Bounded Factorizations for Matrix Groups over Rings, and Applications - an Exposition

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Combinatorial Methods in Enumerative Algebra

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Dedicated To Nikolai Vavilov

So diverse are the topics, I guess

that two weeks at ICTS

are memorable, though short.

A real bonus was to spot

distant cousins of zeta of ess!

This is an expository talk on certain combinatorial aspects of matrix groups over rings.

We discuss different types of bounded factorizations of matrix groups - including Chevalley groups - over certain commutative rings - these include semi-local rings, rings of matrix-valued holomorphic maps on Stein spaces, number rings....

Results are often intimately related to deeper properties such as the congruence subgroup property (CSP), Kazhdan's property T, finiteness of representations etc.

We mention growth (zeta) functions which arise naturally, whose analytic content encodes group theoretic information.

Expressions of a matrix in the group $\mathrm{SL}(2, R)$ over a Euclidean ring R as a product of elementary matrices correspond to continued fractions.

Existence of arbitrary long division chains in $\mathbb{Z}$ shows that the group $\mathrm{SL}(2, \mathbb{Z})$ cannot have bounded width in elementary generators.

We start with the following problem which arises in several independent contexts:

Uni-triangular Factorization

For a commutative ring R, find the shortest factorisation $UU^-UU^- \dots U^\pm$ of an (elementary) Chevalley group $E(\Phi, R)$ (if one exists), in terms of the unipotent radical $U = U(\Phi, R)$ of a standard Borel subgroup $B = B(\Phi, R)$, and the unipotent radical $U^- = U^-(\Phi, R)$ of the opposite Borel subgroup $B^- = B^-(\Phi, R)$.

Over fields, there was a vivid interest in explicit calculation of length.

Gilbert Strang noticed in 1997 that the groups $\mathrm{SL}(n, K)$ over a field K admit unitriangular factorisation

$$\mathrm{SL}(n, K) = U^-(n, K)U(n, K)U^-(n, K)U(n, K)$$

of length 4.

Experts in computational linear algebra call this the LULU-factorisation; amusingly, this result was observed in linear algebra due to applications in computer graphics.

$$\begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & tan(\phi/2) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\sin(\phi) & 1 \end{pmatrix} \begin{pmatrix} 1 & tan(\phi/2) \\ 0 & 1 \end{pmatrix}.$$

This ULU-decomposition into 'shears' is valuable in computer graphics, when a plane figure is to be turned. The rotation is effectively reduced to a series of translations in coordinate directions.

A three-dimensional rotation g is completely determined by its Euler angles (α, β, γ):

$$g = \begin{pmatrix} c(\alpha)c(\beta)c(\gamma) - s(\alpha)s(\gamma) & -c(\alpha)c(\beta)s(\gamma) - s(\alpha)c(\gamma) & c(\alpha)s(\beta) \\ s(\alpha)c(\beta)c(\gamma) + c(\alpha)s(\gamma) & -s(\alpha)c(\beta)s(\gamma) + c(\alpha)c(\gamma) & s(\alpha)s(\beta) \\ -s(\beta)c(\gamma) & s(\beta)s(\gamma) & c(\beta) \end{pmatrix},$$

where $c(\varphi)$ and $s(\varphi)$ denote $\cos(\varphi)$ and $\sin(\varphi)$, respectively.

We have a unitriangular factorization of length 3:

$$g = \begin{pmatrix} 1 & -\tan\left(\frac{\alpha+\gamma}{2}\right) & \cos(\alpha)\tan\left(\frac{\beta}{2}\right) \\ 0 & 1 & \sin(\alpha)\tan\left(\frac{\beta}{2}\right) \\ 0 & 0 & 1 \end{pmatrix} \times$$

$$\begin{pmatrix} 1 & 0 & 0 \\ \sin(\alpha+\gamma) & 1 & 0 \\ -\cos(\gamma)\sin(\beta) & -\frac{\sin\left(\frac{\alpha-\gamma}{2}\right)}{\cos\left(\frac{\alpha+\gamma}{2}\right)}\sin(\beta) & 1 \end{pmatrix} \times \begin{pmatrix} 1 & -\tan\left(\frac{\alpha+\gamma}{2}\right) & \frac{\cos\left(\frac{\alpha-\gamma}{2}\right)}{\cos\left(\frac{\alpha+\gamma}{2}\right)}\tan\left(\frac{\beta}{2}\right) \\ 0 & 1 & -\sin(\gamma)\tan\left(\frac{\beta}{2}\right) \\ 0 & 0 & 1 \end{pmatrix}.$$

To quote Vavilov: "This fact got to the household of linear algebra decades after it became standard in algebraic K-theory, and at that in a much larger generality. However, as we know, the walls between different branches of mathematics are high. Many works addressing the length of unitriangular factorisations show that experts in one field are usually completely unaware of the standard notions and results in another field. It is hard to imagine, how much time and energy could have been saved, should millions of programmers and engineers learn the words *parabolic subgroup* and *Levi decomposition*, rather than persisting in retarded matrix manipulations."

(Over finite fields), the group $U(\Phi, q) = U(\Phi, \mathbb{F}_q)$ is a Sylow p-subgroup of the Chevalley group $G(\Phi, q) = G(\Phi, \mathbb{F}_q)$. Thus, in this case, one seeks to calculate the minimal length of factorisation of a finite simple group of Lie type in terms of its Sylow p-subgroups, in the defining characteristic.

Martin Liebeck and Laszlo Pyber proved that finite Chevalley groups admit unitriangular factorisation

$$G(\Phi, q) = (U(\Phi, q)U^{-}(\Phi, q))^6 U(\Phi, q)$$

of length 13.

Laszlo Babai, Nikolay Nikolov and Laszlo Pyber proved that finite Chevalley groups admit unitriangular factorisation

$$G(\Phi, q) = U(\Phi, q)U^{-}(\Phi, q)U(\Phi, q)U^{-}(\Phi, q)U(\Phi, q)$$

of length 5.

We describe a general result (with Vavilov and Smolensky) for commutative rings of stable rank 1; it generalizes in a uniform way a number of known results such as these.

We shall also discuss results that address the problem over number rings (some of these proved with Alexander Morgan and Andrei Rapinchuk); these have consequences to the so-called bounded generation problem.

A ring R has **stable rank** 1, if for all $x, y \in R$, which generate R as an ideal, there exists a $z \in R$ such that $x + yz$ is invertible. In this case we write $\mathrm{sr}(R) = 1$.

Examples of ring of stable rank 1 are fields, semi-local rings (in particular, finite rings), and the ring of all algebraic integers, ring of entire functions.

More generally, R has stable rank $n+1$ if n is the minimal number such that whenever $(a_0, a_1, \cdots, a_n)$ is a unimodular vector in R^{n+1}, there exist $b_1, \cdots, b_n$ in R such that $(a_1 + b_1 a_0, \cdots, a_n + b_n a_0)$ is a unimodular vector in R^n.

Theorem 1. Let Φ be a reduced irreducible root system and R be a commutative ring such that the stable rank $\text{sr}(R) = 1$. Then the elementary Chevalley group $E(\Phi, R)$ admits unitriangular factorisation

$$E(\Phi, R) = U(\Phi, R)U^-(\Phi, R)U(\Phi, R)U^-(\Phi, R).$$

of length 4. Further, 4 is the minimum possible for such a result to hold good if R has a nontrivial unit.

The last assertion is immediate since $U^-(\Phi, R) \cap B(\Phi, R) = 1$ and, therefore, one has

$$T(\Phi, R) \cap U(\Phi, R)U^-(\Phi, R)U(\Phi, R) = 1.$$

In other words, 1 is the *only* element of the torus $T(\Phi, R)$, that admits a unitriangular factorisation of length < 4.

Theorem 1 is easily proved for the toy case SL_2 (which is the only place where the condition on stable rank 1 is used).

Lemma 1. Let R be a commutative ring of stable rank 1. Then

$$\mathrm{SL}(2,R) = U(2,R)U^-(2,R)U(2,R)U^-(2,R).$$

In particular, $\mathrm{SL}(2,R) = E(2,R)$.

• For a Dedekind ring of arithmetic type R, the simply connected Chevalley groups $G(\Phi, R) = E(\Phi, R)$ of rank ≥ 2 (in contrast with SL_2) have bounded width in elementary generators.

• Wilberd van der Kallen made a striking discovery that in general Chevalley groups of rank ≥ 2 (even $\mathrm{SL}(n, \mathbb{C}[t])$ for $n \geq 4$), may have infinite width in elementary generators. Thus, in general one can hope to establish existence of uni-triangular factorisations only over some very special rings of dimension ≤ 1.

We briefly recall some definitions and notations while studying Chevalley groups.

Let Φ be a reduced irreducible root system of rank l, $W = W(\Phi)$ be its Weyl group and $\mathcal{P}$ be a weight lattice intermediate between the root lattice $\mathcal{Q}(\Phi)$ and the weight lattice $\mathcal{P}(\Phi)$. Further, we fix an order on Φ and denote by $\Pi = \{\alpha_1, \ldots, \alpha_l\}$, Φ^+ and Φ^- the corresponding sets of fundamental, positive and negative roots, respectively.

With this datum, one can associate the Chevalley group $G = G_{\mathcal{P}}(\Phi, R)$, which is the group of R-points of an affine groups scheme $G_{\mathcal{P}}(\Phi, -)$ - the Chevalley-Demazure group scheme.

Since our results, mostly, do not depend on the choice of the lattice $\mathcal{P}$, in the sequel we usually assume that $\mathcal{P} = \mathcal{P}(\Phi)$ and omit any reference to $\mathcal{P}$ in the notation. Thus, $G(\Phi, R)$ will denote the simply connected Chevalley group of type Φ over R.

Fix a split maximal torus $T(\Phi, -)$ of the group scheme $G(\Phi, -)$ and set $T = T(\Phi, R)$. As usual, X_α, $\alpha \in \Phi$, denotes a unipotent root subgroup in G, elementary with respect to T.

Fix isomorphisms $x_\alpha : R \mapsto X_\alpha$ so that $X_\alpha = \{x_\alpha(\xi) \mid \xi \in R\}$, which are interrelated by the Chevalley commutator formulae; the elements $x_\alpha(\xi)$ are called root unipotents, and $E(\Phi, R)$ denotes the elementary subgroup of $G(\Phi, R)$, generated by all root subgroups X_α, $\alpha \in \Phi$.

For $\alpha \in \Phi$ and $\varepsilon \in R^*$, set $h_\alpha(\varepsilon) = w_\alpha(\varepsilon) w_\alpha(1)^{-1}$, where $w_\alpha(\varepsilon) = x_\alpha(\varepsilon) x_{-\alpha}(-\varepsilon^{-1}) x_\alpha(\varepsilon)$; The elements $h_\alpha(\varepsilon)$ are called semisimple root elements.

For a simply connected group one has

$$T = T(\Phi, R) = \langle h_\alpha(\varepsilon),\ \alpha \in \Phi,\ \varepsilon \in R^* \rangle.$$

Let $N = N(\Phi, R)$ denote the algebraic normalizer of the torus $T = T(\Phi, R)$, i. e. the subgroup generated by $T = T(\Phi, R)$ and all elements $w_\alpha(1)$, $\alpha \in \Phi$.
The factor-group N/T is canonically isomorphic to the Weyl group W, and for each $w \in W$ we fix its pre-image n_w in N.

Lemma 2. The elementary Chevalley group $E(\Phi, R)$ is generated by unipotent root elements $x_\alpha(\xi)$, $\alpha \in \pm\Pi$, $\xi \in R$, corresponding to the fundamental and negative fundamental roots.

Further, let $B = B(\Phi, R)$ and $B^- = B^-(\Phi, R)$ be a pair of opposite Borel subgroups containing $T = T(\Phi, R)$, standard with respect to the given order; then B and B^- are semidirect products $B = T \leftthreetimes U$ and $B^- = T \leftthreetimes U^-$, of the torus T and their unipotent radicals:

$$U = U(\Phi, R) = \langle x_\alpha(\xi),\ \alpha \in \Phi^+,\ \xi \in R \rangle,$$
$$U^- = U^-(\Phi, R) = \langle x_\alpha(\xi),\ \alpha \in \Phi^-,\ \xi \in R \rangle.$$

Gauss decomposition over rings of stable rank 1

Let $\Phi, R, G, E, U, U^{-}, T$ etc. be as above.

For the simply connected Chevalley group $G(\Phi, R)$ the torus $T(\Phi, R)$ is contained in the elementary subgroup

$$E(\Phi, R) = \langle U(\Phi, R), U^{-}(\Phi, R) \rangle,$$

generated by $U(\Phi, R)$ and $U^{-}(\Phi, R)$.

In general, when the group is not simply connected or the ring R is not semi-local, elementary subgroup $E(\Phi, R)$ can be strictly smaller than the Chevalley group $G(\Phi, R)$ itself.

For a non simply connected group, even the subgroup

$$H(\Phi, R) = T(\Phi, R) \cap E(\Phi, R)$$

spanned by semi-simple root elements $h_{\alpha}(\varepsilon)$, $\alpha \in \Phi$, $\varepsilon \in R^{*}$, can be strictly smaller than the torus $T(\Phi, R)$.

A slight modification of the same argument used to prove theorem 1, gives the following surprising result, asserting that condition $\mathrm{sr}(R) = 1$ is necessary and *sufficient* for the *elementary* Chevalley group $E(\Phi, R)$ to admit a Gauss decomposition.

Theorem 5. Let Φ be a reduced irreducible root system and R be a commutative ring such that $\mathrm{sr}(R) = 1$. Then the elementary Chevalley group $E(\Phi, R)$ admits a Gauss decomposition

$$E(\Phi, R) = H(\Phi, R)U(\Phi, R)U^{-}(\Phi, R)U(\Phi, R).$$

Conversely, if Gauss decomposition holds for some [elementary] Chevalley group, then $\mathrm{sr}(R) = 1$.

Note that $\mathrm{sr}(R) = 1$ is equivalent to the condition that $R^* \to (R/I)^*$ is surjective for each ideal I.

If R is a commutative ring such that the decomposition $SL(2, R) = U(R)U^-(R)H(R)U(R)$ holds, then for (a, c) unimodular, the matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, R)$.

It is expressible in the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} s & t \\ 0 & s^{-1} \end{pmatrix}$$

which gives $a = s + sxy, c = sy$.

Therefore, $a - xc \in R^*$ thereby showing that $\mathrm{sr}(R) = 1$.

The above Theorem divorces the existence of Gauss decomposition from the triviality of $K_1(\Phi, R)$. In fact, it shows that the stronger stability conditions are needed only to ensure that $G_{\mathrm{sc}}(\Phi, R) = E_{\mathrm{sc}}(\Phi, R)$, but are not necessary for the *elementary* Chevalley group $E(\Phi, R)$ to admit Gauss decomposition!

Two immediate corollaries of the Theorem are:

Corollary 1

Let Φ be a reduced irreducible root system and R be a commutative ring such that $\mathrm{sr}(R) = 1$. Then any element g of the elementary Chevalley group $E(\Phi, R)$ is conjugate to an element of

$$U(\Phi, R)H(\Phi, R)U^{-}(\Phi, R).$$

Corollary 2

Let Φ be a reduced irreducible root system and R be a commutative ring such that $\mathrm{sr}(R) = 1$. Then the elementary Chevalley group $E(\Phi, R)$ admits unitriangular factorisation

$$E(\Phi, R) = U(\Phi, R)U^{-}(\Phi, R)U(\Phi, R)U^{-}(\Phi, R)U(\Phi, R)$$

of length 5.

This corollary is a generalisation of the results on unitriangular factorisations over finite fields etc. obtained by others, with a 'terribly easier' proof.

Of course, Theorem 1, which is proven by the same method, starts with a slightly more precise induction base, and asserts that under condition $\mathrm{sr}(R) = 1$ the elementary Chevalley group $E(\Phi, R)$ admits unitriangular factorisation

$$E(\Phi, R) = U(\Phi, R)U^{-}(\Phi, R)U(\Phi, R)U^{-}(\Phi, R)$$

of length 4.

Tavgen's Idea

The proof of theorem 1 relies on the following reduction of rank result relying on the observation - first used effectively by Oleg Tavgen - that for systems of rank ≥ 2 every fundamental root falls into the subsystem of smaller rank obtained by dropping either the first or the last fundamental root.

Theorem 2. Let Φ be a reduced irreducible root system of rank $\ell \geq 2$, and R be **any** commutative ring. Suppose that for subsystems $\Delta = \Delta_1, \Delta_\ell$ the elementary Chevalley group $E(\Delta, R)$ admits unitriangular factorisation

$$E(\Delta, R) = (U(\Delta, R)U^{-}(\Delta, R))^{L}.$$

Then the elementary Chevalley group $E(\Phi, R)$ admits unitriangular factorisation

$$E(\Phi, R) = (U(\Phi, R)U^{-}(\Phi, R))^{L}.$$

of the same length $2L$.

The general proof of Theorem 1 immediately follows from the toy case Lemma 1, and Theorem 2.

The proof of theorem 2 itself proceeds by considering $Y := (U(\Phi, R)U^{-}(\Phi, R))^{L}$ is a *subset* in $E(\Phi, R)$, and producing a symmetric generating set $X \subseteq G$ satisfying $XY \subseteq Y$, then $Y = G$.

Proof of Theorem 2

Recall that a subset S in Φ is *closed*, if for any two roots $\alpha, \beta \in S$ whenever $\alpha + \beta \in \Phi$, already $\alpha + \beta \in S$.
For closed S, define $E(S) = E(S, R)$ to be the subgroup generated by all elementary root unipotent subgroups X_α, $\alpha \in S$:

$$E(S, R) = \langle x_\alpha(\xi), \quad \alpha \in S, \quad \xi \in R \rangle.$$

In this notation, U and U^- coincide with $E(\Phi^+, R)$ and $E(\Phi^-, R)$, respectively. The groups $E(S, R)$ are particularly important in the case where $S \cap (-S) = \varnothing$. In this case $E(S, R)$ coincides with the *product* of root subgroups X_α, $\alpha \in S$, in some/any fixed order.

Let again $S \subseteq \Phi$ be a closed set of roots; then S can be decomposed into a disjoint union of its *reductive* = *symmetric* part S^r, consisting of those $\alpha \in S$, for which $-\alpha \in S$, and its *unipotent* part S^u, consisting of those $\alpha \in S$, for which $-\alpha \notin S$.
The set S^r is a closed root subsystem, whereas the set S^u is special. Moreover, S^u is an *ideal* of S (i.e., if $\alpha \in S$, $\beta \in S^u$ and $\alpha + \beta \in \Phi$, then $\alpha + \beta \in S^u$).
Levi decomposition asserts that the group $E(S, R)$ decomposes into semidirect product $E(S, R) = E(S^r, R) \leftthreetimes E(S^u, R)$ of its *Levi subgroup* $E(S^r, R)$ and its *unipotent radical* $E(S^u, R)$.

Elementary parabolic subgroups

The main role in the proof of Theorem 2 is played by Levi decomposition for elementary parabolic subgroups.

Denote by $m_k(\alpha)$ the coefficient of α_k in the expansion of α with respect to the fundamental roots:

$$\alpha = \sum m_k(\alpha)\alpha_k, \quad 1 \leq k \leq l.$$

Now, fix any $r = 1, \ldots, \ell$ - in fact, in the reduction to smaller rank it suffices to employ only terminal parabolic subgroups, the ones corresponding to the first and the last fundamental roots, $r = 1, \ell$.

Denote by

$$S = S_r = \{\alpha \in \Phi,\ m_r(\alpha) \geq 0\}$$

the r-th standard parabolic subset in Φ.

As usual, the reductive part $\Delta = \Delta_r$ and the special part $\Sigma = \Sigma_r$ of the set $S = S_r$ are defined as

$$\Delta = \{\alpha \in \Phi,\ m_r(\alpha) = 0\}, \quad \Sigma = \{\alpha \in \Phi,\ m_r(\alpha) > 0\}.$$

The opposite parabolic subset and its special part are defined similarly

$$S^- = S_r^- = \{\alpha \in \Phi,\ m_r(\alpha) \leq 0\}, \quad \Sigma^- = \{\alpha \in \Phi,\ m_r(\alpha) < 0\}.$$

Obviously, the reductive part of S_r^- equals Δ.

Denote by P_r the *elementary* maximal parabolic subgroup of the elementary group $E(\Phi, R)$. By definition,

$$P_r = E(S_r, R) = \langle x_\alpha(\xi),\ \alpha \in S_r,\ \xi \in R \rangle.$$

We have the Levi decomposition,

$$P_r = L_r \ltimes U_r = E(\Delta, R) \ltimes E(\Sigma, R).$$

Recall that

$$L_r = E(\Delta, R) = \langle x_\alpha(\xi), \quad \alpha \in \Delta, \quad \xi \in R \rangle,$$

Whereas

$$U_r = E(\Sigma, R) = \langle x_\alpha(\xi),\ \alpha \in \Sigma,\ \xi \in R \rangle.$$

A similar decomposition holds for the opposite parabolic subgroup P_r^-, whereby the Levi subgroup is the same as for P_r, but the unipotent radical U_r is replaced by the opposite unipotent radical $U_r^- = E(-\Sigma, R)$.

As a matter of fact, we use Levi decomposition in the following form. It will be convenient to slightly change the notation and write $U(\Sigma, R) = E(\Sigma, R)$ and $U^{-}(\Sigma, R) = E(-\Sigma, R)$.

Lemma 3

The group $\langle U^{\sigma}(\Delta, R), U^{\rho}(\Sigma, R)\rangle$, where $\sigma, \rho = \pm 1$, is the semidirect product of its normal subgroup $U^{\rho}(\Sigma, R)$ and the complementary subgroup $U^{\sigma}(\Delta, R)$.

In other words, the subgroup $U^{\pm}(\Delta, R)$ normalizes each of the groups $U^{\pm}(\Sigma, R)$ so that, in particular, one has the following four equalities for products

$$U^{\pm}(\Delta, R)U^{\pm}(\Sigma, R) = U^{\pm}(\Sigma, R)U^{\pm}(\Delta, R).$$

Furthermore, the following four obvious equalities for intersections hold:

$$U^{\pm}(\Delta, R) \cap U^{\pm}(\Sigma, R) = 1.$$

In particular, one has the following decompositions:

$$U(\Phi, R) = U(\Delta, R) \leftthreetimes U(\Sigma, R), \quad U^{-}(\Phi, R) = U^{-}(\Delta, R) \leftthreetimes U^{-}(\Sigma, R).$$

To complete the proof of Theorem 2, note by Lemma 2, that the group G is generated by the fundamental root elements

$$X = \{x_\alpha(\xi) \mid \alpha \in \pm\Pi,\ \xi \in R\}.$$

Thus, it suffices to prove that $XY \subseteq Y$.

Fix a fundamental root unipotent $x_\alpha(\xi)$. Since $\mathrm{rk}(\Phi) \geq 2$, the root α belongs to at least one of the subsystems $\Delta = \Delta_r$, where $r = 1$ or $r = \ell$, generated by all fundamental roots, except for the first or the last one, respectively.

Set $\Sigma = \Sigma_r$ and express $U^{\pm}(\Phi, R)$ in the form

$$U(\Phi, R) = U(\Delta, R)U(\Sigma, R), \quad U^{-}(\Phi, R) = U^{-}(\Delta, R)U^{-}(\Sigma, R).$$

Using Lemma 3 we see that

$$Y = (U(\Delta, R)U^{-}(\Delta, R))^{L}(U(\Sigma, R)U^{-}(\Sigma, R))^{L}.$$

Since $\alpha \in \Delta$, one has $x_\alpha(\xi) \in E(\Delta, R)$, so that the inclusion $x_\alpha(\xi)Y \subseteq Y$ immediately follows from the assumption; the proof of theorem 2 is completed.

Relative Factorizations

Sinchuk and Smolensky also developed the above methods to prove congruence subgroup versions (that is, relative to ideals versions) of the Gauss decomposition in collaboration with Sinchuk.

For an ideal I of R, an I-unimodular row in R^{n+1} is a unimodular row $(a_0, a_1, \cdots, a_n)$ which is congruent to $(1, 0, \cdots, 0)$ modulo I; one calls it stable, if there are $b_1, \cdots, b_n \in I$ so that $(a_0 + b_1 a_n, a_1 + b_2 a_n, \cdots, a_{n-1} + b_n a_n)$ is also I-unimodular.

The stable rank $\mathrm{sr}(I)$ of I is the minimal n for which every I-unimodular row of length $n + 1$ is stable.

Given a simply-connected Chevalley group $G(\Phi, R)$ with respect to an irreducible root system Φ, the normal closure of the group $E(\Phi, I)$ generated by $x_\alpha(s), s \in I, \alpha \in \Phi$ in the principal congruence subgroup modulo I is denoted by $E(\Phi, R, I)$.

Using the same Tavgen idea, Sinchuk and Smolensky prove a relative Gauss decomposition for an ideal I with $\mathrm{sr}(I) = 1$ in any commutative ring:

$$E(\Phi, R, I) = H(\Phi, R, I)U(\Phi^+, I)U(\Phi^-, I)U(\Phi^+, I).$$

The starting point with $SL(2)$ is a toy case that is checked easily.

Further, they also prove using the same approach as in our first paper, that for the ring O_S of S-integers in a number field admitting a real embedding, the principal S-congruence subgroup modulo $I \subset O_S$ of a classical Chevalley group for rank $\Phi \geq 2$, has finite width respect to the so-called Tits-Vaserstein generators $t_{ij}(a)t_{ji}(b)t_{ij}(-a)$ for $a \in O_S$ and $b \in I$ and $i \neq j$.

Recently (2023), in the case of SL_n, Pavel Gvozvedsky has developed the methods to obtain explicit bounds for the width depending on the degree, discriminant and class number.

Covering Number

If G is a group, and C is a conjugacy class, some old results deal with finding the smallest positive integer c (if it exists) such that $C^c = G$.

For instance, for finite simple (nonabelian) groups, there exist C such that $C^2 = G$.

The group $PSL_n(K)$ over a field K of cardinality large enough with respect to n (for example, an algebraically closed field), there exists conjugacy classes C for which $C^2 = G$ (proved in 1993 by A. Lev).

Very recently, Iulian Simion has used the unitriangularization theorem (Theorem 1) to prove that for a simple algebraic group G over algebraically closed fields of good characteristic and for any unipotent conjugacy class C, results of the form $C^c = G$ for appropriate c.

More precisely, he proves that if C is the conjugacy class of a unipotent u, then $C^c = G$ where $c \leq \frac{2^5.3^2 rank(G)}{rank(G)-rank(C_G(u))}$ and also that $c \leq \frac{2^9.3^2 dim(G)}{dim(C)}$.

For distinguished unipotent conjugacy classes (that is, those for which $C_g(u)^0$ is unipotent), he gets a uniform bound $c \leq 72$.

In the most recent issue of Mathematische Annalen, Larsen, Shalev and Tiep have shown that a very general phenomenon occurs in the case of finite simple nonabelian groups.

They prove that if w is any nontrivial word, then either $w(G)^6 = G$ for all finite simple nonabelian groups G, or $w(G_0) = \{1\}$ for some finite, simple nonabelian group G_0.

Rings of holomorphic functions

Very recently, Gaofeng Huang, Frank Kutzschebauch and Josua Schott have applied the unitriangularization theorem in the context of the so-called Gromov-Vaserstein problem (Gromov calls it the Vaserstein problem).

One considers the map of factoring a map f from a topological space X to $SL_n(\mathbb{C})$, which is null-homotopic (i.e., homotopic to a constant map) into a product as

$$f(x) = \begin{pmatrix} 1 & 0 \\ G_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2(x) \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & G_N(x) \\ 0 & 1 \end{pmatrix}$$

where G_i are maps: $X \to \mathbb{C}^{n(n-1)/2}$.

For polynomial maps $f : \mathbb{C} \to SL_n(\mathbb{C})$, such a factorization evidently exists with G_i's polynomials as $\mathbb{C}[X]$ is a Euclidean ring.

On the other hand, polynomials of degree 2 do not always admit such a factorization; a famous example due to P.M. Cohn is $f(X, Y) = \begin{pmatrix} 1 - XY & X^2 \\ -Y^2 & 1 + XY \end{pmatrix}$.

For degrees ≥ 3, they do have a unitriangular factorization by a theorem of Suslin.

Gromov and Vaserstein considered factoring holomorphic maps $f : \mathbb{C}^m \to SL_n(\mathbb{C})$ into finite products of holomorphic maps sending $\mathbb{C}^m$ to unitriangular matrices.

Björn Ivarsson and Frank Kutzschebauch solved the problem (a paper in Annals of Mathematics in 2012) using the so-called Oka principle which is referred by René Thom as the most beautiful principle in analysis - roughly, the principle asserts that if a continuous solution exists, then a holomorphic one also does.

If X is a reduced Stein space, whose smooth part has dimension n, then Alexander Brudnyi proved in 2019 that the stable rank of the ring $O(X)$ of holomorphic functions is $\lfloor dim(X) \rfloor + 1$.

In the 2024 paper, Huang, Kutzschebauch and Schott prove that there exists a minimal upper bound $t(n, d)$ such that every null-homotopic holomorphic map $f : X \to Sp_{2n}(\mathbb{C})$ from a d-dimensional reduced Stein space X factorizes into a product of at most t unitriangular factors.

Also $t(n, d) \leq t(1, d)$, $t(n, 1) = 4$ and $t(n, 2) \leq t(1, 2) \leq 5$ for all $n \geq 1$.

The above-mentioned paper also proves that the path-connected component at identity $Sp_{2n}(O(X))_0$ has Kazhdan's property (T) when $n \geq 2$ - we will talk about property (T) later in the context of bounded generation.

Earlier Ivarsson and Kutzschebauch had also shown that $SL_n(O(X))$ has property (T) when $n \geq 3$. These are examples of non-locally-compact groups with property (T).

Commutator Width

Unitriangular factorizations or Gauss decompositions discussed above are intimately related to another combinatorial problem - one of width with respect to commutators.

A group G is said to have commutator width N if N is the least positive integer such that every element of $[G, G]$ is a product of at the most N commutators; we write $c(G) = N$; this could be infinite or 0 (when G is abelian).

Ore conjecture (which is proved now) asserts that the commutator width of every finite simple, nonabelian group is 1. Also, the symmetric groups on infinite sets have commutator width 1.

Vaserstein and others obtained in the 1990's, bounds for commutator width of classical groups over commutative rings; they proved that $GL(n, R)$ for $n \geq 3$, has commutator width ≤ 2 if $\mathrm{sr}(R) = 1$, and for other classical groups, there are (worse) bounds under more assumptions on R.

If the stable rank is finite, it is possible that $c(GL_n(R)) = \infty$ but, if there is $n \geq \mathrm{sr}(R) + 1$ such that $c(GL_n(R)) < \infty$, then $c(GL_n(R)) \leq 4$ for $n >> 0$.

Andrei Smolensky used the methods we had developed in the earlier works to prove in 2019 that all elementary Chevalley groups of rank ≥ 2 over a ring with $\mathrm{sr}(R) = 1$ have commutator width $\leq 3, 4$ or 5.

In contrast, $SL_n(\mathbb{C}[X])$ has infinite commutator width; so, one cannot generalize the earlier results even to SL_n over rings with stable rank 2.

Smolensky uses the unitriangular factorisation $E(\Phi, R) = U^+(\Phi(R))U^-(\Phi(R))U^+(\Phi(R))U^-(\Phi(R))$ we had proved earlier.

Apart from using the above factorization to carry out induction, he considers weight diagrams, and simple ideas from linear algebra such as proving that given $u \in U^+$, there exists $x \in U^+$ such that $xu\pi x^{-1}$ is a 'companion matrix' where π is a lift of a Coxeter element.

In other words

Images of general word maps (not just commutators) have been studied extensively.

For simple algebraic groups, any nontrivial word is dominant - as shown by Borel.

A more general version of Ore conjecture on commutators has been proved for any nontrivial word map w on a finite simple (nonabelian) group G; it is known that $w(G)w(G) = G$ provided $O(G) >> 0$.

Avni, Gelander, Kassabov and Shalev proved that if p is a prime and $n|(p-1)$ is a proper divisor, then $c(PSL_n(\mathbb{Z}_p)) = 1$.

If O is a local ring, they also prove that every element of $SL_n(O)$ which is outside the congruence subgroup of matrices congruent to a scalar modulo the maximal ideal $\mathfrak{m}$, is a single commutator in case $n < |O/\mathfrak{m}| - 1$.

For general words on p-adic semisimple groups, they prove:

If G is a simply connected, semisimple $\mathbb{Q}$-algebraic group, and w_1, w_2, w_3 are nontrivial words, then $w_1(G(\mathbb{Z}_p))w_2(G(\mathbb{Z}_p))w_3(G(\mathbb{Z}_p)) = G(\mathbb{Z}_p)$ if p is a large enough prime.

Instead of $G(\mathbb{Z}_p)$ for a single prime p, if we consider the adelic group $G(\widehat{\mathbb{Z}})$, then an analogue of the above result due to Dan Segal says that the image is open.

Some more old results on $c(G)$

We have $c(GL_n(F)) = 1$ if F is an algebraically closed field; $c(GL_n(F)) \leq 2$ for any real-closed field F; and $c(SL_n(F)) \leq n$ for any infinite field F.

For any connected compact topological group G in which $[G, G]$ is dense (these include all connected compact semi-simple Lie groups), $c(G) = 1$.

If G_i $(1 \leq i \leq n)$ are nonabelian, finitely presented groups, then the free product $G = G_1 * G_2 * \cdots * G_n$ has $c(G) \geq \sum_{i=1}^{n} c(G_i) \geq n$.

If G varies over finite groups with $[G, G]$ cyclic, the values of $c(G)$ are unbounded, but if G varies only over nilpotent groups with this property, then $c(G) = 1$.

For the ring R of continuous functions from $\mathbb{R}$ to itself, we have $c(SL_n(R)) < \infty$ if $n \geq 3$ and ∞ if $n = 2$.

The universal cover $\widetilde{SL_2(\mathbb{R})}$ has infinite commutator width.

For any PID R, we have $c(SL_n(R)) \leq 6$ for $n >> 0$ in case $c(SL_3(A)) < \infty$ (note that the latter is infinite for the PID $\mathbb{C}[X]$).

Number rings

Number-theoretic rings are usually more complicated as they have stable rank > 1; over the ring $\mathbb{Z}[1/p]$, we proved in the unitriangular factorization paper:

Theorem. Let p be a prime. The elementary Chevalley group $E\big(\Phi, \mathbb{Z}\big[\frac{1}{p}\big]\big)$ admits unitriangular factorisation

$$E\Big(\Phi, \mathbb{Z}\Big[\frac{1}{p}\Big]\Big) = \Big(U\Big(\Phi, \mathbb{Z}\Big[\frac{1}{p}\Big]\Big) U^{-}\Big(\Phi, \mathbb{Z}\Big[\frac{1}{p}\Big]\Big)\Big)^{3}$$

of length 6.

The theorem was deduced from the one below for SL_2 which has the stronger bound of 5:

$$SL_2\Big(\mathbb{Z}\Big[\frac{1}{p}\Big]\Big) = UU^-UU^-U = U^-UU^-UU^-.$$

Later, with Morgan and Rapinchuk, we have dealt with all rings of S-integers which have infinitely many units; these imply a property of bounded generation with respect to cyclic subgroups, and we briefly digress to discuss this notion.

Bounded generation

An abstract group G is said to be boundedly generated of degree $\leq n$ if there exists a sequence of (not necessarily distinct) elements $g_1, \cdots, g_n$ such that

$$G = < g_1 >< g_2 > \cdots < g_n >$$

that is,

$$G = \{g_1^{a_1} g_2^{a_2} \cdots g_n^{a_n} : a_i \in \mathbf{Z}\}.$$

A free, non-abelian group (and therefore, $SL_2(\mathbf{Z})$ also) is not boundedly generated.

On the other hand, a group like $SL_n(\mathbb{Z})$ for $n \geq 3$, is boundedly generated by elementary matrices (an elementary proof of this can be given using Dirichlet's theorem on primes in arithmetic progressions).

In fact, Bogdan Nica has shown more recently that if $n \geq 3$ and F is a finite field, the group $SL_n(F[X])$ is boundedly generated by $29 + \frac{n(3n-1)}{2}$ elementary matrices, and the proof over $F[X]$ is based on the following analogue of Dirichlet's theorem due to Kornblum and Artin:

If $f, g \in F[X]$ and $g \neq 0$, then there are infinitely many primes congruent to g mod f. Further, such a prime can have arbitrary degree provided the degree is sufficiently large.

Finitely generated matrix groups over polynomial rings arise naturally also in other contexts; for example, the wreath product $\mathbb{Z} \wr \mathbb{Z}$ is a finitely generated subgroup of $GL(2, \mathbf{Z}[t, 1/t])$.

This group can be realized as the group of 2×2 matrices

$$\begin{pmatrix} t^m & t^n f(t) \\ 0 & 1 \end{pmatrix}$$

where f is any polynomial with integer coefficients and m, n are any integers.

It has an infinitely generated abelian subgroup but, is itself, generated by just two matrices:

$\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

This group does not have bounded generation (a more general result proved with Nikolay Nikolov shows that $A \wr B$ can have BG if, and only only if, A has BG and B is finite; this uses a beautiful combinatorial group theoretic theorem of B H Neumann which asserts that when a group is written as a union of finitely many left cosets of subgroups, those subgroups which are of infinite index can be dropped!

Carter and Keller proved BG for $SL_n(O)$ for rings of integers O in number fields and $n \geq 3$ - they used the properties of so-called Mennicke symbols.

Tavgen proved bounded generation of arithmetic groups in rank > 1 split and quasi-split groups.

Bounded generation of S-arithmetic subgroups in isotropic, but not necessarily quasi-split, orthogonal groups of quadratic forms over number fields was established (under some natural assumptions) by Igor Erovenko and Andrei Rapinchuk.

Bounded generation for arithmetic groups, when valid, has deep consequences to the congruence subgroup problem and finiteness of character variety etc.

For instance, Rapinchuk proved that if G is an abstract group with bounded generation and has the Fabulous property FAb (H_{ab} is finite for each H of finite index), then it has only finitely many completely reducible representations in any given dimension (in positive characteristic, this is due to Abert, Lubotzky and Pyber).

Here is a quick indication of the idea of proof when $G = SL(n, \mathbb{Z})$ with $n \geq 3$.

Let $\rho : G \to GL(r, \mathbf{C})$ and consider its restriction to the upper triangular unipotent subgroup U of G.

As U is solvable, there is a normal subgroup V of finite index (d say) in U such that $\rho(V)$ is upper triangular; then $\rho([V, V])$ is unipotent.

Since $X_{ij}^{d^2} = [X_{ik}(d), X_{kj}(d)] \in [V, V]$, we have $\rho(X_{ij}(1))^{d^2}$ is unipotent for all $i \neq j$.

Since $X_{ij}(1)$ are mutually conjugate for $i \neq j$, the Zariski-closure of $\rho(X_{ij}(1))$ has dimension ≤ 1.

As $G =< g_1 > \cdots < g_k >$ with each g_i some $X_{ij}(1)$, the Zariski closure of $\rho(G)$ has dimension $\leq k$; from this, the semisimple-rigidity asserted can be obtained.

A profinite group G is said to be boundedly generated as a profinite group if there exists a sequence of (not necessarily distinct) elements $g_1, \cdots, g_n$ such that

$$G = \overline{< g_1 >< g_2 >} \cdots \overline{< g_n >}.$$

It follows from Lazard's deep work on p-adic Lie groups and the solution to the restricted Burnside problem that a pro-p group has bounded generation (as a profinite group) if and only if it is a p-adic compact Lie group; this can be thought of as an analogue of Hilbert's 5th problem for the p-adic case.

If an abstract group has bounded generation, then so do its pro-p completions for each prime p (as does the full profinite completion).

This gives a sufficient criterion for an abstract group to have a faithful linear representation - viz., if it has bounded generation and is virtually residually-p.

We can use this fact to show, in particular, that the automorphism group of a free group does not have bounded generation (as it does not have a faithful finite-dimensional representation).

A characterization such as the one above for pro-p groups by Lazard, does not hold good for general profinite groups.

For instance, even though the profinite completion of $SL(n, \mathbb{Z})$ is boundedly generated for $n \geq 3$, the Sylow pro-p subgroups of are not even finitely generated.

The profinite completion of an S-arithmetic group is boundedly generated if, and only if, it has the S-congruence subgroup property.

Very recently, Corvaja, Rapinchuk, Ren and Zannier proved stunningly surprising results, which show that co-compact S-arithmetic lattices NEVER have bounded generation (even if their profinite completions do).

The last-mentioned work which proves that S-arithmetic groups in a K-anisotropic, simple algebraic group over a number field K cannot have bounded generation - note that all elements of $G(K)$ must be semi-simple - actually proves something more general:

Let $\Gamma \leq GL_n(K)$ be any linear group over a field K of characteristic zero. Assume Γ is not virtually solvable. Then, in any possible BG-representation for Γ, at least two of the elements must be **non-semi-simple**.

The above result is not a necessary and sufficient criterion, but they are able to deduce the following criterion:

A linear group $\Gamma \leq GL_n(K)$ over a field of characteristic zero tnat consists entirely of semisimple elements, has (BG) if, and only if, it is finitely generated and virtually abelian.

Over any field of positive characteristic, it turns out - as shown by Abert, Lubotzky and Pyber - that the only possible linear groups that have bounded generation are the virtually abelian ones.

In 1999, Yehuda Shalom noticed a connection of Kazhdan's property (T) with bounded generation; recall that property (T) means the trivial representation is isolated in the space of all unitary representations.

An easy consequence of property (T) for a discrete group is that it is finitely generated and has finite abelianization.

For the groups $SL(n, \mathbb{Z})$ with $n \geq 3$ (using explicit bounded generation), Shalom obtained explicit Kazhdan constants.

More interestingly, he showed that for $SL(n, \mathbb{Z}[X_1, \cdots, X_m])$ has property T if we know that it has bounded generation with respect to the elementary matrices.

The above groups are finitely generated and are generated by the elementary matrices, as proved by Suslin.

It is an open question whether the above groups $SL(n, \mathbb{Z}[X_1, \cdots, X_m])$ (or the groups $SL(n, \mathbb{Q}[X]))$ have bounded generation or not.

It has been proved by van der Kallen that $SL(n, F[X])$ does NOT have bounded generation by elementaries if F is a field with infinite transcendence degree over $\mathbf{Q}$.

The proof is K-theoretic and depends strongly on the fact that F is a field; so, it doesn't work for rings like the p-adic integers.

The subgroups $SL(n, \mathbb{Z}[X_1, \cdots, X_m])$ are sometimes called 'universal lattices' because many lattices in $SL(d, K)$ for various fields K, are images of these groups.

Kassabov and Nikolov have proved in 2006 that these universal lattices have a weaker property called property tau; a residually finite group has property τ if the trivial representation is isolated in the space of unitary representations with finite images.

In 2006, property (T) itself was proved for the groups $SL(n, \mathbb{Z}[X_1, \cdots, X_m])$ for $n \geq m + 3$ by Shalom; this gives uniform Kazhdan constants for $SL(n, O)$ for many rings of integers in different number fields, independent of the field.

As mentioned earlier, bounded generation is still open for these universal lattices but Shalom circumvented this cleverly.

Basically, Shalom uses the fact that if R is a ring with stable range $n+1$, then the elementary group $E_{n+1}(R)$ has finite width with respect to the set of conjugate subgroups of $E_n(R)$ (which is embedded as the lower right-hand corner subgroup).

In addition, he notices the following bounded generation lemma:
Assume that a group G has finite width with respect to a family of subgroups $\{H_i\}$. If an isometric action of G on a Hilbert space admits a fixed point for each H_i, then G fixes a point.

Note that $R = \mathbb{Z}[X_1, \cdots, X_m]$ has stable range $m+3$ and by Suslin, $E_n = SL_n$ over R.

Common conjecture of CSP and CmSP

We briefly mention a very exciting result by Yehuda Shalom and George Willis that generalizes the Margulis normal subgroup theorem.

They study something called the commensurator subgroup problem (CmSP) using bounded generation.

An arithmetic group like $SL(2, \mathbb{Z})$ is commensurated by a larger S-arithmetic subgroup $SL(2, \mathbb{Z}[1/p])$.

The CmSP is the question as to whether any S-arithmetic subgroup Γ of $G(K)$ (K a number field and S-rank of G is at least 2) has the property that the only infinite subgroups it commensurates are S_1-arithmetic subgroups for a subset S_1 of S.

A related property is the inner commensurator-normaliser property: each commensurated subgroup is commensurable with a normal subgroup. One calls it the outer commensurator-normalizer property if the above property holds under any homomorphism.

Both the commensurator subgroup property and the outer commensurator-normalizer property for $G(O_K)$ are proved for higher rank G (assuming bounded generation is by unipotents). Shalom-Willis go on to formulate a conjecture which generalizes both the CmSP and the CSP simultaneously!

In this context, here is an interesting open question:

Do there exist constants $a, b > 0$ such that each matrix in $SL(2, \mathbf{Z})$ is a product of at the most a elementary matrices in $SL(2, \mathbf{Q})$ whose denominators are bounded in absolute value by b?

$SL(2, O)$ - work with Morgan and Rapinchuk

Let $\mathbb{O}$ be the ring of S-integers in a number field k whose group of units $\mathbb{O}^\times$ is infinite.
We show that every matrix in $\Gamma = \mathrm{SL}_2(\mathbb{O})$ is a product of at most 9 elementary matrices.
As a consequence, we obtain that Γ is boundedly generated as an abstract group by $9[k : \mathbb{Q}] + 10$ cyclic subgroups. We can also deduce that for $n \geq 3$, the abstract group $SL_n(\mathbb{O})$ can be boundedly generated by $4 + \frac{n(3n-1)}{2}$ elementary generators.

The key notion we use is that of $\mathbb{Q}$-split prime: we say that a prime $\mathfrak{p}$ of a number field k is $\mathbb{Q}$-split if it is non-dyadic and its local degree over the corresponding rational prime is 1.
Some simple properties of $\mathbb{Q}$-split primes are listed as:
Let $\mathfrak{p}$ be a $\mathbb{Q}$-split prime in $\mathbb{O}$, and for $n \geq 1$ let $\rho_n \colon \mathbb{O} \to \mathbb{O}/\mathfrak{p}^n$ be the corresponding quotient map. Then:
(a) the group of invertible elements $(\mathbb{O}/\mathfrak{p}^n)^\times$ is cyclic for any n;
(b) if $c \in \mathbb{O}$ is such that $\rho_2(c)$ generates $(\mathbb{O}/\mathfrak{p}^2)^\times$ then $\rho_n(c)$ generates $(\mathbb{O}/\mathfrak{p}^n)^\times$ for any $n \geq 2$.

Among other things, we prove and use the following refinement of Dirichlet's theorem:

Let $\mathbb{O}$ be the ring of S-integers in a number field k for some finite $S \subset V^k$ containing V_∞^k. If nonzero $a, b \in \mathbb{O}$ are relatively prime (i.e., $a\mathbb{O} + b\mathbb{O} = \mathbb{O}$), then there exist infinitely many principal $\mathbb{Q}$-split prime ideals $\mathfrak{p}$ of $\mathbb{O}$ with a generator π such that $\pi \equiv a$ (modulo $b\mathbb{O}$) and $\pi > 0$ in all real completions of k.

Bounded Presentations

We mentioned that bounded generation for arithmetic groups has bearing on the congruence subgroup property; here, we go the other way and show how the CSP plays a role in bounded presentations for finite simple groups.

The group $SL_2(\mathbb{Z}[1/2])$ has the congruence subgroup property as Mennicke proved in 1967; using this, he proved in collaboration with Behr in 1968 that for any odd prime p, the group $PSL_2(\mathbb{F}_p)$ has the presentation

$$< S, T | S^p, T^2, (ST)^3, (S^2 T S^{(p+1)/2} T)^3 > .$$

CSP implies that for odd m, the normal subgroup generated by $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ is the full congruence subgroup of level m.

Guralnick, Kantor, Kassabov and Lubotzky proved some very surprising results regarding finite simple groups.

Without even using CFSG, they show that there is an absolute bound $C \leq 100$ such that every finite simple group with the possible exception of the Ree groups ${}^2G_2(q)$ has a presentation with 2 generators and $\leq C$ relations; one can take $C = 8$ for all A_n's and all S_n's.

They use the bounded presentation for $PSL_2(p)$ along with its usual embedding in S_{p+1} in order to obtain a bounded presentation for S_{p+2} for primes $p > 3$. Gluing together such presentations for two copies of S_{p+2}, they obtain bounded presentations for all S_n.

For $n \geq 6$, the groups $SL_n(\mathbb{Z})$ can be presented by 4 generators and 16 relations.

The (non-linear) group $Aut(F_n)$ for $n \geq 3$ can be generated by 5 generators and 18 relations.

B H Neumann

Here is the result proved with Nikolov that was mentioned earlier:
If A and B are groups then $A \wr B$ has bounded generation if and only if A has bounded generation and B is finite.

The proof depends on the following elementary but beautiful group theoretic result of B H Neumann which asserts that when a group is written as a union of finitely many left cosets of subgroups, those subgroups which are of infinite index can be dropped!

Subgroup Growth

Lubotzky, Mann, and Segal proved the remarkable theorem that a finitely generated, residually finite group G has PSG if, and only if, it is virtually solvable, and of finite rank. Equivalently, these things happen if, and only if, G is virtually solvable and is linear over $\mathbb{Q}$.

Amazingly, this needs an analogue of the Hilbert 5th problem for p-adic groups; in fact, they show that a finitely generated, pro-p group G is a p-adic Lie group if, and only if, G has PSG (as a profinite group).

The main point of the proof is that for finitely generated, linear groups which are not virtually solvable, $a_n \geq n^{c \log n / \log \log n}$ for some constant $c > 0$.

In particular, there is a gap in the possible growths of finitely generated, linear groups.

The proof also involves the prime number theorem.

Fritz Grunewald and Marcus du Sautoy proved for a finitely generated, infinite, nilpotent group that:

(i) the abscissa of convergence $\alpha(G)$ of $\zeta_G(s)$ is a rational number and $\zeta_G(s)$ can be meromorphically continued to $Re(s) > \alpha(G) - \delta$ for some $\delta > 0$. The continued function is holomorphic on the line $Re(s) = \alpha(G)$ except for a pole at $s = \alpha(G)$;

(ii) there exist a nonnegative integer $b(G)$ and real numbers c, c' such that

$$s_n \sim c n^{\alpha(G)} (log n)^{b(G)}$$

$$s_n^{\alpha(G)} \sim c' (log n)^{b(G)+1}$$

as $n \to \infty$.

Lubotzky related subgrowth for arithmetic groups (in characteristic 0) with the congruence subgroup property; he proved that the growth of congruence subgroups in characteristic 0 satisfies the inequalities

$$n^{c_1 logn/loglogn} \leq c_n \leq n^{c_2 logn/loglogn}.$$

This implies that if CSP does not hold good, then $s_n \geq n^{clogn}$ for some c and infinitely many n.

In other words, the growth rate of all subgroups of finite index is strictly greater than the growth rate of the congruence of subgroups among them.

For PSG groups, one defines the degree $\deg(G)$ of subgroup growth as the 'smallest' positive real number c such that $a_n(G) = O(n^{c+\epsilon})$ for all $\epsilon > 0$.

Shalev showed that the degree cannot lie in the intervals $(0,1),(1,3/2),(3/2,5/3)$ - note that the Heisenberg group has degree $3/2$.

It is not known whether the degree could be irrational and, whether the set of degrees forms a countable set.

Marcus du Sautoy proved the deep result that the zeta function of a compact p-adic Lie group is a rational function of p^{-s}, and that the degree in this case is always rational.

Representation Zeta Functions

There are other types of zeta functions defined; for instance, one on which there have been lectures already in the first week, and there will be more talks in this conference encodes the number of representations of a given dimension which is called the representation zeta function.

Liebeck and Shalev followed by Larsen and Lubotzky in 2008, and later Avni, Klopsch, Onn and Voll have proved deep results on the representation zeta functions of p-adic analytic groups and, of arithmetic groups.

If $r_n(G)$ denotes the number of irreducible representations of dimension n for a group G, then for groups for which $r_n(G) < \infty$ for all n (this includes all arithmetic groups satisfying the CSP), the representation zeta function is defined as $\zeta_G^{rep}(s) = \sum_{n \geq 1} r_n(G) n^{-s}$.

Lubotzky and Martin showed that $r_n(\Gamma)$ is polynomially bounded (i.e., polynomial representation growth holds) if, and only if, Γ satisfies the CSP.

If G is an absolutely simple group defined over a global field K, then Larsen and Lubotzky proved the following results; they showed that for any infinite linear group Γ, the abscissa of convergence $\rho(\Gamma) \geq 1/15$.

For $G(\mathbb{C})$ itself, they proved that ρ is r/κ where r is the absolute rank and $\kappa = |\Phi^+|$, the number of absolute roots; note that $r/\kappa = 2/h$, where h is the Coxeter number. For every finite place v, they show $\rho(G(O_v)) \geq r/\kappa$. For a division algebra of degree d over a local field K, the abscissa of convergence is $2/d$.

Larsen-Lubotzky proved also that the representation zeta function of arithmetic groups admits an Euler product. They also conjectured some general results for arithmetic groups, and proved it for the special case of products of SL_2's. They proved: if $G = \prod_{i=1}^{\ell} SL_2(K_i)$ where K_i are local fields of characteristics different from 2 and $\ell \geq 2$, any irreducible lattice Γ (which is hence S-arithmetic) satisfying the CSP (always conjectured to be so), they proved $\rho(\Gamma) = 2$.

Avni had shown $\rho(\Gamma)$ is a rational number when Γ satisfies the CSP.

Avni, Klopsch, Onn and Voll proved quantitative results on $\rho(\Gamma)$ for arithmetic groups, proving, in particular, the Larsen-Lubotzky conjecture in many cases.

Their methods involve the Kirillov orbit method, p-adic integrals and mathematical logic as well.

More precisely, they prove that for any irreducible root system Φ, there exists a constant ρ_Φ such that $\rho(G(O_S)) = \rho_\Phi$ if $G(O_S)$ has the CSP, where O_S is the ring of S-integers in a number field K (with S finite), and where G is an affine group scheme over O_S whose generic fiber is connected, simply connected, absolutely almost simple, with absolute root system Φ.

The conjecture of Larsen-Lubotzky proved by these four authors asserts that ρ is the same for different irreducible lattices (in the same semisimple group of characteristic 0) which satisfy the CSP.

Avni proved $\rho_{A_2} = 1$; nothing much is known about general ρ_Φ.

$2, 3$-generation etc.

A $(2, 3)$-generated group is a group generated by an element of order 2 and an element of order 3; therefore, these are precisely the quotient groups of the modular group $PSL_2(\mathbb{Z})$.

The problem of classifying all such groups is hopeless; there are 2^{N_0} isomorphism classes of infinite simple (2,3)-generated groups.

We briefly mention some results for classical matrix groups over $\mathbb{Z}$ and also some conjectures.

We mention in passing an important result for FINITE groups:

Theorem (Liebeck, Shalev, 1996). Let G run through some infinite set of finite classical groups, other than $PSp_4(p^k)$). Then

$$lim_{|G| \to \infty} Prob(x^2 = y^3 = 1; G =< x; y >) = 1.$$

Further, such a result remains true if we either fix the field and let the rank tend to infinity or if we fix the type and let the size of the field tend to infinity.

For $PSp_4(p^k)$ with $p \geq 5$, the probability tends to $1/2$ as k tends to infinity, and it tends to 0 if $p = 2$ or 3.

These methods involve intricate estimations of the number of maximal subgroups and their indices etc. and do not provide any set of generators explicitly.

Nevertheless, these tools provide remarkable combinatorial applications - for instance, to the question of isomorphism classes of simple, primitive subgroups of S_n.

The latter type of question has number-theoretic interpretations of independent interest - for example, can a given positive integer n be expressed as a binomial coefficient $\binom{m}{k}$ in more than $O(1)$ ways?

Classical groups over $\mathbb{Z}$

$SL_n(Z)$ and $GL_n(Z)$ are (2,3)-generated precisely when $n \geq 5$ - a combination of results proved by Tamburini, Vsemirnov et al.

Note that $SL_2(Z)$ is NOT $(2,3)$-generated as it contains no non-central involution; $SL_4(Z)$ and $GL_4(Z)$ are NOT (2,3)-generated as $SL_4(2) = GL_4(2) \cong A_8$.

Symplectic case

Theorem (Vasiliev, Vsemirnov, 2008-2011).
$Sp_2(Z), Sp_4(Z)$, and $Sp_6(Z)$ are NOT (2,3)-generated.
$Sp_8(Z), Sp_{10}(Z)$ are (2,3)-generated.
$Sp_{2n}(Z)$ is (2,3)-generated for $2n \geq 50$.

The cases $12 \leq 2n \leq 48$ remain open; the answer is expected to be positive.

Problem 1 Does the group $SL_n(\mathbf{Z}[X_1, \cdots, X_m])$ have bounded generation with respect to elementary matrices if n is large enough compared to n?

Problem 2 Under assumption $\mathrm{sr}(R) = 1$ prove that any element of $E_{\mathrm{ad}}(\Phi, R)$ is a product of ≤ 2 commutators in $G_{\mathrm{ad}}(\Phi, R)$.

Problem 3 Under assumption $\mathrm{sr}(R) = 1$ prove that any element of $E(\Phi, R)$ is a product of ≤ 3 commutators in $E(\Phi, R)$.

Problem 4 If the stable rank $\mathrm{sr}(R)$ of R is finite, and for some $m \geq 2$ the elementary linear group $E(m, R) = E_{\mathrm{sc}}(A_{m-1}, R)$ has bounded word length with respect to elementary generators, then for all Φ of sufficiently large rank one has

$$E(\Phi, R) = (U(\Phi, R)U^{-}(\Phi, R))^3.$$

Problem 5 If the stable rank $\mathrm{sr}(R)$ of R is finite, and for some $m \geq 2$ the elementary linear group $E(m, R)$ has bounded word length with respect to elementary generators, then for all Φ of sufficiently large rank any element of $E(\Phi, R)$ is a product of ≤ 4 commutators in $E(\Phi, R)$.

Problem 6 Let R be a Dedekind ring of arithmetic type with infinite multiplicative group. Prove that any element of $E_{\mathrm{ad}}(\Phi, R)$ is a product of ≤ 3 commutators in $G_{\mathrm{ad}}(\Phi, R)$.

Problem 7 Find for a Chevalley group of rank ≥ 2 the minimal L such that

$$G(\Phi, \mathbb{Z}) = (U(\Phi, \mathbb{Z})U^{-}(\Phi, \mathbb{Z}))^{L}.$$

Problem 8 Is it true that $U^{-}UU^{-}UU^{-} = UU^{-}UU^{-}U$?

Problem 9 Calculate the width of the elementary Chevalley group $E(\Phi, R)$ over a semilocal ring R in terms of unipotent radicals U_P and U_P^- of two opposite parabolic subgroups.

Problem 10 Calculate the width of the elementary subgroup $E(R)$ of an isotropic reductive group $G(R)$ over a semilocal ring R, in terms of unipotent radicals U_P and U_P^- of two opposite parabolic subgroups.

Problem 11 Prove that the elementary subgroup $E(R)$ of an isotropic reductive group $G(R)$ of relative rank ≥ 2 has bounded width with respect to the unipotent radicals U_P and U_P^- of two opposite parabolic subgroups, in the case where $R = \mathcal{O}_S$ is a Dedekind ring of arithmetic type.

References

- H. Bass, J. Milnor, J.-P. Serre, *Solution of the congruence subgroup problem for* SL_n $(n \geq 3)$ *and* Sp_{2n} $(n \geq 2)$. Publ. Math. Inst. Hautes Etudes Sci. 33 (1967), 59–137.
- N. A. Vavilov, *Parabolic subgroups of Chevalley groups over a commutative ring*. — J. Soviet Math **116** (1984), 1848–1860.
- N. A. Vavilov, S. S. Sinchuk, *Dennis—Vaserstein type decomposition*. — Zapiski Nauchn. Semin. POMI **375** (2010), 48–60 (in Russian).
- N. A. Vavilov, S. S. Sinchuk, *Parabolic factorisations of the split classical groups*. — Algebra and Analysis **23** (2011), No. 4, 1–30 (in Russian).

- A. Yu. Luzgarev, A. K. Stavrova, *Elementary subgroup of an isotropic reductive group is perfect*, — Algebra and Analysis **23** (2011).
- V. A. Petrov, A. K. Stavrova, *Elementary subgroups of isotropic reductive groups*. — St. Petersburg Math. J. **20** (2008), no. 3, 160–188.
- J.-P. Serre, *Le problème des groupes de congruence pour* SL_2. — Ann. Math. **92** (1970), 489–527.
- R. Steinberg, *Lectures on Chevalley groups*, Yale University, 1967.

- O. I. Tavgen, *Finite width of arithmetic subgroups of Chevalley groups of rank* ≥ 2. — Soviet Math. Doklady **41** (1990), no.1, 136–140.
- O. I. Tavgen, *Bounded generation of Chevalley groups over rings of S-integer algebraic numbers*. — Izv. Acad. Sci. USSR **54** (1990), no.1, 97–122.
- E. Abe, *Chevalley groups over local rings*. — Tôhoku Math. J. **21** (1969), No. 3, 474–494.
- E. Abe, K. Suzuki, *On normal subgroups of Chevalley groups over commutative rings*. — Tôhoku Math. J. **28** (1976), No. 1, 185–198.

L. Babai, N. Nikolov, L. Pyber, *Product growth and mixing in finite groups*. — In: 19th Annual ACM–SIAM Symposium on Discrete Algorithms, ACM–SIAM (2008), pp. 248–257.

H. Bass, K-*theory and stable algebra*. — Publ. Math. Inst. Hautes Études Sci. No. 22 (1964), 5–60.

D. Carter, G. Keller, *Bounded elementary generation of* $SL_n(\mathcal{O})$. — Amer. J. Math. **105** (1983), 673–687.

D. Carter, G. Keller, *Elementary expressions for unimodular matrices*. — Commun. Algebra **12** (1984), 379–389.

- D. Carter, G. E. Keller, E. Paige, *Bounded expressions in* $\mathrm{SL}(2, \mathcal{O})$. — Preprint Univ. Virginia (1983).
- R. W. Carter, *Simple groups of Lie type*. Wiley, London et al., 1972.
- Chen Baoquan, A. Kaufman, $3\mathrm{D}$ *volume rotation using shear transformations*. — Graph. Models **62** (2000), 308–322.
- Chen Huanyin, Chen Miaosen, *On products of three triangular matrices over associative rings* — Linear Algebra Applic. **387** (2004), 297–311.

V. Chernousov, E. Ellers, N. Gordeev, *Gauss decomposition with prescribed semisimple part: short proof*. — J. Algebra **229** (2000), 314–332.

P. M. Cohn, *On the structure of the* GL_2 *of a ring*. — Publ. Math. Inst. Hautes Études Sci. No. 30 (1967), 5–53.

G. Cooke, P. J. Weinberger, *On the construction of division chains in algebraic number rings, with applications to* SL_2. — Commun. Algebra **3** (1975), 481–524.

R. K. Dennis, L. N. Vaserstein, *On a question of M. Newman on the number of commutators*. — J. Algebra **118** (1988), 150–161.

E. Ellers, N. Gordeev, *On the conjectures of J. Thompson and O. Ore*. — Trans. Amer. Math. Soc. **350** (1998), 3657–3671.

I. V. Erovenko, A. S. Rapinchuk, *Bounded generation of some S-arithmetic orthogonal groups*. — C. R. Acad. Sci. **333** (2001), No. 5, 395–398.

F. J. Grunewald, J. Mennicke, L. N. Vaserstein, *On the groups* $\mathrm{SL}_2(\mathbb{Z}[x])$ *and* $\mathrm{SL}_2(K[x,y])$. — Israel J. Math. **86** (1994), Nos. 1–3, 157–193.

R. M. Guralnick, G. Malle, *Products of conjugacy classes and fixed point spaces*. `arXiv: 1005.3756`.

- Hao Pengwei, *Customizable triangular factorizations of matrices*. — Linear Algebra Applic. **382** (2004), 135–154.
- W. van der Kallen, $SL_3(\mathbb{C}[x])$ *does not have bounded word length*. — Springer Lect. Notes Math. **966** (1982), 357–361.
- T. J. Laffey, *Expressing unipotent matrices over rings as products of involutions*. — Preprint Univ. Dublin (2010).
- T. J. Laffey, *Lectures on integer matrices*. — Beijing (2010), 1–38.

Lei Yang, Hao Pengwei, Wu Dapeng, *Stabilization and optimization of PLUS factorization and its application to image coding*. — J. Visual Communication & Image Representation **22** (2011), No. 1.

M. Larsen, A. Shalev, *Word maps and Waring type problems*. — J. Amer. Math. Soc. **22** (2009), 437–466.

H. W. Lenstra (jr.), P. Moree, P. Stevenhagen, *Character sums for primitive root densities*. — (2011) (to appear).

M. Liebeck, A. Shalev, *Classical groups, probabilistic methods, and the* $(2,3)$*-generation problem*. — Ann. Math. **144** (1996), No. 1, 77-125.

- M. Liebeck, A. Shalev, *Diameteres of finite simple groups: sharp bounds and applications*. — Ann. Math. **154** (2001), 383–406.
- M. Liebeck, N. Nikolov, A. Shalev, *Groups of Lie type as products of* SL_2 *subgroups*. — J. Algebra **326** (2011), 201–207.
- M. Liebeck, E. A. O'Brien, A. Shalev, Pham Huu Tiep, *The Ore conjecture*. — J. Europ. Math. Soc. **12** (2010), 939–1008.
- M. Liebeck, E. A. O'Brien, A. Shalev, Pham Huu Tiep, *Products of squares in finite simple groups*. — Proc. Amer. Math. Soc. (2011).

- M. Liebeck, L. Pyber, *Finite linear groups and bounded generation*. — Duke Math. J. **107** (2001), 159–171.
- B. Liehl, *Beschränkte Wortlänge in* SL_2. — Math. Z. **186** (1984), 509–524.
- D. W. Morris, *Bounded generation of* $\mathrm{SL}(n, A)$ (*after D. Carter, G. Keller, and E. Paige*) — New York J. Math. **13** (2007), 383–421.
- K. R. Nagarajan, M. P. Devaasahayam, T. Soundararajan, *Products of three triangular matrices over commutative rings*. — Linear Algebra Applic. **348** (2002), 1–6.

N. Nikolov, *A product decomposition for the classical quasisimple groups*. — J. Group Theory **10** (2007), 43–53.

N. Nikolov, L. Pyber, *Product decomposition of quasirandom groups and a Jordan type theorem*. arXiv:math/0703343 (2007).

A. Paeth, *A fast algorithm for general raster rotation*. — In: Graphics Gems, Acad. Press (1990), pp. 179–195.

A. Shalev, *Commutators, words, conjugacy classes, and character methods*. — Turk. J. Math. **31** (2007), 131–148.

- A. Shalev, *Word maps, conjugacy classes, and a noncommutative Waring-type theorem*. — Ann. Math. **170**, No. 3 (2009), 1383–1416.
- R. W. Sharpe *On the structure of the Steinberg group* St(Λ). — J. Algebra **68** (1981), 453–467.
- She Yiyuan, Hao Pengwei, *On the necessity and sufficiency of PLUS factorizations*. — Linear Algebra Applic. **400** (2005), 193–202.
- S. Sinchuk, N. Vavilov, *Parabolic factorisations of exceptional Chevalley groups* (to appear).

A. Sivatski, A. Stepanov, *On the word length of commutators in* $\mathrm{GL}_n(R)$. — K-theory **17** (1999), 295–302.

M. R. Stein, *Surjective stability in dimension* 0 *for* K_2 *and related functors*. — Trans. Amer. Math. Soc. **178** (1973), 176–191.

A. Stepanov, N. Vavilov, *On the length of commutators in Chevalley groups*. — Israel Math. J. (2011), 1–20.

G. Strang, *Every unit matrix is a LULU*. — Linear Algebra Applic. **265** (1997), 165–172.

- O. I. Tavgen, *Bounded generation of normal and twisted Chevalley groups over the rings of S-integers*. — Contemp. Math. **131** (1992), No. 1, 409–421.
- T. Toffoli, *Almost every unit matrix is a ULU*. — Linear Algebra Applic. **259** (1997), 31–38.
- T. Toffoli, J. Quick, *Three dimensional rotations by three shears*. — Graphical Models & Image Processing **59** (1997), 89–96.
- L. N. Vaserstein, *Bass's first stable range condition*. — J. Pure Appl. Algebra **34** (1984), Nos. 2–3, 319–330.

L. N. Vaserstein, E. Wheland, *Commutators and companion matrices over rings of stable rank* 1. — Linear Algebra Appl. **142** (1990), 263–277.

N. Vavilov, *Structure of Chevalley groups over commutative rings*. — In: Proc. Conf. Non-associative algebras and related topics (Hiroshima – 1990) , World Sci. Publ., London et al. (1991), pp. 219–335.

N. Vavilov, E. Plotkin, *Chevalley groups over commutative rings.* I. *Elementary calculations*. — Acta Applicandae Math. **45** (1996), 73–115.

A. Smolensky, B. Sury, N. A. Vavilov, *Unitriangular factorizations of Chevalley groups,* J. Math. Sci. (N. Y.) 183(5) (2012) 584599.

- N. A. Vavilov, E. B. Plotkin, *Net subgroups of Chevalley groups.* II. *Gauss decomposition*. — J. Soviet Math. **114** (1984), 2874–288.
- N. A. Vavilov, E. I. Kovach, SL_2*-factorisations of Chevalley groups*. — Zapiski Nauchn. Semin. POMI (2011).
- I. V. Erovenko, A. S. Rapinchuk, *Bounded generation of S-arithmetic subgroups of isotropic orthogonal groups over number fields*, J. Number Theory **119**(2006), No. 1, 28-48.
- V.K. Murty, *Bounded and finite generation of arithmetic groups. (English summary)*, Number theory (Halifax, NS, 1994), CMS Conf. Proc. **15**, Amer. Math. Soc., Providence, RI, (1995), 249-261.

- Y. Shalom, G. Willis, *Commensurated subgroups of arithmetic groups, totally disconnected groups and adelic rigidity*, Geom. Funct. Anal., **23**(2013), No. 5, 1631-1683.
- M. Vsemirnov, *Short unitriangular factorizations of* $\mathrm{SL}_2(\mathbb{Z}[1/p])$, Quart. J. Math., **65** (2014), 279-290.
- N. Avni, B. Klopsch, U. Onn and C. Voll, *Arithmetic groups, Base change, and representation growth*, Geom. Funct. Anal. DOI: 10.1007/s00039-016-0359-6.
- M. Larsen and A. Lubotzky, *Representation growth of linear groups*, Journal of the European Mathematical Society, (10) (2008), 351390.

- M.W. Liebeck and A. Shalev, *Character degrees and random walks in finite groups of Lie type*, Proceedings of the London Mathematical Society, (90) (2005), 6186.
- A. Lubotzky and B. Martin, *Polynomial representation growth and the congruence subgroup problem*, Israel Journal of Mathematics, (144) (2004), 293316.
- A. Lubotzky and N. Nikolov, *Subgroup growth of lattices in semisimple Lie groups*, Acta Mathematica, (193) (2004), 105139.
- P. Corvaja,A. S. Rapinchuk, J. Ren and U. M. Zannier, *Non-virtually abelian anisotropic linear groups are not boundedly generated*, Invent. math. https://doi.org/10.1007/s00222-021-01064-y.

- M. Abért, A. Lubotzky, L. Pyber, *Bounded generation and linear groups,* Inernat. J. Algebra Comput. **13** (2003), no. 4, 401–413.
- M. Kassabov, *Kazhdan constants for* $\mathrm{SL}_n(\mathbb{Z})$, Internat. J. Algebra Comput. **15** (2005), no. 5–6, 971–995.
- M. Lazard, *Groupes analytiques p-adiques,* Inst. Hautes Études Sci. Publ. Math. **26** (1965), 389–603.
- A. Lubotzky, *Dimension function for discrete groups,* London Math. Soc. Lecture Note Ser. **121** (1986), 254–262.

- A. Lubotzky, *Subgroup growth and congruence subgroups,* Invent. Math. **119** (1995), no. 2, 267–295.
- B.H. Neumann, *Groups covered by permutable subsets*, J. London Math. Soc. **29** (1954), 236–248.
- Y. Shalom, *Bounded generation and Kazhdan's property (T),* Inst. Hautes Études Sci. Publ. Math. **90** (1999), 145–168.
- V.P. Platonov and A.S. Rapinchuk, *Abstract properties of S-arithmetic groups and the congruence problem*, Russian Acad. Sci. Izv. Math. **40** (1993), no. 3, 455–476.

- L. Pyber, D. Segal, *Finitely generated groups with polynomial index growth*, Journal fur die Reine und Angewande Mathematik **612** (2007), 173-211.
- A. Smolensky, *Commutator width of Chevalley groups over rings of stable rank 1*, J. Group Theory 22 (2019), 83101 DOI 10.1515/jgth-2018-0035.
- N. Avni, T. Gelander, M. Kassabov and A. Shalev, *Word values in p-adic and adelic groups*, Bull. Lond. Math. Soc. 45 (2013), no. 6, 13231330.
- A. Smolensky, B. Sury and N. A. Vavilov, *Gauss decomposition for Chevalley groups revisited,* Int. J. Group Theory 1 (2012), no. 1, 316.

- R. C. Thompson, *Commutators in the special and general linear groups,* Trans. Amer. Math. Soc. 101 (1961), 1633.
- M. R. Stein, *Generators, relations, and coverings of Chevalley groups over commutative rings,* Amer. J. Math. 93 (1971) 9651004.
- A. W. Mason, W. W. Stothers, *On subgroups of GL(n, A) which are generated by commutators*, Invent. Math. 23 (1974) 327346.
- B. Ivarsson and F. Kutzschebauch, *Holomorphic factorization of mappings into* $SL_n(\mathbb{C})$, Annals of Mathematics 175 (2012), 45-69 http://dx.doi.org/10.4007/annals.2012.175.1.3.

B. Ivarsson and F. Kutzschebauch, *On the number of factors in the unipotent factorization of holomorphic mappings into $SL_2(\mathbb{C})$*, Proc. Amer. Math. Soc., Available at http://dx.doi.org/10.1090/S0002-9939-2011-11025-6.

G. Huang, F. Kutzschebauch and J. Schott, *Factorization of holomorphic matrices and Kazhdan's property (T)*, Bull. Sci.math.190(2024)103376.

A. V. Morgan, A. S. Rapinchuk and B. Sury, *Bounded generation of SL_2 over rings of S-integers with infinitely many units*, Algebra Number Theory 12, 19491974.

N. Nikolov and B. Sury, *Bounded generation of wreath products*, J. Group Theory 18(6), 951959 (2015).

B. Sury, *Bounded generation does not imply finite presentation*, Comm. Alg., Vol.25 (1997) P.1673-1683.

A. S. Rapinchuk, *On the finite dimensional unitary representations of Kazhdan groups,* Proc. Amer. Math. Soc. 127 (1999), 1557-1562.

M. du Sautoy and F. Grunewald, *Analytic Properties of Zeta Functions and Subgroup Growth*, Annals of Mathematics, Second Series, Vol. 152, No. 3 (Nov., 2000), pp. 793-833.

M. Larsen, A. Shalev and P. H. Tiep, *Characteristic covering numbers of finite simple groups*, Mathematische Annalen (2024) 388:167189 https://doi.org/10.1007/s00208-022-02520-7.

I. I. Simion, *Covering Numbers of Unipotent Conjugacy Classes in Simple Algebraic Groups*, Transformation Groups (2024) 29:11991211 https://doi.org/10.1007/s00031-022-09756-2.

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