GW energy-momentum tensor and GW propagation

According to GR, any form of energy contributes to space-time curvature
Are GWs a source of space-time curvature?

- One needs to go beyond linearisation over Minkowski, otherwise one excludes from the beginning the presence of any background space-time curvature

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x)\,, \qquad\qquad |h_{\mu\nu}(x)| \ll 1$$

- In this new setting, how to decide what is the background and what is the fluctuation?

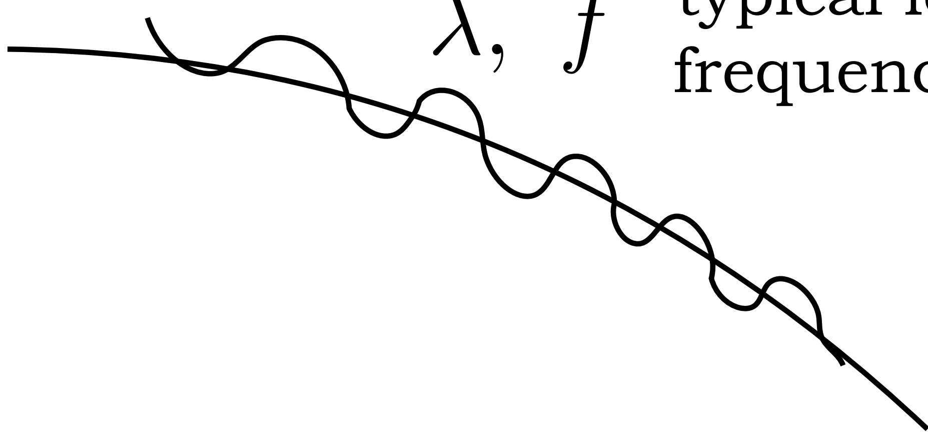
    1. The background space-time has a clear symmetry (static, FLRW…)
    2. It is possible to resort to a clear separation of scales/frequencies

"Gravitational Waves", M. Maggiore, Oxford University Press 2008

E.E. Flanagan and S.A. Hughes, "The basics of GW theory", arXiv:gr-qc/0501041

R.A. Isaacson, Physical Review, Volume 166, number 5, pages 1263 and 1272, 1968

GW energy-momentum tensor and GW propagation



$\lambda, \ f$ typical length-scale / frequency of the GWs $\qquad \lambda = \frac{c}{f}$

$L_B, \ f_B$ typical scale of spatial variations / frequency of time variations of the background (needn't be related among themselves)

- There are two expansions in the game:

$$1. \quad |h_{\mu\nu}| \ll 1 \qquad\qquad 2. \quad \frac{\lambda}{L_B} \ll 1, \quad \frac{f_B}{f} \ll 1$$

- In order to effectively implement the distinction among background and GWs, one needs to average physical quantities

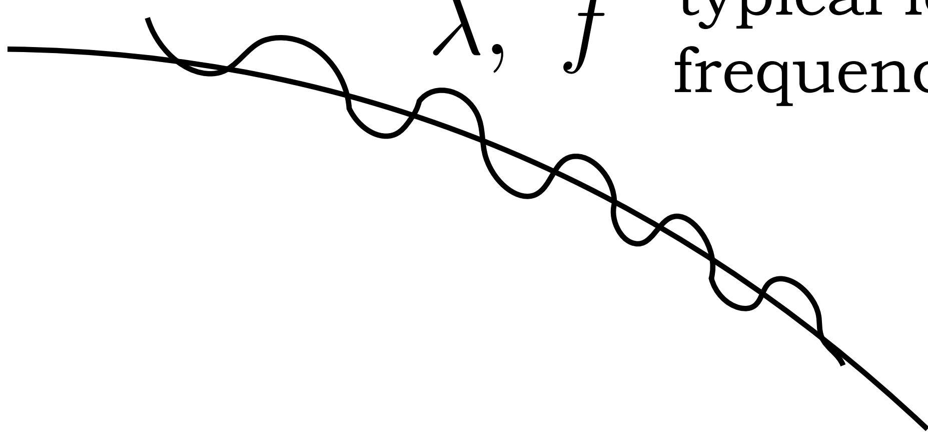
$$\lambda \ll \bar{\ell} \ll L_B \qquad\qquad f_B \ll \bar{f} \ll f$$

$$\bar{g}_{\mu\nu} \equiv \langle g_{\mu\nu} \rangle$$

$$\langle h_{\mu\nu} \rangle = 0$$

"Gravitational Waves", M. Maggiore, Oxford University Press 2008

GW energy-momentum tensor and GW propagation



$\lambda, \; f$ typical length-scale / frequency of the GWs $\qquad \lambda = \frac{c}{f}$

$L_B, \; f_B$ typical scale of spatial variations / frequency of time variations of the background

(needn't be related among themselves)

By expanding the Einstein equations to second order in $|h_{\mu\nu}| \ll 1$ and separating the background and first order components by averaging, one finds

- The expression for the GW energy momentum tensor (how GWs influence the background)
- The equation representing GW propagation on a curved background

GW energy-momentum tensor and GW propagation

Expand up to second order in $|h_{\mu\nu}| \ll 1$ $\qquad R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$

The linear term averages to zero

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu}$$

The quadratic term can influence the background, as it contains both high and low modes

$\langle \ldots \rangle = [\ldots]^{\text{low}}$

Background Einstein equation

$$\bar{R}_{\mu\nu} = [-R^{(2)}_{\mu\nu}]^{\text{low}} + 8\pi G \left[T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right]^{\text{low}}$$

GWs sourcing the bckg curvature

Matter sourcing the bckg curvature

$\mathcal{O}\left(\frac{1}{L_B}\right)^2$ $\qquad \mathcal{O}\left(\frac{h}{\lambda}\right)^2$

$$h \lesssim \frac{\lambda}{L_B}$$

Necessary condition for GW to make sense

GW energy-momentum tensor and GW propagation

Rearranging the Einstein equations and performing the average leads to:

$$\underbrace{\bar{G}_{\mu\nu}}_{\text{Dynamics of the bckg space-time}} = \langle R_{\mu\nu}\rangle - \frac{1}{2}\bar{g}_{\mu\nu}\langle R\rangle = 8\pi G(\underbrace{\langle T_{\mu\nu}\rangle}_{\text{Low-mode part of the matter component}} + \underbrace{T^{\mathrm{GW}}_{\mu\nu}}_{\text{GWs}})$$

not separately conserved!

GW energy-momentum tensor

$$T^{GW}_{\mu\nu} = -\frac{1}{8\pi G}\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)}\rangle$$

calculating $R^{(2)}_{\mu\nu}$ and reducing to the TT gauge

$$T^{GW}_{\mu\nu} = \frac{1}{32\pi G}\langle\nabla_\mu h_{\alpha\beta}\nabla_\nu h^{\alpha\beta}\rangle$$

GW energy density:

$$\rho_{GW} = \frac{\langle\dot{h}_{ij}\dot{h}^{ij}\rangle}{32\pi G}$$

C.W. Misner, K.S. Thorne, J.A. Wheeler, "Gravitation", Freeman 1997 (chapter 35.15)

GW energy-momentum tensor and GW propagation

Expand up to second order in $|h_{\mu\nu}| \ll 1$ $\quad R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$

Focus on the linear term: $\quad R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu}$

Perturbed Einstein equation

$$R^{(1)}_{\mu\nu} = [-R^{(2)}_{\mu\nu}]^{\text{high}} + 8\pi G \left[T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right]^{\text{high}}$$

$$\mathcal{O}\left(\frac{h}{\lambda^2}\right) \qquad \mathcal{O}\left(\frac{h}{\lambda}\right)^2$$

Matter possibly sourcing GWs

Negligible
(non-linear interaction of the wave with itself)

GW energy-momentum tensor and GW propagation

Expand up to second order in $|h_{\mu\nu}| \ll 1$ $\quad R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$

Focus on the linear term: $\quad R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu}$

Perturbed Einstein equation

$$R^{(1)}_{\mu\nu} - \frac{1}{2}(\bar{g}_{\mu\nu} R^{(1)} + h_{\mu\nu} \bar{R}) \simeq 8\pi G \left[T_{\mu\nu} \right]^{\rm high}$$

Evolution of GWs on a curved but smooth / slowly evolving background such as gravitational redshift and lensing

Possible source of GWs

calculating $R^{(1)}_{\mu\nu}$:

$$-\frac{1}{2}\Box \bar{h}_{\mu\nu} + R^{\lambda}{}_{\mu\nu}{}^{\sigma} \bar{h}_{\lambda\sigma} + \nabla_{(\nu} \nabla^{\sigma} \bar{h}_{\mu)\sigma} - \frac{1}{2} \bar{g}_{\mu\nu} \nabla^{\alpha} \nabla^{\beta} \bar{h}_{\alpha\beta} +$$

$$+ R^{\alpha\beta} \left[\frac{1}{2} \bar{g}_{\mu\nu} \bar{h}_{\alpha\beta} - \frac{1}{2} \bar{h}_{\mu\nu} \bar{g}_{\alpha\beta} + \bar{g}_{\beta(\mu} \bar{h}_{\nu)\alpha} \right] = 8\pi \, G \, \delta T_{\mu\nu}$$

GW energy-momentum tensor and GW propagation

Expand up to second order in $|h_{\mu\nu}| \ll 1$ $\quad R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$

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In a FLRW universe, equation of sourcing and propagation of GWs

$$ds^2 = -dt^2 + a^2(t)\,(\delta_{ij} + h_{ij})\, dx^i dx^j$$

Neglecting scalar and vector perturbations

$$\partial_i h_{ij} = h_{ii} = 0$$

$$\ddot{h}_{ij}(\mathbf{x}, t) + 3\, H\, \dot{h}_{ij}(\mathbf{x}, t) - \frac{\nabla^2}{a^2} h_{ij}(\mathbf{x}, t) = 16\pi G\, \Pi_{ij}(\mathbf{x}, t)$$

GW propagation equation in FLRW

COMMENTS

- In the rest of the course, we will be dealing with solutions of the above equation
- It can be derived also from cosmological perturbation theory, here I presented the connection with a more general approach
- In cosmology, the FLRW space-time is homogeneous and isotropic, so tensor modes can be defined also when $\lambda \sim L_B$ (exemple: horizon re-entry after inflation), but one cannot say these are GWs, unless modes are well within the horizon ($\lambda \ll L_B$)

GW propagation equation in FLRW

$$\ddot{h}_{ij}(\mathbf{x},t) + 3\,H\,\dot{h}_{ij}(\mathbf{x},t) - \frac{\nabla^2}{a^2}\,h_{ij}(\mathbf{x},t) = 16\pi G\,\boxed{\Pi_{ij}(\mathbf{x},t)}$$

Source: tensor anisotropic stress

Perfect fluid

$$T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu}$$

In the cosmological context:
energy momentum tensor of the matter content of the universe (background + perturbations)

$$\delta T_{ij} = \bar{p}\,\delta g_{ij} + a^2[\delta p\,\delta_{ij} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)\sigma + 2\partial_{(i}v_{j)}\boxed{+\,\Pi_{ij}]}$$

$$(\partial_i v_i = 0,\ \partial_i \Pi_{ij} = 0,\ \Pi_{ii} = 0)$$

NO GWs FROM THE HOMOGENEOUS MATTER COMPONENT

GW propagation equation in FLRW

$$\ddot{h}_{ij}(\mathbf{x},t) + 3\,H\,\dot{h}_{ij}(\mathbf{x},t) - \frac{\nabla^2}{a^2}\,h_{ij}(\mathbf{x},t) = 16\pi G\,\Pi_{ij}(\mathbf{x},t)$$

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energy momentum tensor of the matter content of the universe (background + perturbations)

One exploits the translational invariance and performs a F.T. in space

$$h_{ij}(\mathbf{x},t) = \sum_{r=+,\times} \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, h_r(\mathbf{k},t)\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, e^r_{ij}(\hat{\mathbf{k}})$$

$$\Pi_{ij}(\mathbf{x},t) = \sum_{r=+,\times} \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, \Pi_r(\mathbf{k},t)\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, e^r_{ij}(\hat{\mathbf{k}})$$

GW propagation equation in FLRW

$$\ddot{h}_{ij}(\mathbf{x},t) + 3\,H\,\dot{h}_{ij}(\mathbf{x},t) - \frac{\nabla^2}{a^2}\,h_{ij}(\mathbf{x},t) = 16\pi G\,\Pi_{ij}(\mathbf{x},t)$$

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not just plane waves as before

The evolution equation decouples for each polarisation mode

$$h_r''(\mathbf{k},\eta) + 2\,\mathcal{H}\,h_r'(\mathbf{k},\eta) + k^2\,h_r(\mathbf{k},\eta) = 16\pi G\,a^2\,\Pi_r(\mathbf{k},\eta)$$

conformal time, Hubble factor and comoving wavenumber

GW propagation equation in FLRW

$$h_r''(\mathbf{k},\eta) + 2\,\mathcal{H}\,h_r'(\mathbf{k},\eta) + k^2\,h_r(\mathbf{k},\eta) = 16\pi G\,a^2\,\Pi_r(\mathbf{k},\eta)$$

Solution of the homogeneous equation

Power-law scale factor $a(\eta) = a_n \eta^n$

Covering matter (n=2) and radiation domination (n=1), and De Sitter inflation n=-1)

$$h_r(\mathbf{k},\eta) = \frac{A_r(\mathbf{k})}{a_n \eta^{n-1}}\,j_{n-1}(k\eta) + \frac{B_r(\mathbf{k})}{a_n \eta^{n-1}}\,y_{n-1}(k\eta)$$

Two notable limiting cases: sub-Hubble and super-Hubble modes

$$H_r(\mathbf{k},\eta) = a\,h_r(\mathbf{k},\eta) \qquad H_r''(\mathbf{k},\eta) + \left(k^2 - \frac{a''}{a}\right) H_r(\mathbf{k},\eta) = 0$$

$$a''/a \propto \mathcal{H}^2$$

GW propagation equation in FLRW

CASE 1: Sub-Hubble modes, relevant for propagation after the source stops

$$k^2 \gg \mathcal{H}^2 \qquad h_r(\mathbf{k},\eta) = \frac{A_r(\mathbf{k})}{a(\eta)}\, e^{ik\eta} + \frac{B_r(\mathbf{k})}{a(\eta)}\, e^{-ik\eta}$$

In this limit, GWs are plane waves with redshifting amplitude

What are the coefficients $A_r(\mathbf{k})$ and $B_r(\mathbf{k})$ from the initial condition?

Suppose the source operates in a time interval η_{fin} - η_{in} in the radiation dominated era

$$H_r^{\rm rad}(\mathbf{k},\eta<\eta_{\rm fin}) = \frac{16\pi G}{k}\int_{\eta_{\rm in}}^{\eta} d\tau\, a(\tau)^3\, \sin[k(\eta-\tau)]\, \Pi_r(\mathbf{k},\tau)$$

Matching at η_{fin} with the homogeneous solution to find the GW signal today

$$H_r^{\rm rad}(\mathbf{k},\eta>\eta_{\rm fin}) = A_r^{\rm rad}(\mathbf{k})\cos(k\eta) + B_r^{\rm rad}(\mathbf{k})\sin(k\eta)$$

$$A_r^{\rm rad}(\mathbf{k}) = \frac{16\pi G}{k}\int_{\eta_{\rm in}}^{\eta_{\rm fin}} d\tau\, a(\tau)^3\, \sin(-k\tau)\, \Pi_r(\mathbf{k},\tau)$$

$$B_r^{\rm rad}(\mathbf{k}) = \frac{16\pi G}{k}\int_{\eta_{\rm in}}^{\eta_{\rm fin}} d\tau\, a(\tau)^3\, \cos(k\tau)\, \Pi_r(\mathbf{k},\tau)$$

GW propagation equation in FLRW

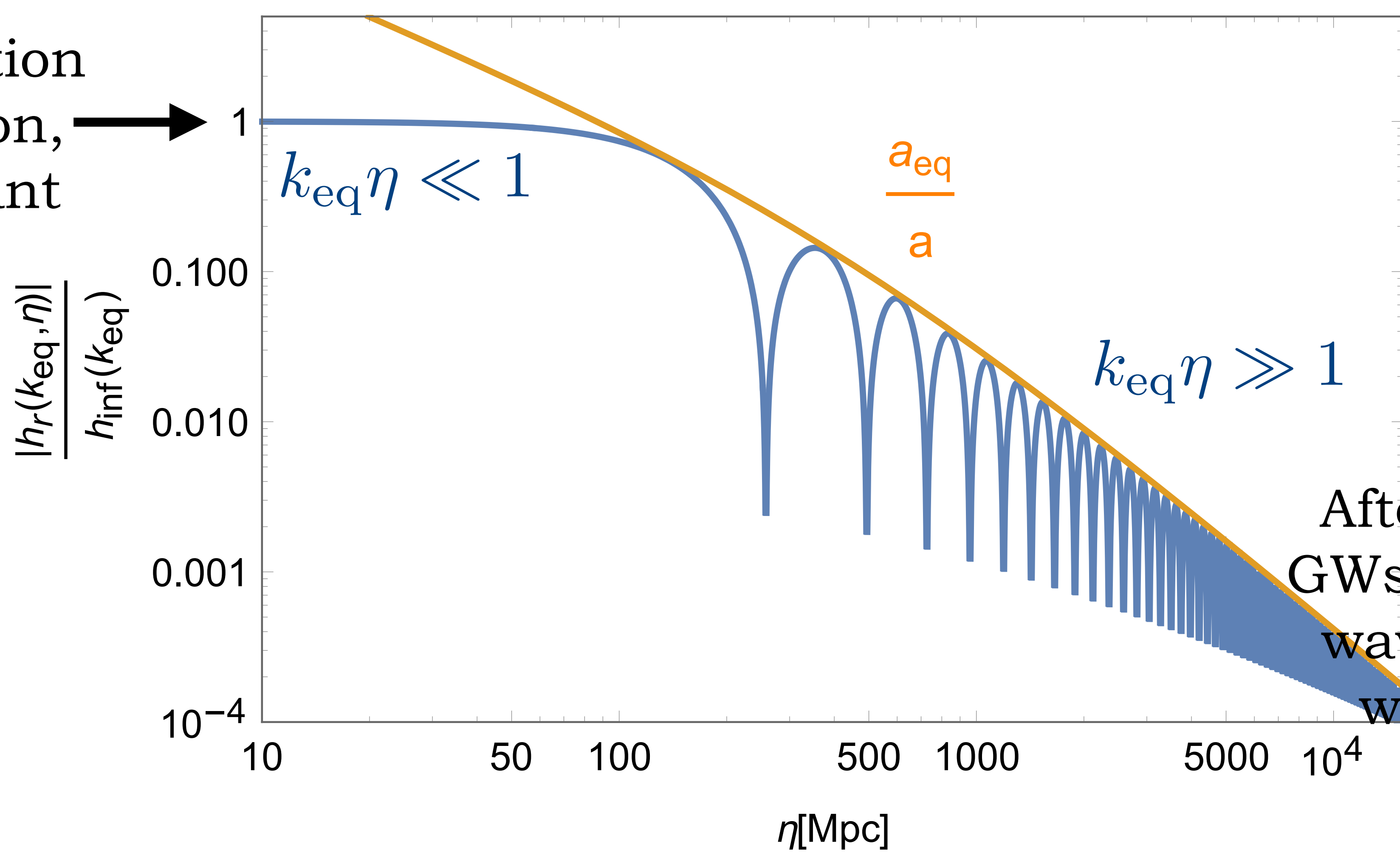
CASE 2: Super-Hubble modes, relevant for inflationary tensor perturbations

$$k^2 \ll \mathcal{H}^2 \qquad h_r(\mathbf{k},\eta) = A_r(\mathbf{k}) + B_r(\mathbf{k}) \int^{\eta} \frac{d\eta'}{a^2(\eta')}$$

→ Decaying mode, negligible

Full solution with inflationary initial conditions
Hubble re-entry at the radiation-matter transition

Initial condition from inflation, $A_r(\mathbf{k})$ constant →



After Hubble re-entry, GWs oscillating as plane waves and redshifting with the expansion

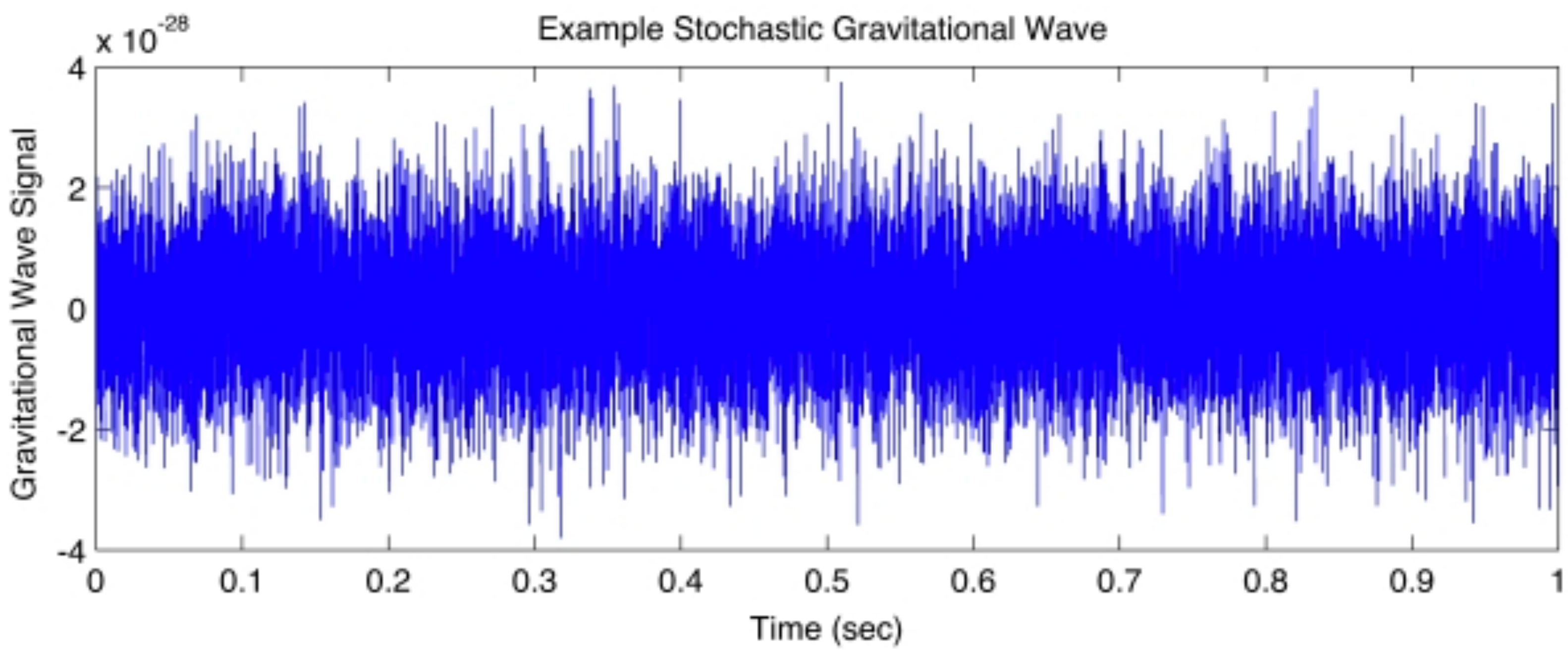
Summary up to here:

- We have defined GWs and GW energy density without ambiguity in the FLRW spacetime, making the connection with linearised gravity
- After their generation by some sourcing process, GWs in the FLRW space-time oscillate and decay with the expansion of the universe
- The sourcing process can be connected to the presence of anisotropic stresses at first order in cosmological perturbation theory, and/or to an inflationary phase
- We have gone as far as we could in all generality; to continue solving the equation one needs to specify more the characteristics of the sourcing process
- However, before analysing examples of GW sourcing processes in the early universe, we proceed with presenting some general features of signals from the early universe, and with describing present and future GW detectors with a particular focus on PTAs

Why sources in the early universe produce SGWBs?

A stochastic GW background is a signal for which *only the statistical properties can be accessed* because it is given by the incoherent superposition of sources that cannot be individually resolved

LIGO website



- For example, the superposition of deterministic GW signals from astrophysical binary sources with too low signal-to-noise ratio, or too much overlap in time and frequency -> confusion noise (Examples: LVK, LISA, PTAs…)

- Early universe GW sources produce SGWBs because they are homogeneously and isotropically distributed over the entire universe, and/or correlated on scales much smaller than the detector resolution

Why sources in the early universe produce SGWBs?

A GW source acting at time t_* in the early universe cannot produce a signal correlated on length/time scales larger than the causal horizon at that time

$$\ell_* \leq H_*^{-1}$$

ℓ_* characteristic length-scale of the source
(typical size of variation of the tensor anisotropic stresses)

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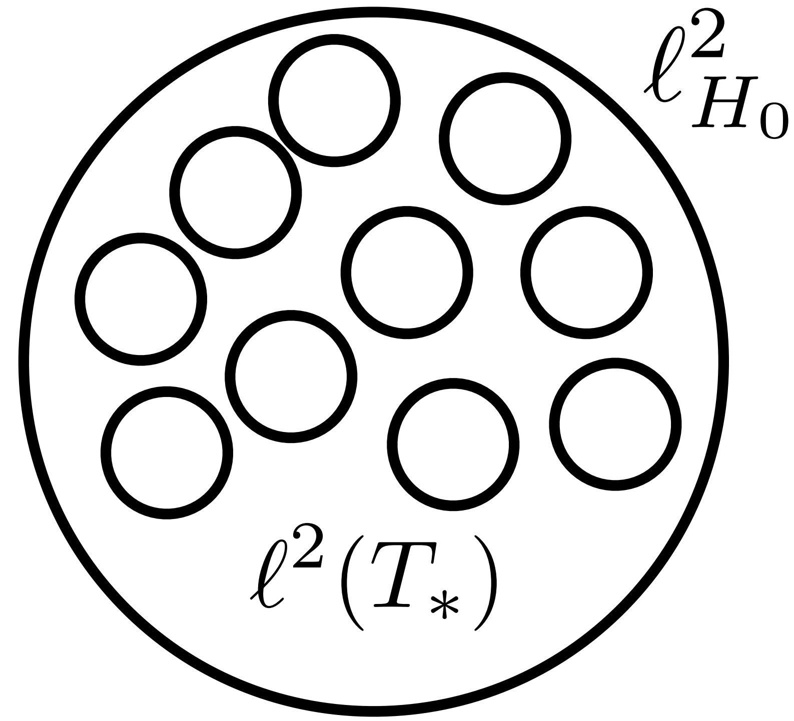
$$\ell_* \leq H_*^{-1}$$



Angular size on the sky today of a region in which the SGWB signal is correlated

$$\Theta_* = \frac{\ell_*}{d_A(z_*)}$$

Angular diameter distance

Number of uncorrelated regions accessible today $\sim \Theta_*^{-2}$

Suppose a GW detector angular resolution of 10 deg $\longrightarrow$ $z_* \lesssim 17$

$$\Theta(z_* = 1090) \simeq 0.9\,\text{deg} \qquad \Theta(T_* = 100\,\text{GeV}) \simeq 10^{-12}\text{deg}$$

Only the statistical properties of the signal can be accessed

Why sources in the early universe produce SGWBs?

- We access today the GW signal from many independent horizon volumes: $h_{ij}(\mathbf{x},t)$ must be treated as a random variable, only its statistical properties can be accessed, e.g. its correlator $\langle h_r(\mathbf{x}, \eta_1)\, h_s(\mathbf{y}, \eta_2) \rangle$

  where <…> is an ensemble average

- The universe is homogeneous and isotropic, so the GW source is operating everywhere at the same time with the same average properties ("a-causal" initial conditions from Inflation)

- Under the ergodic hypothesis, the ensemble average can be substituted with volume / time averages: we identify this average with the volume / time one necessary to define the GW energy momentum tensor

- Notable exception: *SGWB from Inflation* (intrinsic quantum fluctuations that become classical (stochastic) outside the horizon)

Characterisation of a primordial SGWB

The SGWB is in general homogenous and isotropic, unpolarised and Gaussian

As the FLRW space-time

$$\langle h_{ij}(\mathbf{x},\eta_1)\, h_{lm}(\mathbf{y},\eta_2)\rangle = F_{ijlm}(|\mathbf{x}-\mathbf{y}|,\eta_1,\eta_2)$$

Certainly some *induced anisotropy*, e.g. the dipole with respect to the cosmological frame

More challenging to detect than the "monopole"

If the sourcing process preserves parity

$$\langle h_{+2}(\mathbf{k},\eta)h_{+2}(\mathbf{k},\eta) - h_{-2}(\mathbf{k},\eta)h_{-2}(\mathbf{k},\eta)\rangle = \langle h_{+}(\mathbf{k},\eta)h_{\times}(\mathbf{k},\eta)\rangle = 0$$

Helicity basis $e_{ij}^{\pm 2} = \dfrac{e_{ij}^{+} \pm i\, e_{ij}^{\times}}{2}$

There are exceptions!

Central limit theorem: the signal comes from the superposition of many independent regions

Characterisation of a primordial SGWB

Power spectrum of the GW amplitude $h_c(k,t)$

$$\langle h_r(\mathbf{k},\eta)\, h_p^*(\mathbf{q},\eta)\rangle = \frac{8\pi^5}{k^3}\,\delta^{(3)}(\mathbf{k}-\mathbf{q})\,\delta_{rp}\,h_c^2(k,\eta)$$

Statistical homogeneity and isotropy

Unpolarised

Gaussianity: the two-point correlation function is enough to fully describe the SGWB

$$\langle h_{ij}(\mathbf{x},\eta)\, h_{ij}(\mathbf{x},\eta)\rangle = 2\int_0^{+\infty}\frac{dk}{k}\,h_c^2(k,\eta)$$

Related to the variance of the GW amplitude in real space

For *freely propagating sub-Hubble modes*, and taking the time-average:

$$h_r(\mathbf{k},\eta) = \frac{A_r(\mathbf{k})}{a(\eta)}\,e^{ik\eta} + \frac{B_r(\mathbf{k})}{a(\eta)}\,e^{-ik\eta}$$

$$\langle h_r(\mathbf{k},\eta)\, h_p^*(\mathbf{q},\eta)\rangle = \frac{1}{a^2(\eta)}\left[\langle A_r(\mathbf{k})\,A_p^*(\mathbf{q})\rangle + \langle B_r(\mathbf{k})\,B_p^*(\mathbf{q})\rangle\right]$$

$$h_c(k,\eta) \propto \frac{1}{a^2(\eta)}$$