Entanglement dynamics from universal low-lying modes

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2407.16763 with Sanjay Moudgalya

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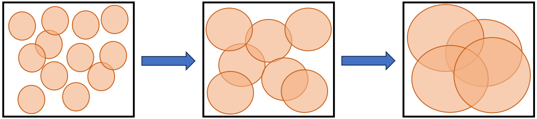
- Holographic CFTs are an example of highly chaotic systems. Thermalization in the CFT corresponds to black hole formation in the bulk.

- With local interactions, increasingly large regions are thermalized as time evolves:



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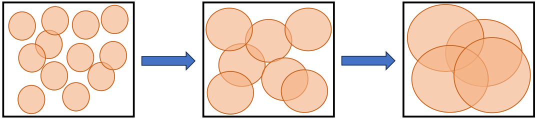

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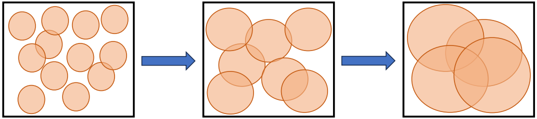

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- One way to probe this process is by considering time-evolution of the Renyi entropies,

$$S_{n,A}(t) = -\frac{1}{n-1} \log \mathrm{Tr}[\rho_A(t)^n] \, , \quad n = 1, 2, \ldots$$

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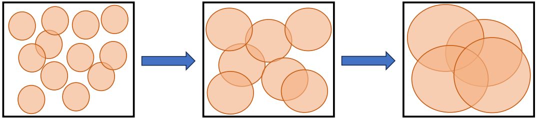

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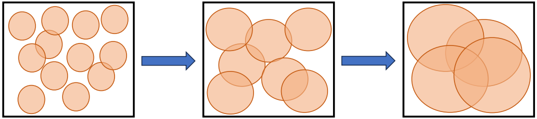

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- Due to thermalization, these quantities will approach extensive thermal values at late times:

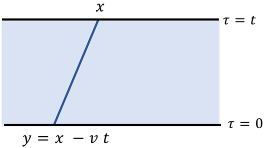
$$\lim_{t\to\infty} S_{n,A}(t) = s_{\mathrm{eq}} V_A\,.$$

Universality in approach to equilibrium

- We expect there to be universality not only in the late-time saturation values of various quantities during thermalization, but also in the way in which they are approached.
- Evolution of entanglement entropy in generic chaotic time-evolutions is very difficult to study analytically.
- But the few analytically tractable examples we can study suggest a remarkable universality.
- In both random circuits and holographic CFTs, the evolution of entanglement entropy at late times can be expressed in terms of a membrane formula.
- Conjectured to hold universally in Jonay, Huse, Nahum.

Membrane picture for entanglement growth

- In one spatial dimension, suppose we want to find S_n of the region to the left of some x at time t.
- Extend the system in time direction from $\tau = 0$ to $\tau = t$, and consider lines with different velocities v:



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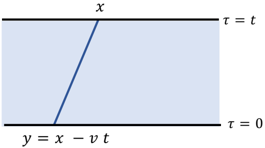
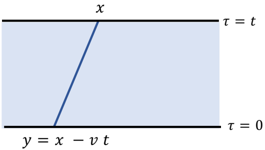


$$S_n(x, t) = \min_v \left[s_{\mathrm{eq}}\, \mathcal{E}_n(v) t + S_n(y, t = 0) \right]$$

Membrane picture for entanglement growth

- In one spatial dimension, suppose we want to find the entanglement entropy of the left half-line at time t.

- Extend the system in time direction from $\tau = 0$ to $\tau = t$, and consider all possible curves:



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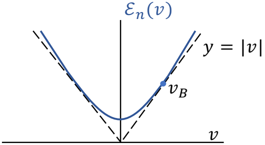
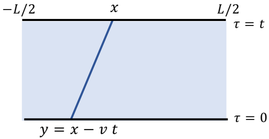
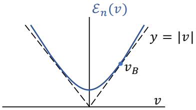


Physical consequences





$$S_n(x,t) = \min_v \left[s_{\rm eq}\,\mathcal{E}_n(v)\,t + S_n(y, t=0)\right]$$

- Consider an initial state with volume law entanglement entropy, with coefficient s:

$$S_n(y,t=0) = s\ \times\ (y + L/2), \quad 0 < s < s_{\rm eq}\,.$$

- The membrane formula gives an s-dependent growth rate of $S_n(x,t)$:

$$S_n(x,t) = S_n(x,t=0) + s_{\rm eq}\,\Gamma_n(s)\,t$$

  where Γ is related to $\mathcal{E}$ by Legendre transform.

- Constraints on membrane tension are equivalent to the condition that

$$\Gamma_n(s_{\rm eq}) = 0\,.$$

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  - Mezei showed that for large system size and time, we can get rid of the radial direction in the bulk, and reduce the HRT formula to a minimization problem in the boundary.

  - The resulting membrane tension satisfies non-trivial constraints from Jonay, Huse, Nahum.

Questions

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2. Is there an underlying structure in terms of low-lying modes, which we could look for in a continuum theory such as a holographic CFT?
3. How does the structure of $\mathcal{E}(v)$ depend on the Renyi index n?

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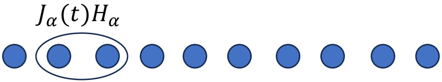
$$H(t) = \sum_{\alpha} J_\alpha(t) H_\alpha,$$

where the H_α are local operators, and $J_\alpha(t)$ are random numbers, uncorrelated for different times and different α.



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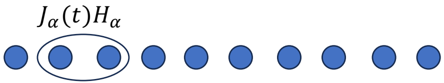

- Previously, such models have allowed a derivation of diffusion in two-point functions Moudgalya and Motrunich; Ogunnaike, Feldmeier, Lee.

From Lorentzian to Euclidean time-evolution

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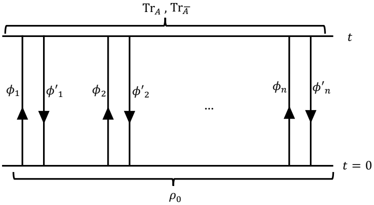

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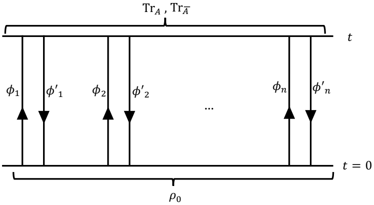

- The key simplification in Brownian models is that the Lorentzian evolution on $2n$ copies can be replaced with a Euclidean evolution:

$$\overline{(U(t) \otimes U(t)^*)^{\otimes n}} = e^{-P_{2n} t}$$

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$$= 1 - \epsilon P_2 + O(\epsilon^2) \quad \approx \quad e^{-\epsilon P}$$

where

$$P_2 = \sum_{\alpha} (H_{a,\alpha} - H^T_{b,\alpha})^2$$

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- Approach to equilibrium is determined by low energy eigenstates, which have a universal structure.

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- The low-energy excitations include a "one-particle" band approximately given by:

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- This structure leads to the membrane picture.

- The one-particle excitations have a gapped dispersion relation $E(k)$. The entanglement growth rate is given by

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- For the third Renyi entropy, we have an analogous set of low-energy eigenstates. In addition to these, competition from another set of eigenstates leads to phase transitions in $\mathcal{E}_3(v)$ as a function of v.

Plan

- Introduce expression for the second Renyi entropy as a transition amplitude, and the definition of $|\uparrow\rangle$ and $|\downarrow\rangle$.
- Derive the low-energy excitations in a simplifying limit.
- Discuss how the structure remains robust more generally.
- Discuss qualitatively new features of the third Renyi entropy.

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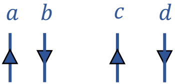

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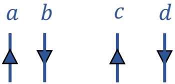

- Let us introduce the following "spins" on four copies of a single site in particular, Zhou and Nahum

$$|\uparrow\rangle = |\mathrm{MAX}\rangle_{ab} |\mathrm{MAX}\rangle_{cd}, \quad |\downarrow\rangle = |\mathrm{MAX}\rangle_{ad} |\mathrm{MAX}\rangle_{bc}$$

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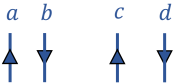

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- Evolution of second Renyi entropy is given by

$$e^{-S_{2,A}(t)} = \langle D_{\Sigma_A}|(U \otimes U^*)^2 |\rho_0\rangle |\rho_0\rangle$$

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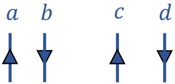
- Second Renyi entropy involves two forward and two backward copies of U:

$$e^{-S_{2,A}(t)} = \mathrm{Tr}_A \left(\mathrm{Tr}_{\bar{A}} U \rho_0 U^\dagger \right)^2$$



- Let us introduce the following "spins" on four copies of a single site in particular, Zhou and Nahum

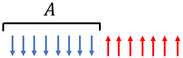
$$|\uparrow\rangle = |\mathrm{MAX}\rangle_{ab} |\mathrm{MAX}\rangle_{cd}\,, \quad |\downarrow\rangle = |\mathrm{MAX}\rangle_{ad} |\mathrm{MAX}\rangle_{bc}$$

- Evolution of second Renyi entropy is given by

$$e^{-S_{2,A}(t)} = \langle D_{\Sigma_A} | (U \otimes U^*)^2 |\rho_0\rangle |\rho_0\rangle$$

where

$$\langle D_{\Sigma_A} | = \bigotimes_{i\in A} \langle\downarrow| \bigotimes_{i\in\bar{A}} \langle\uparrow|$$



Equilibrium value in models without conserved quantities

- P_4 generally has exactly two zero energy eigenstates:

$$|\uparrow .... \uparrow\rangle\,, \quad |\downarrow ... \downarrow\rangle$$

- This gives the Page value for the entropy of pure state at late times:

$$\lim_{t\to\infty} S_{2,A}(t) = \min(\log d_A, \log d_{\bar{A}})$$

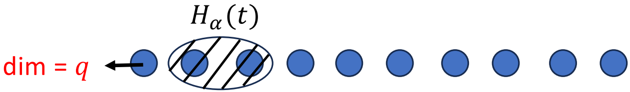
- We would now like to understand the approach to this value using the low-energy modes of P_4.

Low energy excitations: GUE model

GUE model

- Take each $H_\alpha(t)$ to be an i.i.d. random Hermitian matrix on adjacent sites drawn from the GUE ensemble:

$$H_\alpha(t) = H_{i,i+1}^{(\mathrm{GUE})}(t)$$



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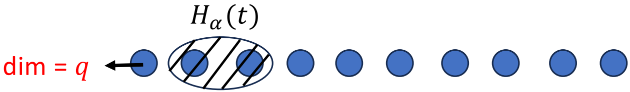


- Using the average over these random matrices, P_4 can be expressed entirely in terms of $|\uparrow\rangle, |\downarrow\rangle$.

Analytically solvable large q limit

- In the large q limit, P_4 is exactly solvable, and has a very simple action on a single domain wall

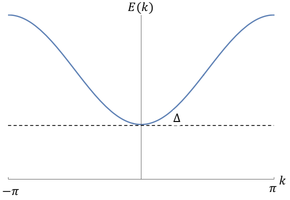
$$\langle D_x| \equiv \langle \downarrow \downarrow \ldots \downarrow_x \uparrow_{x+1} \uparrow \ldots \uparrow|$$

$$\langle D_x| P_4 = \langle D_x| - \frac{1}{q}(\langle D_{x-1}| + \langle D_{x+1}|)$$

- This leads to the following band of lowest excited states:

$$\langle \psi_k| = \sum_x e^{ikx} \langle D_x|$$

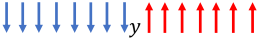
$$E(k) = 1 - \frac{2}{q} \cos k$$



Second Renyi entropy for half-line region

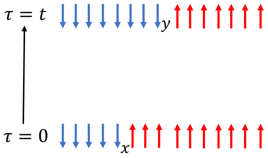
- Let us return to the second Renyi entropy of a half-line region:

$$e^{-S_2(y,t)} = \langle D_y | e^{-P_4 t} | \rho_0, e \rangle$$



- Since $\langle D_y |$ only evolves to a superposition of $\langle D_x |$ at other locations,

$$\begin{aligned} e^{-S_2(y,t)} &= \sum_x \langle D_y | e^{-P_4 t} | \bar{D}_x \rangle \langle D_x | \rho_0, e \rangle \\ &= \sum_x \langle D_y | e^{-P_4 t} | \bar{D}_x \rangle \, e^{-S_2(x, t=0)} \end{aligned}$$



Membrane picture from one domain wall band

- Using one-particle eigenstates in domain wall propagator:

$$\langle D_y | e^{-P_4 t} | \bar{D}_x \rangle = \sum_k e^{ik(x-y)} e^{-E(k)t}$$

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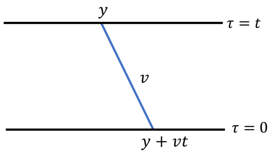
$$S_2(y,t) = \min_v \left[s_{\mathrm{eq}} \, \mathcal{E}(v) \; t + S_2(y+vt, t=0) \right]$$

where

$$\mathcal{E}(v) = \frac{E(k_v) - ik_v v}{s_{\mathrm{eq}}}, \quad k_v \text{ is solution to } E'(k_v) = iv \,.$$



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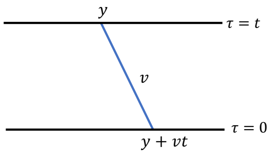

- We can also check that for an initial state with volume law entropy with coefficient s,

$$\Gamma(s) = E(k = is)/s_{\mathrm{eq}}\,.$$

Finite q in Brownian GUE model

- Away from large q limit, interactions can cause domain walls to split, so the eigenstates and eigenvalues are modified.

$$\langle D_x| P_4 = \langle D_x| - \frac{1}{q}(\langle D_{x-1}| + \langle D_{x+1}|) + \frac{1}{q^2} \langle D_{x-1,x,x+1}|$$

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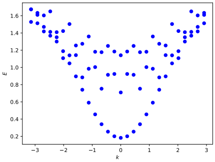
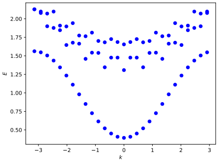
- From numerical diagonalization of P_4:

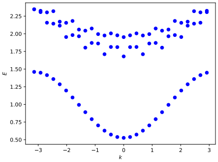
$q = 2$



$q = 3$



$q = 4$



Gapped spectrum in all cases, which implies $v_E \neq 0$.

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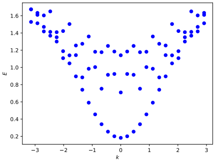

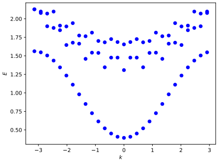

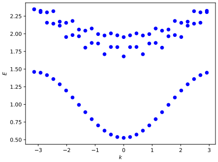

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- Is the structure of the eigenstates robust?

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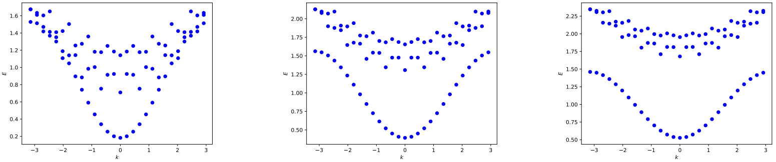

Gapped spectrum in all cases, which implies $v_E \neq 0$.

- Is the structure of the eigenstates robust?
- Is there still a well-defined one-particle band within the continuum for $q=2$?

Structure of eigenstates at finite q

- Let us consider a variational ansatz for the eigenstates:

$$|\psi_k\rangle = \sum_x e^{-ikx} |\downarrow \dots \downarrow_x\rangle |\phi_{x+1,\dots,x+\Delta}\rangle |\uparrow_{x+\Delta+1} \cdots \uparrow\rangle$$

Structure of eigenstates at finite q

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- We can increase the value of Δ, and at each Δ, minimize

$$E_{\rm var}(k) = \langle \psi_k | P_4 | \psi_k \rangle$$

over all choices of $|\phi\rangle$.

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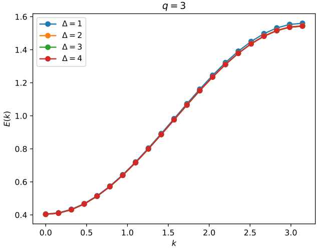
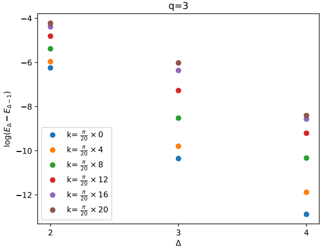
- Rapid convergence of $E_{\rm var}(k)$ with Δ would tell us that the eigenstates are well-approximated by $|\psi_k\rangle$ for $O(1)$ Δ. Haegeman, Spyridon, Michalakis, Nachtergaele, Osborne, Schuch, Verstraete

Variational results for $q = 3$

From minimizing $\langle\psi_k|A|\psi_k\rangle$ over all choices of $|\phi\rangle$ for various Δ:





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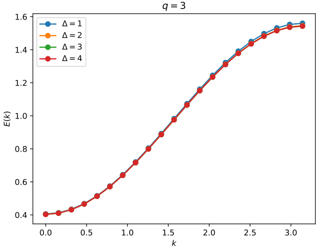
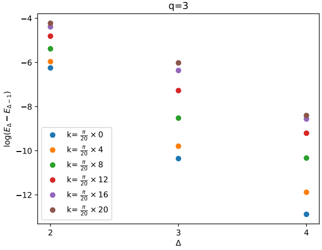
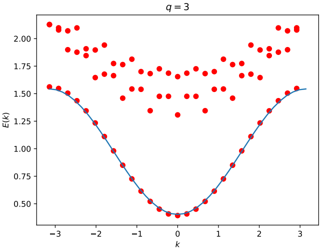


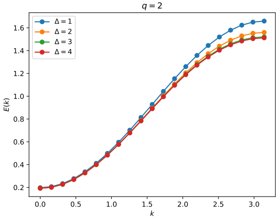


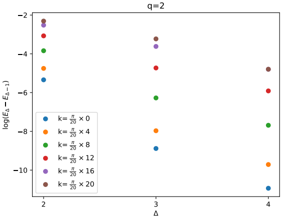
Good agreement with exact diagonalization results:



Variational results for $q = 2$

Still very good convergence with Δ:





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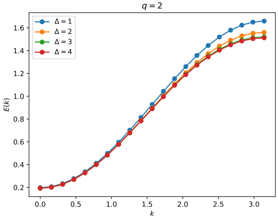
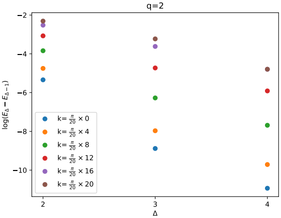
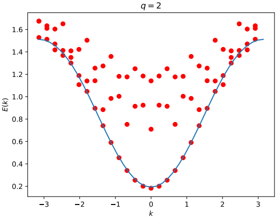


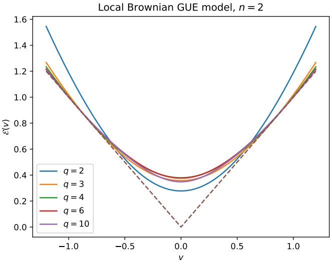


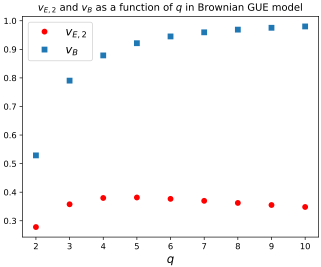
- Confirms that there is still a well-defined domain wall band within the continuum. ($\langle D_x|$ will only have significant overlap with this band.)



Membrane picture at finite q

- In the limit of large system sizes and late times, the local dressing with $O(1)$ Δ can be neglected.
- We still get the membrane formula at finite q, with $\mathcal{E}(v)$ given by Legendre transform of exact dispersion relation.
- $\mathcal{E}(v)$ can be found numerically from $E(k)$, and in particular we can check that the constraints on the membrane tension are satisfied:





Brownian mixed-field Ising model

- Let us now consider a spin-1/2 system, with the following time-dependent Hamiltonian.

$$H(t) = \sum_i J_Z(t)\ Z_i + J_X(t)\ X_i + J_{ZZ}(t)\ Z_i Z_{i+1}$$

where the variances are proportional to g_Z, g_X, g_{ZZ}.

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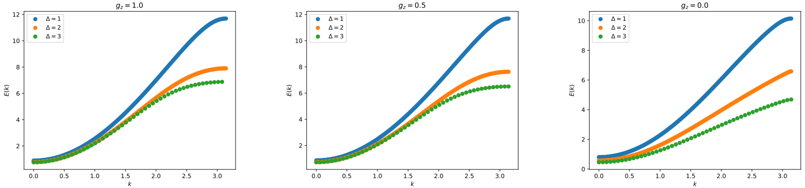
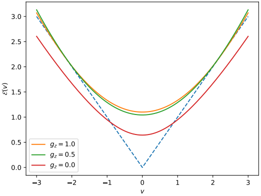
- Again, let us consider a variational ansatz

$$|\phi_k\rangle = \sum_x e^{-ikx} \left|\downarrow \ldots . \downarrow_x\right\rangle \left|\phi_{x+1,\ldots,x+\Delta}\right\rangle \left|\uparrow_{x+\Delta+1} \cdots \uparrow\right\rangle$$

where now, $|\phi_{x+1,\ldots,x+n}\rangle$ is an arbitrary state in a 16^Δ-dimensional Hilbert space.

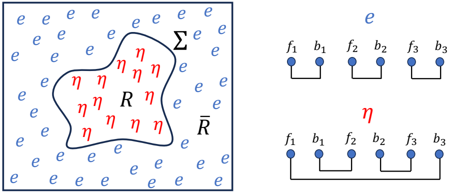


- Generic values of $g_z \neq 0$ should correspond to chaotic systems.
- For $g_z = 0$, $H(t)$ can be rewritten in terms of free fermions. The gap of P_4 vanishes and the ground state subspace is larger. Swann, Bernard, Nahum
- We see better convergence of $E(k)$ with Δ for $g_z \neq 0$.
- Membrane tensions from $\Delta = 3$ dispersion relations:



Higher Renyi entropies in Brownian GUE model

The n-th Renyi entropy can again be written as a transition amplitude, now with the final state:



$\bigotimes_{i\in R} \langle\eta| \bigotimes_{i\in R} \langle e|$

where e is associated with identity permutation, and η with the cyclic permutation $(n\, n-1 \ldots 1)$.

- For the superhamiltonian in the Brownian GUE model, we now have an $n!$-dimensional Hilbert space at each site.

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- Let us make the variational ansatz that there are eigenstates of the form

$$|\psi_k\rangle = \sum_x e^{-ikx} |\eta ... \eta_x\rangle |\phi_{x+1,...,x+\Delta}\rangle |e_{x+\Delta+1} ... e\rangle$$

  where now $|\phi\rangle$ is an arbitrary state in an $(n!)^\Delta$-dimensional Hilbert space.

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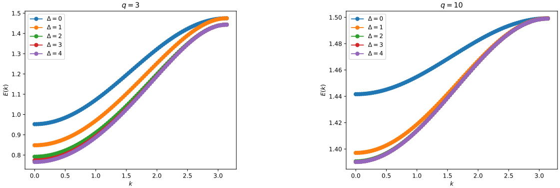
- For $n = 3$, we can use the variational method up to $\Delta = 4$.
- We find good convergence of $E_3(k)$ with Δ, indicating that we do have low-energy eigenstates of this form.



- By plugging this form of the eigenstates into the time-evolution of S_n for initial state with entropy density s, we now get

$$e^{-(n-1)S_n(x,t)} = \int_{-\infty}^{\infty} dv \int_{-\pi}^{\pi} \frac{dk}{2\pi}\ e^{-[E_n(k)+ikv-(n-1)sv]t}$$

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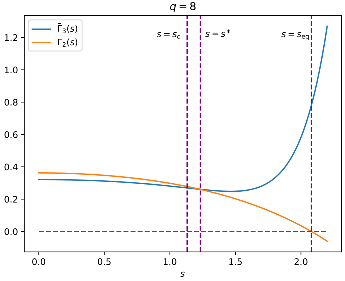

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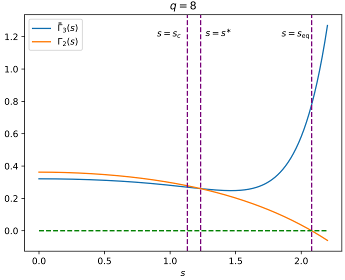
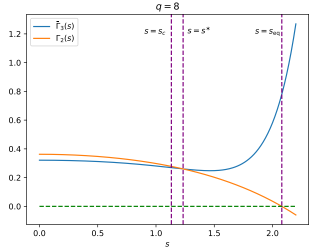


- This seems to give the unphysical prediction that the $\Gamma_3(s_{\mathrm{eq}}) \neq 0$!

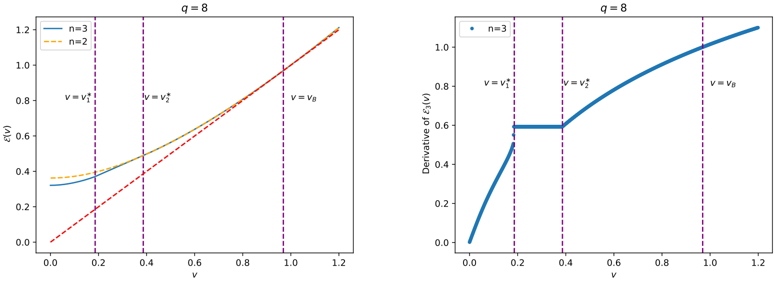
- Unphysical prediction must be corrected by contributions to e^{-2S_3} from some other set of eigenstates of P_6.
- We can argue that there is another natural set of eigenstates of P_6, such that

$$e^{-2S_3(x,t)} \propto e^{-2s_{\rm eq}\bar{\Gamma}_3(s)t} + e^{-2s_{\rm eq}\Gamma_2(s)t}$$

$$\Rightarrow \quad \Gamma_3(s) = \min(\ \bar{\Gamma}_3(s), \Gamma_2(s)\ )$$



- $\mathcal{E}(v)$ is related to $\Gamma(s)$ by Legendre transformation. We find that $\mathcal{E}_3(v)$ has two phase transitions, of first and second order respectively.



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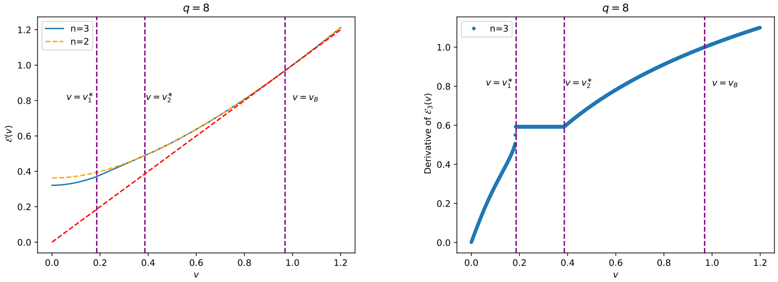


- For $v > v_2^*$, $\mathcal{E}_3(v) = \mathcal{E}_2(v)$.

  In particular v_B is the same for $n = 2$ and $n = 3$. (Independent hints from holography that it should be the same for all n.)

  Is there a physical reason for this?

Summary and further questions

- In Brownian models without conserved quantities, the membrane picture is a result of gapped low-energy modes that resemble plane waves of domain walls between permutations.

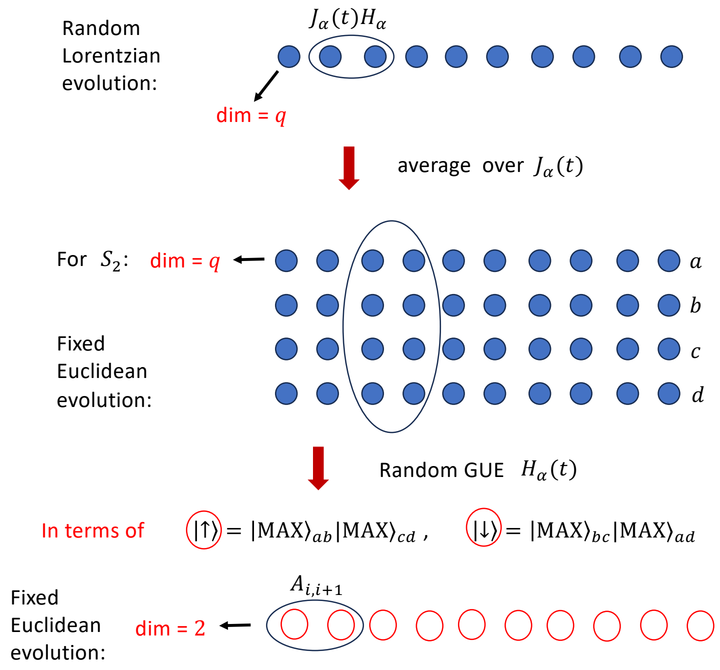
Questions:

- How does this picture generalize to finite temperature?
  - How does the picture change in Brownian circuits with conserved quantities? work in progress with Sanjay Moudgalya
  - How can a similar set of modes emerge in systems without random averaging? Can they be seen in holographic CFTs? work in progress with Mark Mezei and Zhencheng Wang
- What is the physical interpretation of the phase transitions in the higher Renyi membrane tensions?
- Can we quantitatively analyse the higher-dimensional case?
- Can these modes be used to formulate an effective field theory of hydrodynamics for entanglement?

Thank you!

$$\mathcal{E}(v) = \max_s \left(\frac{vs}{s_{\mathrm{eq}}} + \Gamma(s) \right)$$

$$\mathcal{E}_3(v) = \begin{cases} \bar{\mathcal{E}}_3(v) & v \leq v_1^* \\ \bar{\Gamma}_3(s_*) + \frac{s^*}{s_{\mathrm{eq}}}\ v & v_1^* \leq v \leq v_2^* \\ \mathcal{E}_2(v) & v \geq v_2^* \end{cases}$$



Random Lorentzian evolution:
$J_\alpha(t)H_\alpha$
dim = q
average over $J_\alpha(t)$
For S_2:
dim = q
a
b
c
d
Fixed Euclidean evolution:
Random GUE $H_\alpha(t)$
In terms of $|\uparrow\rangle = |\mathrm{MAX}\rangle_{ab}|\mathrm{MAX}\rangle_{cd}$, $|\downarrow\rangle = |\mathrm{MAX}\rangle_{bc}|\mathrm{MAX}\rangle_{ad}$
$A_{i,i+1}$
Fixed Euclidean evolution:
dim = 2