

Reflection Intertwiners

$$\tau_1^\vee = T_1 + t^{-\frac{1}{2}} \frac{(1-t)}{(1-Y^{-2})}, \qquad \tau_\pi^\vee = XT_1 \in \tilde{H}$$

Then in the polynomial representation



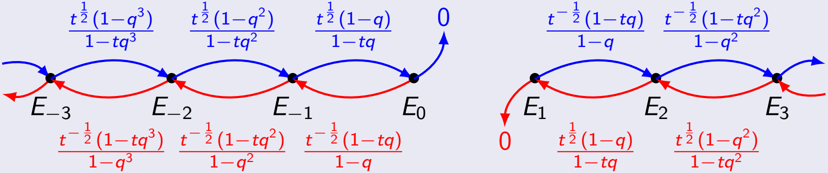
$E_0(X) = 1$

Translation Intertwiners

$$\eta = \tau_\pi^\vee t^{\frac{1}{2}} \frac{(1 - Y^{-2})}{(1 - tY^{-2})} \tau_1^\vee \qquad \eta^{-1} = t^{\frac{1}{2}} \frac{(1 - Y^{-2})}{(1 - tY^{-2})} \tau_1^\vee \tau_\pi^\vee .$$

Proposition

There exists elements $\eta, \eta^{-1} \in \tilde{H}$, *such that in* $\mathbb{C}[X^{\pm 1}]\mathbf{1}_Y$



Action of translation intertwiners

$$\eta^{-j}\eta^{\ell-j}E_m(X)\mathbf{1}_Y = t^{-\frac{1}{2}(\ell-2j)}\mathrm{ev}_m(G^+_{-j,\ell-j}(Y))E_{m+\ell-2j}(X)\mathbf{1}_Y$$
$$\eta^{\ell-j}\eta^{-j}E_{-m}(X)\mathbf{1}_Y = t^{\frac{1}{2}(\ell-2j)}\mathrm{ev}_m(G^-_{\ell-j,-j}(Y))E_{-m+\ell-2j}(X)\mathbf{1}_Y$$

where $G^+_{-j,\ell-j}(Y)$, $G^-_{\ell-j,-j}(Y)$ are products:

$$G^+_{-j,\ell-j}(Y) = \frac{(t^{-1}Y^{-2}q^{\ell-2j};q)_j}{(Y^{-2}q^{\ell-2j};q)_j}\cdot\frac{(Y^{-2};q)_{\ell-j}}{(t^{-1}Y^{-2};q)_{\ell-j}}$$
$$G^-_{\ell-j,-j}(Y) = \frac{(t^{-1}Y^{-2}q^{-(\ell-2j)+1};q)_{\ell-j}}{(Y^{-2}q^{-(\ell-2j)+1};q)_{\ell-j}}\frac{(Y^{-2}q;q)_j}{(t^{-1}Y^{-2}q;q)_j}$$

Computing $E_\ell(X)P_m(X)\mathbf{1}_Y$

$$E_\ell(X)\mathbf{1}_0 = \sum_{j=0}^{\ell-1} \eta^{\ell-2j} D_j^{(\ell-1)}(Y)\mathbf{1}_0 \in \tilde{H}\,,$$

$$\begin{aligned}
E_\ell(X)P_m(X)\mathbf{1}_Y &= E_\ell(X)t^{\frac{1}{2}}\mathbf{1}_0 E_m(X)\mathbf{1}_Y \\
&= t^{\frac{1}{2}}\Big(\sum_{j=0}^{\ell-1} \eta^{\ell-2j} D_j^{(\ell-1)}(Y)\Big)\mathbf{1}_0 E_m(X)\mathbf{1}_Y \\
&= \Big(\sum_{j=0}^{\ell-1} \eta^{\ell-2j} D_j^{(\ell-1)}(Y)\Big)P_m(X)\mathbf{1}_Y \\
&= \Big(\sum_{j=0}^{\ell-1} \eta^{\ell-2j} D_j^{(\ell-1)}(Y)\Big)\Big(E_{-m}(X) + t\frac{(1-q^m)}{(1-q^m t)}E_m(X)\Big)\mathbf{1}_Y.
\end{aligned}$$

Computing $E_\ell(X)P_m(X)\mathbf{1}_Y$

$$E_\ell(X)P_m(X)\mathbf{1}_Y = \Big(\sum_{j=0}^{\ell-1}\eta^{\ell-2j}D_j^{(\ell-1)}(Y)\Big)\Big(E_{-m}(X) + t\frac{(1-q^m)}{(1-q^m t)}E_m(X)\Big)\mathbf{1}_Y.$$

$$\eta^{\ell-2j}D_j^{(\ell-1)}(Y)E_{-m}(X)\mathbf{1}_Y = \mathrm{ev}_m(D_j^{(\ell-1)}(Y^{-1}))\eta^{\ell-j}\eta^{-j}E_{-m}(X)\mathbf{1}_Y$$
$$= \mathrm{ev}_m\Big(D_j^{(\ell-1)}(Y^{-1})t^{\frac{1}{2}(\ell-2j)}G^{-}_{\ell-j,-j}(Y)\Big)E_{-m+\ell-2j}(X)\mathbf{1}_Y$$
$$= \mathrm{ev}_m(B_j^{(\ell)}(Y))E_{-m+\ell-2j}(X)\mathbf{1}_Y,$$

Computing $E_\ell(X)P_m(X)\mathbf{1}_Y$

$$E_\ell(X)P_m(X)\mathbf{1}_Y = \Big(\sum_{j=0}^{\ell-1}\eta^{\ell-2j}D_j^{(\ell-1)}(Y)\Big)\Big(E_{-m}(X) + t\frac{(1-q^m)}{(1-q^m t)}E_m(X)\Big)\mathbf{1}_Y.$$

$$\begin{aligned}\eta^{\ell-2j}D_j^{(\ell-1)}(Y)E_{-m}(X)\mathbf{1}_Y &= \mathrm{ev}_m(D_j^{(\ell-1)}(Y^{-1}))\eta^{\ell-j}\eta^{-j}E_{-m}(X)\mathbf{1}_Y \\ &= \mathrm{ev}_m\Big(D_j^{(\ell-1)}(Y^{-1})t^{\frac{1}{2}(\ell-2j)}G^{-}_{\ell-j,-j}(Y)\Big)E_{-m+\ell-2j}(X)\mathbf{1}_Y \\ &= \mathrm{ev}_m(B_j^{(\ell)}(Y))E_{-m+\ell-2j}(X)\mathbf{1}_Y,\end{aligned}$$

Similar computations $\implies$

$$\eta^{\ell-2j}D_j^{(\ell-1)}(Y)t\frac{(1-q^m)}{(1-q^m t)}E_m(X)\mathbf{1}_Y = \mathrm{ev}_m(A_j^{(\ell)}(Y))E_{m+\ell-2j}(X)\mathbf{1}_Y$$

Computing $E_\ell(X)P_m(X)\mathbf{1}_Y$

$$E_\ell(X)P_m(X)\mathbf{1}_Y = \Big(\sum_{j=0}^{\ell-1}\eta^{\ell-2j}D_j^{(\ell-1)}(Y)\Big)\Big(E_{-m}(X) + t\frac{(1-q^m)}{(1-q^m t)}E_m(X)\Big)\mathbf{1}_Y.$$

$$\begin{aligned}\eta^{\ell-2j}D_j^{(\ell-1)}(Y)E_{-m}(X)\mathbf{1}_Y &= \mathrm{ev}_m(D_j^{(\ell-1)}(Y^{-1}))\eta^{\ell-j}\eta^{-j}E_{-m}(X)\mathbf{1}_Y \\ &= \mathrm{ev}_m\Big(D_j^{(\ell-1)}(Y^{-1})t^{\frac{1}{2}(\ell-2j)}G^{-}_{\ell-j,-j}(Y)\Big)E_{-m+\ell-2j}(X)\mathbf{1}_Y \\ &= \mathrm{ev}_m(B_j^{(\ell)}(Y))E_{-m+\ell-2j}(X)\mathbf{1}_Y,\end{aligned}$$

Similar computations $\implies$

$$\eta^{\ell-2j}D_j^{(\ell-1)}(Y)t\frac{(1-q^m)}{(1-q^m t)}E_m(X)\mathbf{1}_Y = \mathrm{ev}_m(A_j^{(\ell)}(Y))E_{m+\ell-2j}(X)\mathbf{1}_Y$$

So finally,

$$\begin{aligned}&E_\ell(X)P_m(X)\mathbf{1}_Y \\ &= \sum_{j=0}^{\ell-1}\mathrm{ev}_m(A_j^{(\ell)}(Y))E_{m+\ell-2j}(X)\mathbf{1}_Y + \sum_{j=0}^{\ell-1}\mathrm{ev}_m(B_j^{(\ell)}(Y))E_{-m+\ell-2j}(X)\mathbf{1}_Y\end{aligned}$$

Theorem

In $\tilde{H}$

$$E_{\ell}(X)\mathbf{1}_0 = \sum_{j=0}^{\ell-1} \eta^{\ell-2j} D_j^{(\ell-1)}(Y)\mathbf{1}_0\,,$$

$$E_{-\ell}(X)\mathbf{1}_0 = \sum_{j=0}^{\ell+1} \eta^{-\ell+2j} D_j^{(-\ell)}(Y)\mathbf{1}_0\,,$$

$$\mathbf{1}_0 E_{\ell}(X)\mathbf{1}_0 = \sum_{j=0}^{\ell} \mathbf{1}_0 \eta^{\ell-2j} K_j^{(\ell)}(Y)$$

where $D_j^{(\ell-1)}(Y)\,, D_j^{(-\ell)}(Y)\,, K_j^{(\ell)}(Y)$ are 'products'.

Universal Formulas

$$D_j^{(\ell)}(Y) = t^{-\frac{1}{2}(\ell+1)} t^{\ell-j} \begin{bmatrix} \ell \\ j \end{bmatrix}_{q,t} \binom{\ell}{j}_Y \frac{(1-tq^{\ell-j})}{(1-tq^{\ell})} \frac{(1-tY^{-2}q^{\ell-j})}{(1-tY^{-2}q^{\ell-2j})},$$

$$D_j^{(-\ell)}(Y) = t^{-\frac{1}{2}\ell} (qt)^j \begin{bmatrix} \ell \\ j \end{bmatrix}_{q,t} \binom{\ell}{\ell-j}_Y \frac{(1-tq^{\ell-j})}{(1-tq^{\ell})} \frac{(1-t^{-1}Y^{-2}q^{-(\ell-j)})}{(1-t^{-1}Y^{-2}q^{-(\ell-2j)})},$$

$$K_j^{(\ell)}(Y) = t^{-\frac{1}{2}(\ell-1)} t^{\ell-1-j} \begin{bmatrix} \ell \\ j \end{bmatrix}_{q,t} \binom{\ell}{j}_Y \frac{(1-Y^{-2}q^{\ell-2j})}{(1-t^{-1}Y^{-2}q^{\ell-2j})} \frac{(1-t^{-1}Y^{-2})}{(1-Y^{-2})},$$

where

$$\binom{\ell}{j}_Y = \frac{(t^{-1}Y^{-2}q^{-(j-1)};q)_{\ell-j}(tY^{-2}q^{\ell-2j};q)_j}{(Y^{-2}q;q)_{\ell-j}(Y^{-2}q^{-j};q)_j}.$$

Thank You!