QCD at non-zero temperature and heavy quarks: Lecture 2

Peter Petreczky

Brookhaven National Laboratory

Recap from lecture 1:

QCD transitions at non-zero temperature and densities, relation to QCD symmetries (chiral , Z(N)), Polyakov loop, Polyakov loop correlators, free energy of static quarks, color octet contribution (color deconfinement), color screening of static quarks;in 2+1 flavor QCD at physical quark mases there is remnant of deconfinement, which happens at the chiral transition;
Free energy of static (anti) quark at high
Temperature can be understood in terms of weak coupling EFTs: pNRQCD, EQCD

In this lecture: the chiral transition, HRG, equation of state, fluctuations of conserved charges and charm degrees of freedom across T_c

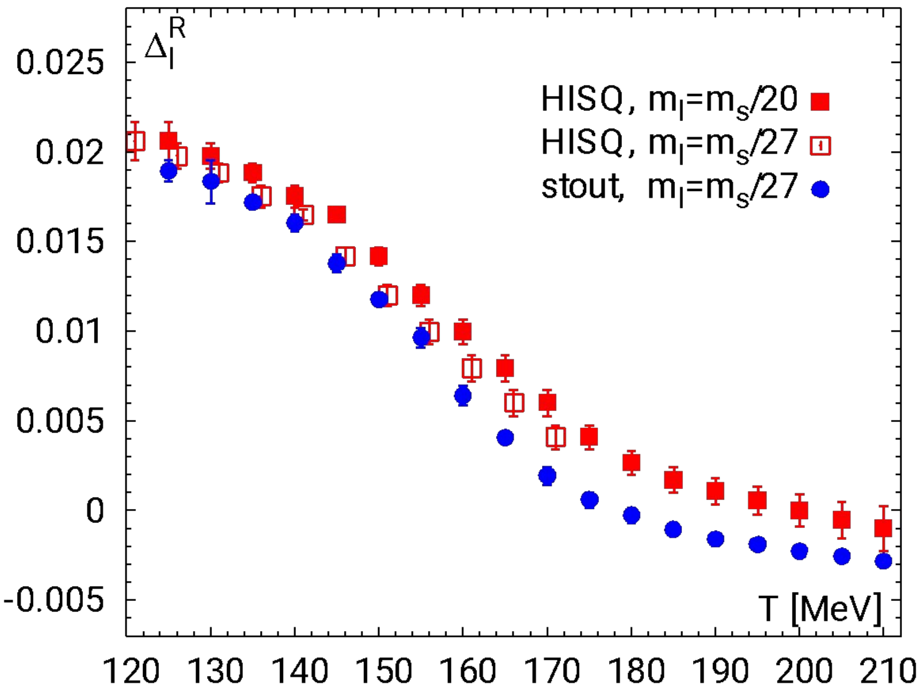
Hard probes in non-equilibrium QCD matter, ICTS, Bangalore, March 16-27,2026

The chiral transition at non-zero temperature

Renormalized chiral condensate

$$\langle\bar{\psi}\psi\rangle \Rightarrow \Delta_l^R(T) =$$
$$= m_s r_1^4(\langle\bar{\psi}\psi\rangle_T - \langle\bar{\psi}\psi\rangle_{T=0}) + d,$$
$$d = m_s r_1^4 \langle\bar{\psi}\psi\rangle_{T=0}^{m_q=0}, \quad r_1 = 0.3106\text{fm}$$

HotQCD, PRD85 (2012) 054503;
Bazavov, PP, PRD 87(2013)094505,
Borsányi et al, JHEP 1009 (2010) 073

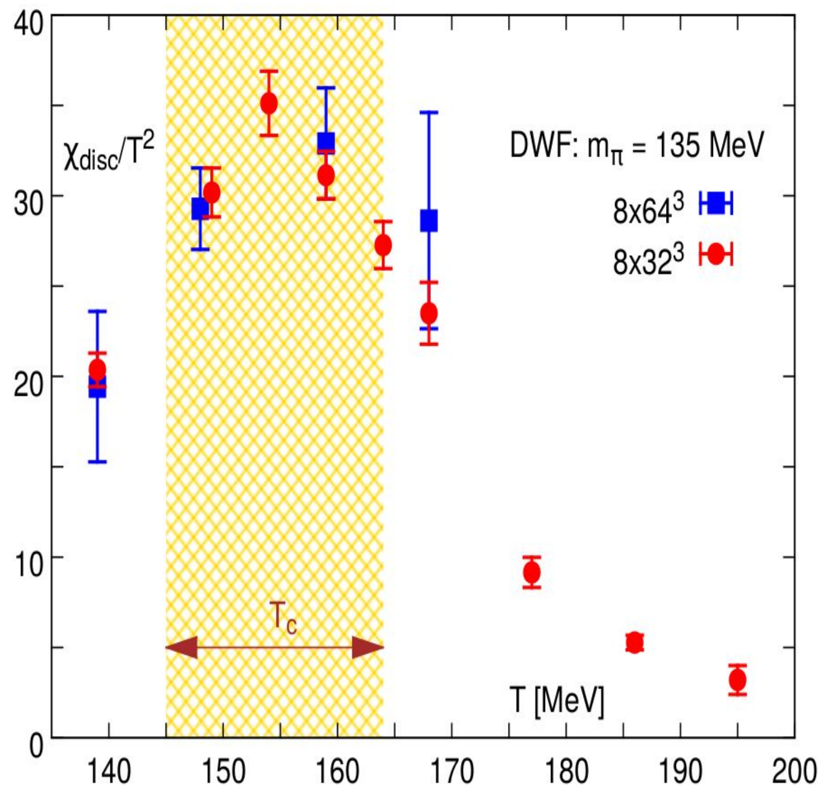


$T_c = (154 \pm 8 \pm 1(scale))$MeV

Fluctuations of the order parameter:

$$\chi_{disc} = VT^{-1}\left(\langle(\bar{\psi}\psi)^2\rangle - \langle\bar{\psi}\psi\rangle^2\right)$$

HotQCD, PRL 113 (2014)082001



$T_c = (155 \pm 8 \pm 1)$MeV

No increase with the volume

$\Rightarrow$ Crossover transition

O(4) scaling and the chiral transition temperature

$$SU_A(2) \times SU_V(2) \sim O(4)$$

For sufficiently small m_l and in the vicinity of the transition temperature:

$$f(T, m_l) = -\frac{T}{V} \ln Z = f_{reg}(T, m_l) + f_s(t, h),\ t = \frac{1}{t_0}\left(\frac{T - T_c^0}{T_c^0} + \kappa\frac{\mu_q^2}{T^2}\right),\ H = \frac{m_l}{m_s}, h = \frac{H}{h_0}$$

governed by universal *O(4)* scaling $M = -\frac{\partial f_s(t,h)}{\partial H} = h^{1/\delta} f_G(z),\ \ z = t/h^{1/\beta\delta}$

$$\langle q\bar{q}\rangle = T(\partial \ln Z)/\partial m_f$$

$T_c^{\,0}$ is critical temperature in the mass-less limit, h_0 and t_0 are scale parameters

Pseudo-critical temperatures for non-zero quark mass are defined as peaks in the response functions (susceptibilities) :

$$\chi_{m,l} = \frac{T}{V}\frac{\partial^2 \ln Z}{\partial m_l^2} \sim m_l^{1/\delta - 1} \qquad \chi_{t,l} = \frac{T}{V}\frac{\partial^2 \ln Z}{\partial m_l \partial t} \sim m_l^{\frac{\beta-1}{\beta\delta}} \qquad \chi_{t,t} = \frac{T}{V}\frac{\partial^2 \ln Z}{\partial t^2} \sim |t|^{-\alpha}$$

$$T_{m,l} \quad = \quad T_{t,l} \quad = \quad T_{t,t} \quad = \quad \boldsymbol{T_c^{\,0}}$$

in the zero quark mass limit

$$\frac{\chi_{l,m}}{T^2} = \frac{T^2}{m_s^2}\left(\frac{1}{h_0} h^{1/\delta - 1} f_\chi(z) + reg.\right)$$

universal scaling function has a peak at $z=z_p$

$$T_c(H) = T_{m,l} = T_c^0 + T_c^0 \frac{z_p}{z_0} H^{1/(\beta\delta)} + \ldots$$

Caveat : staggered fermions *O(2)*

$m_l \to 0,\ a > 0,$

proper limit $a \to 0$, before $m_l \to 0$

The chiral cross-over temperature for physical masses

Chiral order parameter:

$$\Sigma = \frac{1}{f_K^4}\left[m_s\langle \bar{u}u + \bar{d}d\rangle - (m_u + m_d)\langle \bar{s}s\rangle\right] \qquad \langle q\bar{q}\rangle = T(\partial \ln Z)/\partial m_f$$

and the corresponding susceptibilities:

$$\chi^{\Sigma} = m_s\left(\frac{\partial}{\partial m_u} + \frac{\partial}{\partial m_d}\right)\Sigma \qquad \chi = \frac{m_s^2}{f_K^4}\left[\langle(\bar{u}u + \bar{d}d)^2\rangle - (\langle \bar{u}u\rangle + \langle \bar{d}d\rangle)^2\right]$$

For non-zero chemical potential we use Taylor expansion

$$\Sigma(T,\mu_X) = \sum_{n=0}^{\infty}\frac{C_{2n}^{\Sigma}(T)}{(2n)!}\left(\frac{\mu_X}{T}\right)^{2n} \quad \chi(T,\mu_X) = \sum_{n=0}^{\infty}\frac{C_{2n}^{\chi}(T)}{(2n)!}\left(\frac{\mu_X}{T}\right)^{2n} \qquad C_0^{\Sigma} = \Sigma, \quad C_0^{\chi} = \chi$$

Derivatives in μ_X^2 are similar to derivatives in T $\quad$ $e.g.\ \partial_T C_0^{\chi} \sim C_2^{\chi}$

$\Rightarrow$ the following quantities will peak at T_c

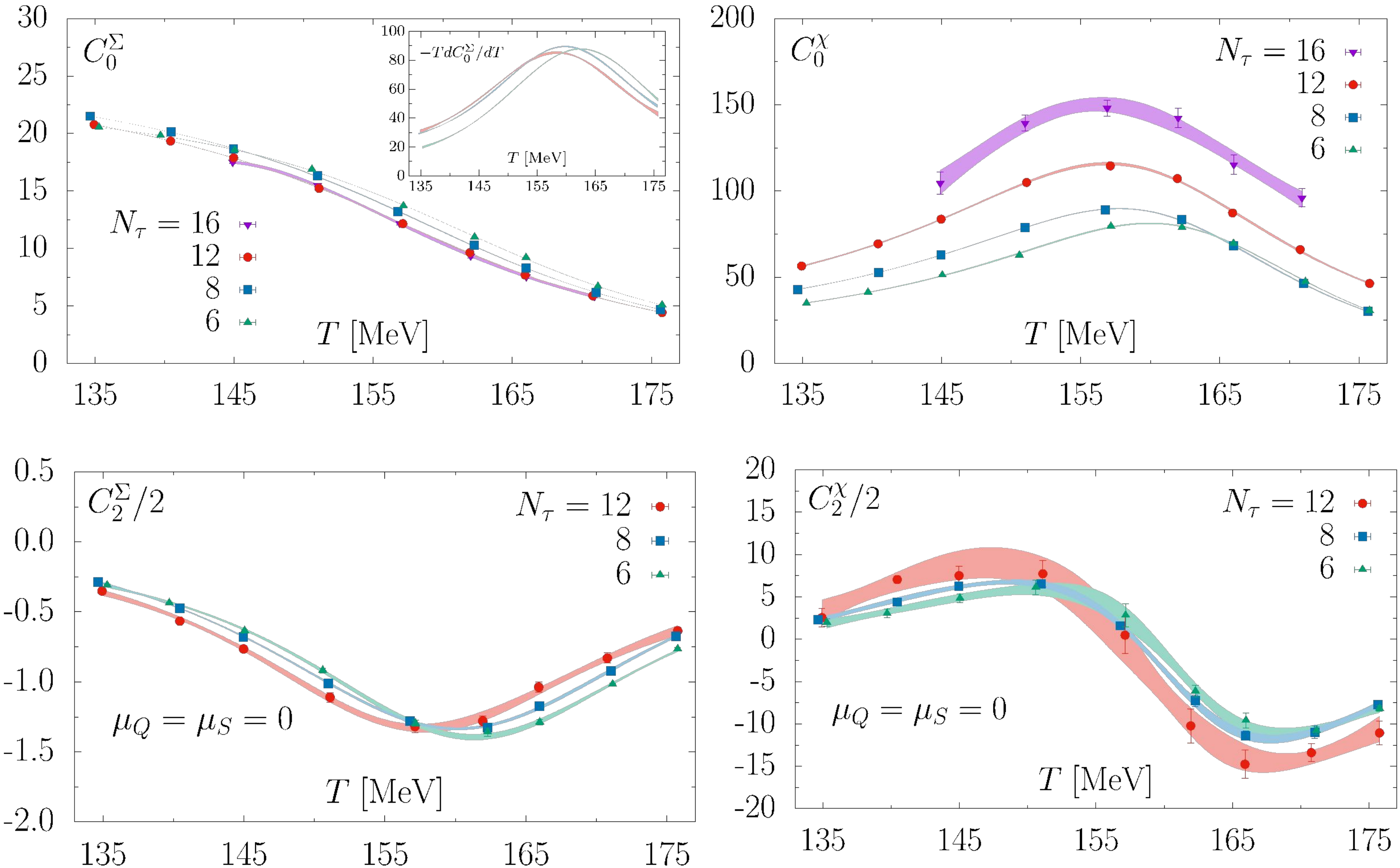
$\chi^{\Sigma},\ C_0^{\chi}(T) \sim \chi_{l,m} \qquad \partial_T C_0^{\Sigma},\ C_2^{\Sigma}(T) \sim \chi_{t,m}$ $\qquad$ HotQCD, PLB795 (2019) 15

5 different definitions of T^{pc}:

$$\partial_T C_0^{\chi} = 0,\ \partial_T \chi^{\Sigma} = 0\ ,C_2^{\chi} = 0 \qquad \partial_T^2 C_0^{\Sigma} = 0,\ \partial_T C_2^{\Sigma} = 0$$

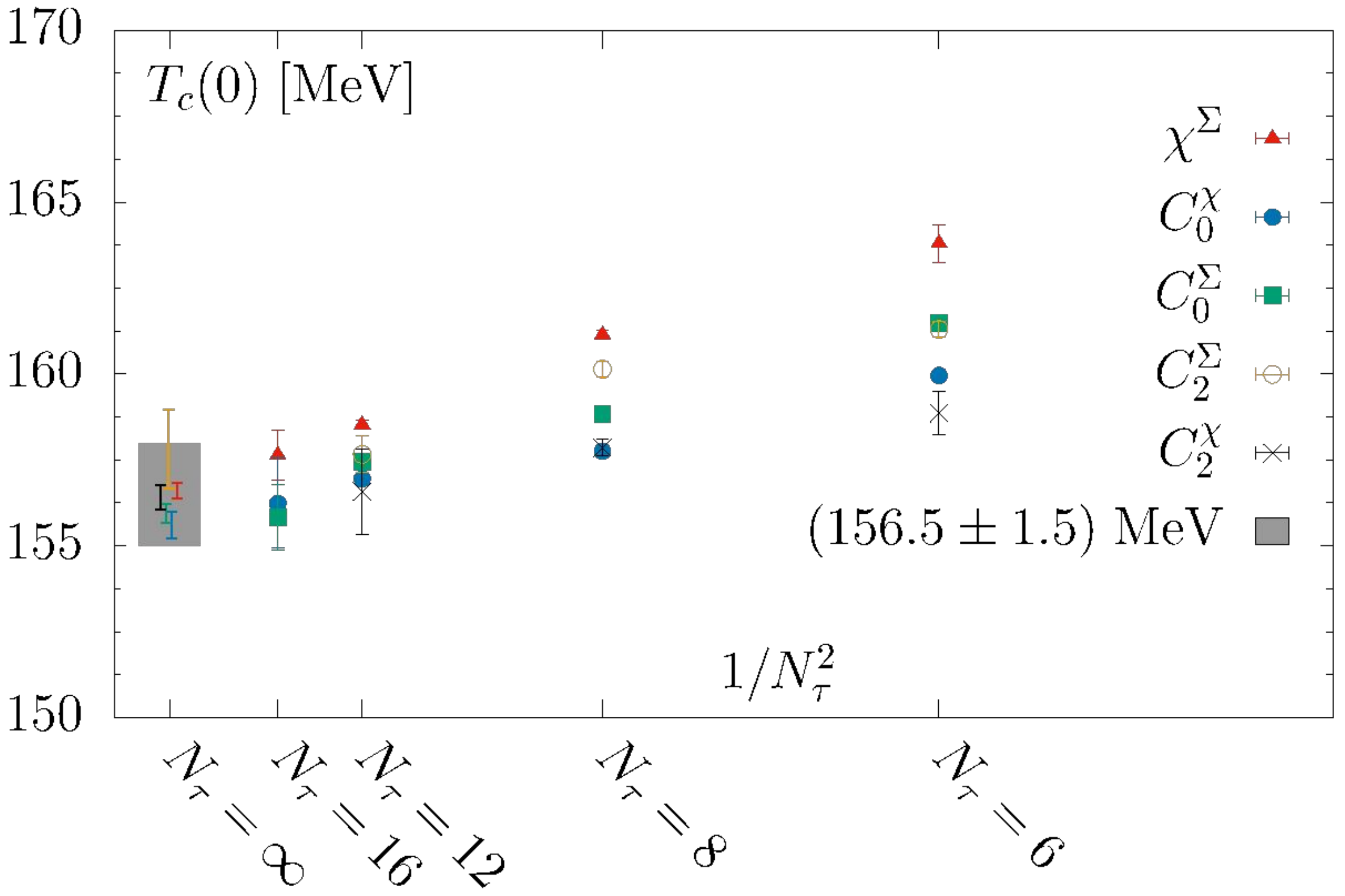
The 5 different T_c values reduce to $T_{l,m}$ and $T_{l,t}$ if regular part is zero

Lattice calculations based on 100K - 500 K configurations, $N_\tau = 6 - 12$, and 4K configurations for $N_\tau = 16$



HotQCD, PLB795 (2019) 15

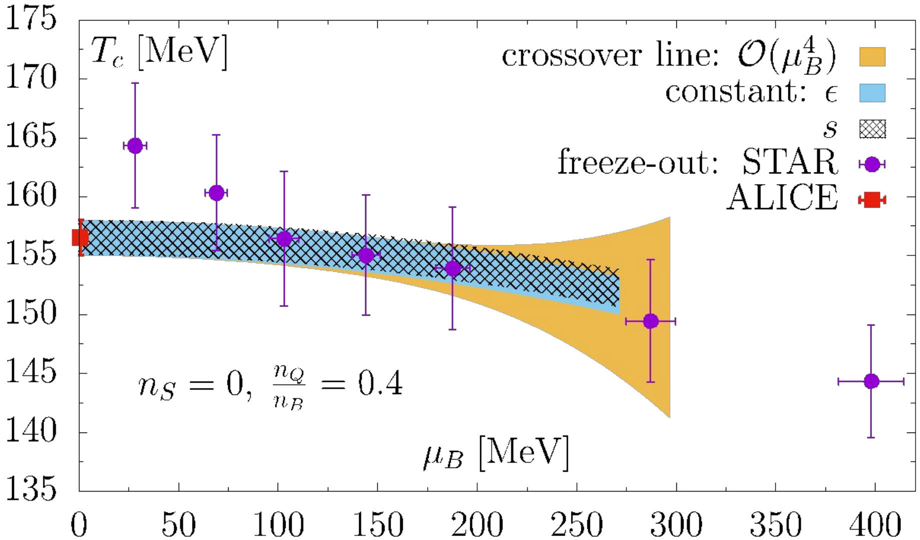
Different definitions of T_c surprisingly agree in the continuum limit and we for zero chemical potential we get $T_c = 156 \pm 1.5$ MeV



HotQCD, PLB795 (2019) 15

The chiral cross-over temperature at non-zero density

$$T_c(\mu_B) = T_c(0)\left[1 - \kappa_2^B\left(\frac{\mu_B}{T_c(0)}\right)^2 - \kappa_4^B\left(\frac{\mu_B}{T_c(0)}\right)^4\right]$$

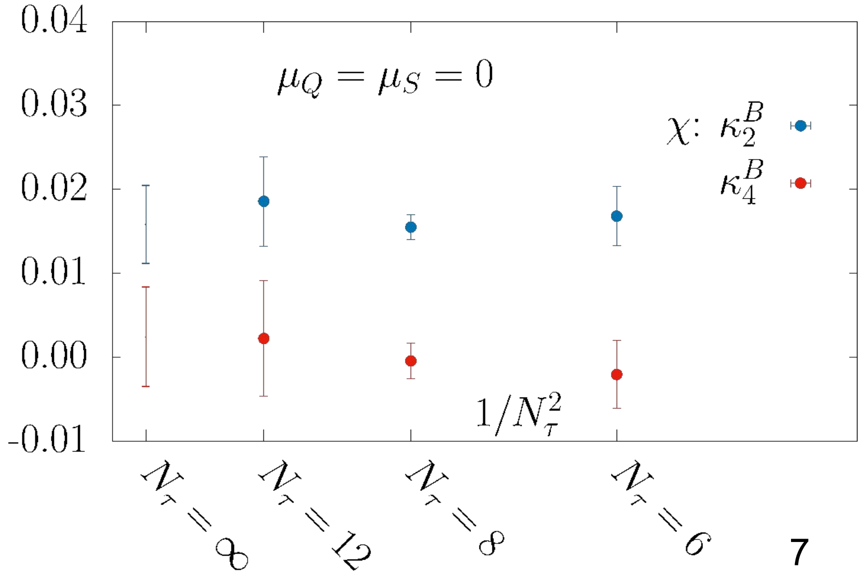


The freeze-out condition corresponding to constant energy density or constant entropy density agrees with the crossover line within errors

The μ_B dependence of T_c is small

$$\kappa_2^{B,\chi} \simeq \kappa_2^{B,\Sigma}$$

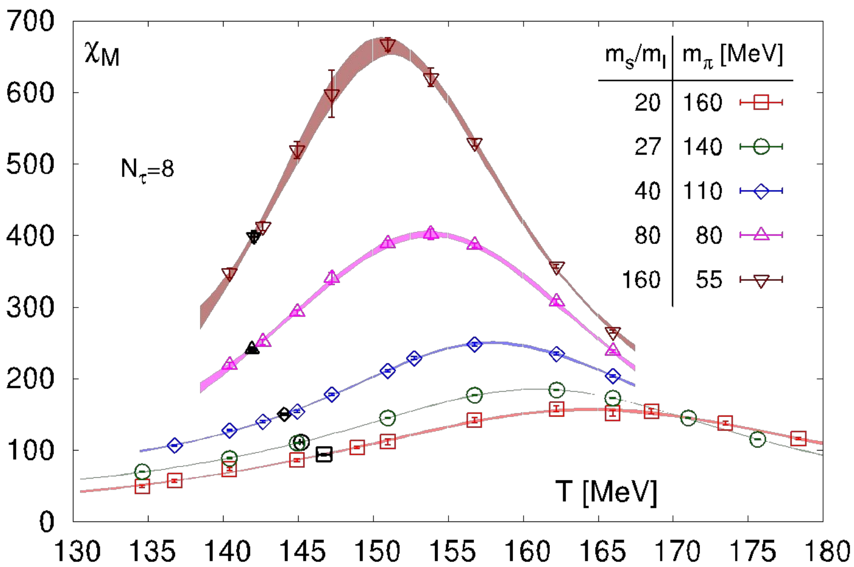
HotQCD, PLB795 (2019) 15

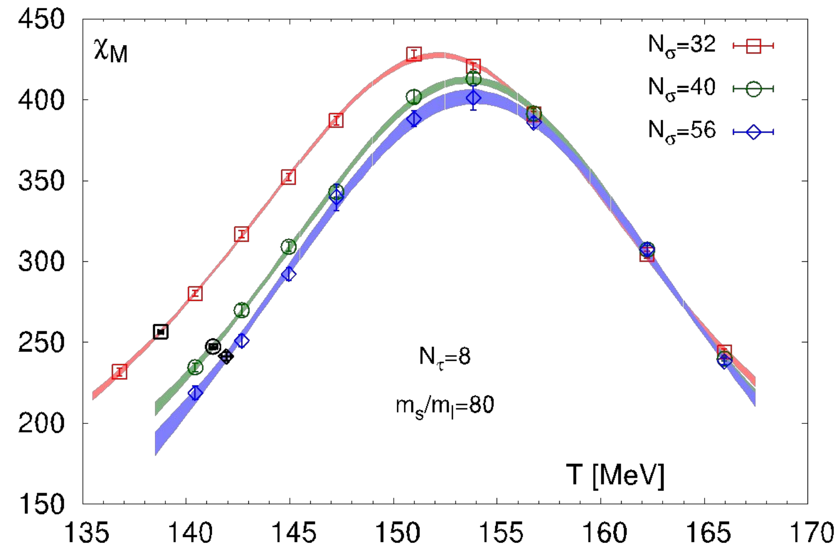


Chiral phase transition in 2+1 flavor QCD

What is the nature of the chiral transition in 2+1 flavor QCD for fixed m_s and $m_l \to 0$?

HotQCD, PRL 123 (2019) 062002



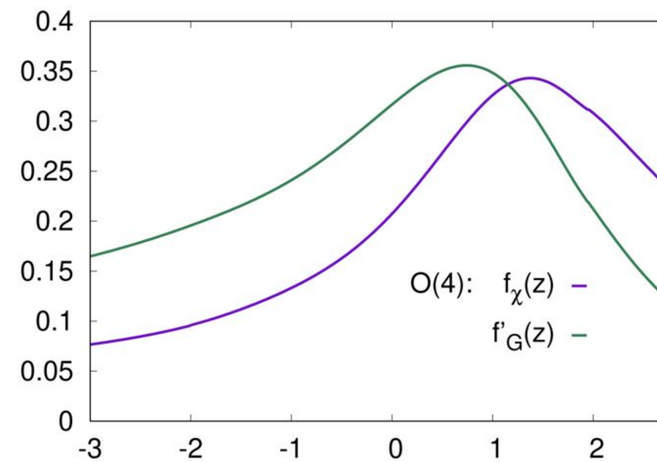


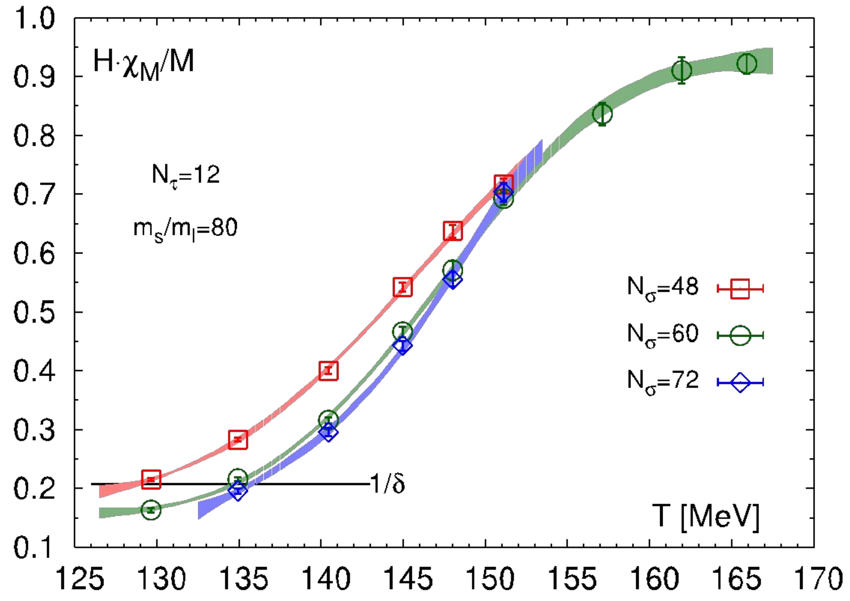
$$M = h^{1/\delta} f_G(z, z_L) + f_{sub}(T, H, L)$$

$$\chi_M = h_0^{-1} h^{1/\delta - 1} f_\chi(z, z_L) + \tilde{f}_{sub}(T, H, L)$$

$$T_p(H, L) = T_c^0 \left(1 + \frac{z_p(z_L)}{z_0} H^{1/\beta\delta}\right) + \text{sub leading}$$

$$T_X(H, L) = T_c^0 \left(1 + \left(\frac{z_X(z_L)}{z_0}\right) H^{1/\beta\delta}\right)$$



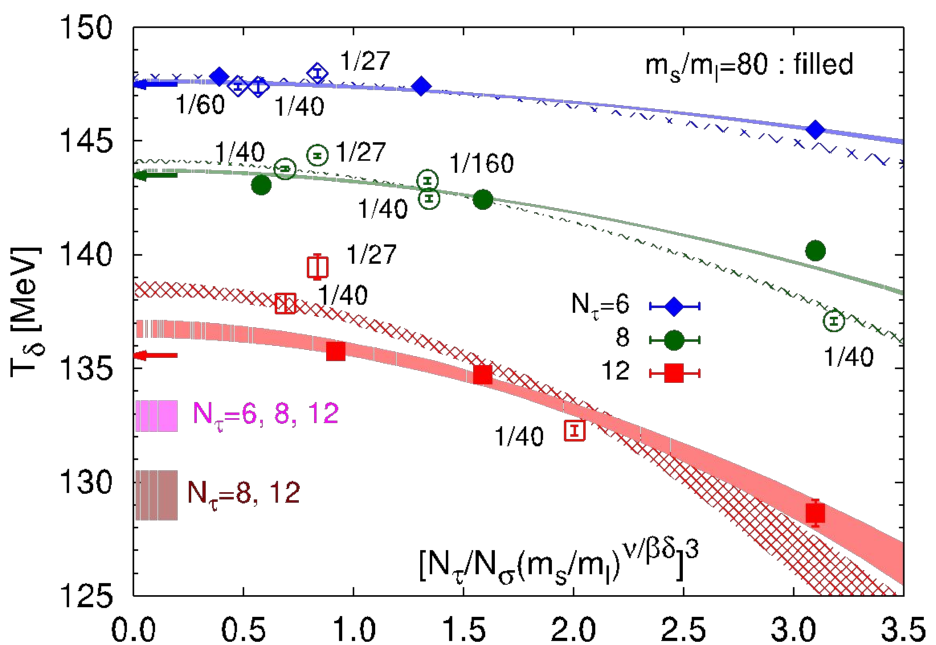


$$\frac{H\chi_M(T_\delta, H, L)}{M(T_\delta, H, L)} = \frac{1}{\delta} ;$$

$$\chi_M(T_{60}, H) = 0.6\chi_M^{max} .$$

$$T_X(H, L) = T_c^0 \left(1 + \left(\frac{z_X(z_L)}{z_0}\right) H^{1/\beta\delta}\right) + c_X H^{1-1/\delta+1/\beta\delta} , \quad X = \delta,\ 60$$

$$z_{60} \simeq z_\delta \simeq 0$$



Use $O(4)$ fits for m_l and volume dependence

HotQCD, PRL 123 (2019) 062002

Continuum extrapolations:

$$T_c^0 = 132^{+3}_{-6} \text{ MeV}$$

$$T^{CEP} < 132 \text{ MeV}$$

Relativistic Virial Expansion and Hadron Resonance Gas

Chiral perturbation theory is limited to pion gas. Other hadrons, resonances ?
Relativistic virial expansion : compute thermodynamic quantities in terms
as a gas of non-interacting particles and *S* – matrix
Dashen, Ma, Bernstein, Phys. Rev. 187 (1969) 345

Free gas of stables hadros: *π, K, N*

interactions

$$\ln Z = \ln Z_0 + \sum_{i_1,i_2} e^{\mu_{i_1}/T} e^{\mu_{i_2}/T} b(i_1,i_2)$$

$$b(i_1,i_2) = \frac{V}{4\pi i}\int \frac{d^3p}{(2\pi)^3}\int dE\, e^{-(p^2+E^2)^{1/2}/T}\sum_{final}\left[AS(S^{-1}\frac{\partial S}{\partial E} - \frac{\partial S^{-1}}{\partial E}S)\right]$$

(anti) symmetrization (spin-statistics)

Elastic scattering only (final state = initial state)

$$S(E) = \sum_{l,I}{}'(2l+1)(2I+1)\exp(2i\delta_l^I(E))$$

Partial wave decomposition

perform the integral over the 3-momentum

$$b_2 = \frac{T}{2\pi^3}\int_M^\infty dE\, E^2 K_2(E/T)\sum{}'_{l,I}(2l+1)(2I+1)\frac{\partial\delta_l^I(E)}{\partial E}$$

of the pair at threshold invariant mass

Use experimental phase shifts to determine b_2 , Venugopalan, Prakash, NPA546 (1992) 718

After summing all the channels only resonance contributions survives in

$$\sum_{l,I}{}^{'}(2l+1)(2I+1)\frac{\partial \delta_l^I(E)}{\partial E}$$

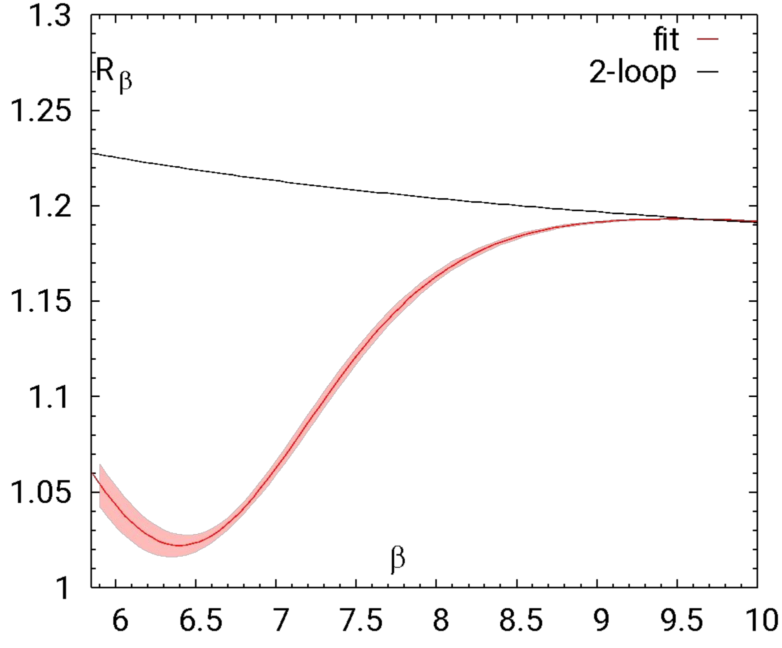
Interacting hadron gas = non-nteracting gas of hadrons and resonances

QCD trace anomaly and the integral method

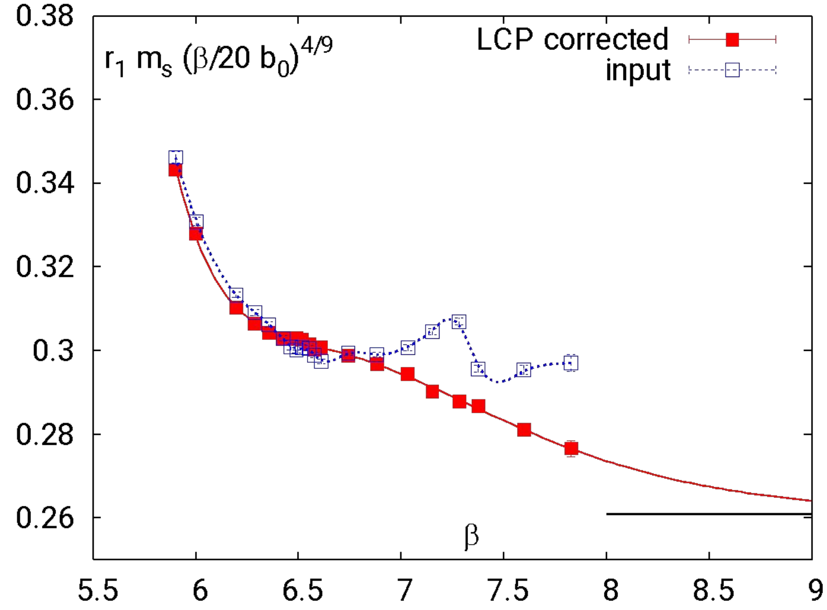
$$\frac{\Theta^{\mu\mu}(T)}{T^4} = \frac{\varepsilon - 3p}{T^4} = T\frac{d}{dT}\left(\frac{p}{T^4}\right) \Rightarrow \frac{p(T)}{T^4} - \frac{p(T_0)}{T_0^4} = \int_{T_0}^{T} dT' \frac{\Theta^{\mu\mu}(T')}{T'^5},$$

$$\frac{\Theta^{\mu\mu}(T)}{T^4} = \frac{\epsilon - 3p}{T^4} = R_\beta \{\langle S_G\rangle_0 - \langle S_G\rangle_T\} - R_\beta R_m \{2m_l(\langle\bar{q}q\rangle_0 - \langle\bar{q}q\rangle) + m_s(\langle\bar{s}s\rangle_0 - \langle\bar{s}s\rangle_T)\}$$

$$R_\beta(\beta) = -a\frac{d\beta}{da},\ R_m = \frac{1}{m_q(\beta)}\frac{dm_q(\beta)}{d\beta},\ \beta = 10/g^2$$



Bazavov, PP, Weber, PRD97 (2018) 014510

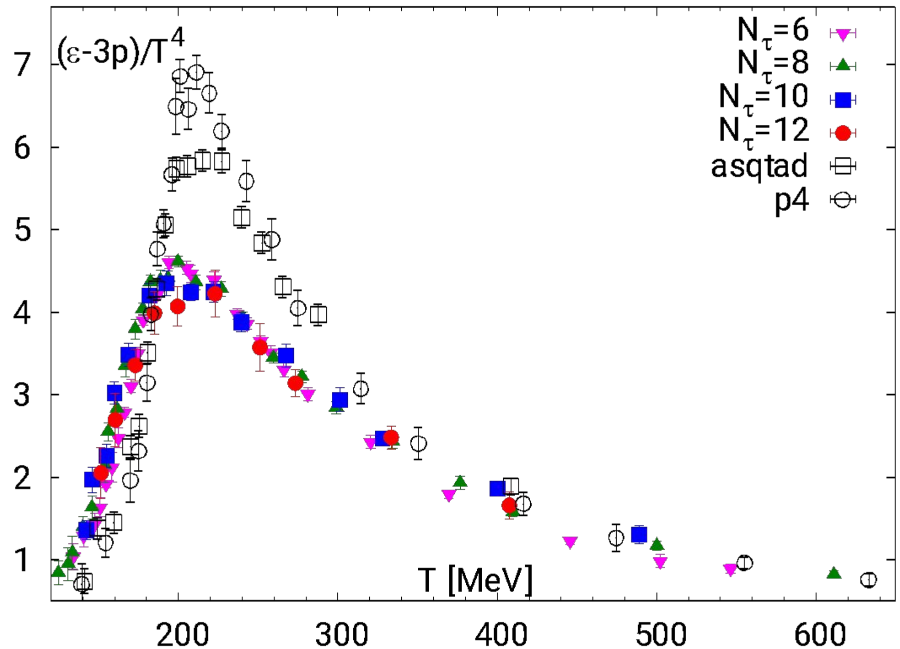


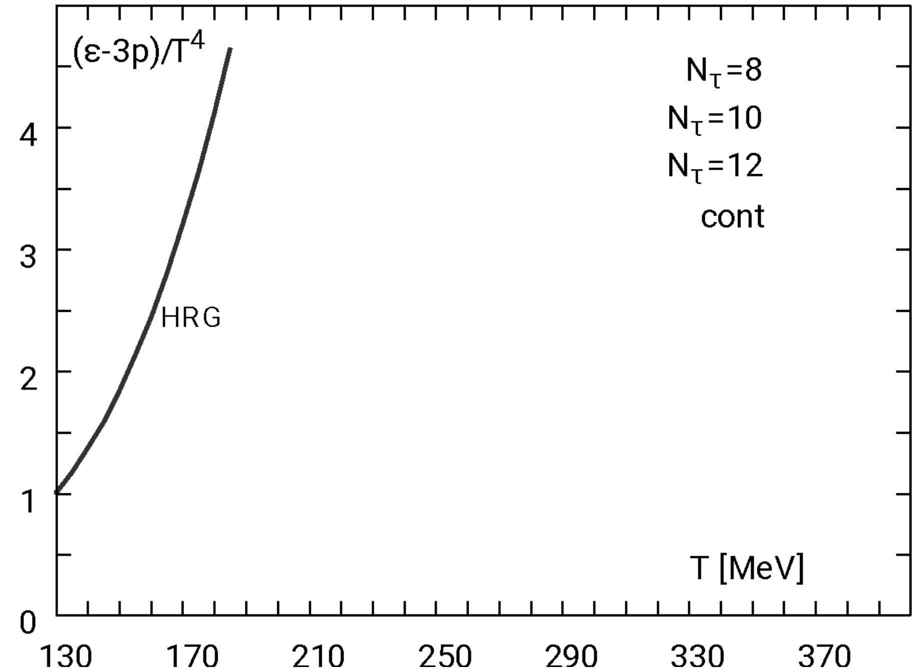
HotQCD, PRD 90 (2014) 094503

QCD results on the trace anomaly

2+1 flavor QCD calculations with almost physical light and strange quark masses

HotQCD, PRD 90 (2014) 094503





The peak height is much reduced compared to the asqtad and p4 $N_\tau=8$ calculations

Agreement with p4 and asqtad calculations for $T>350$ MeV

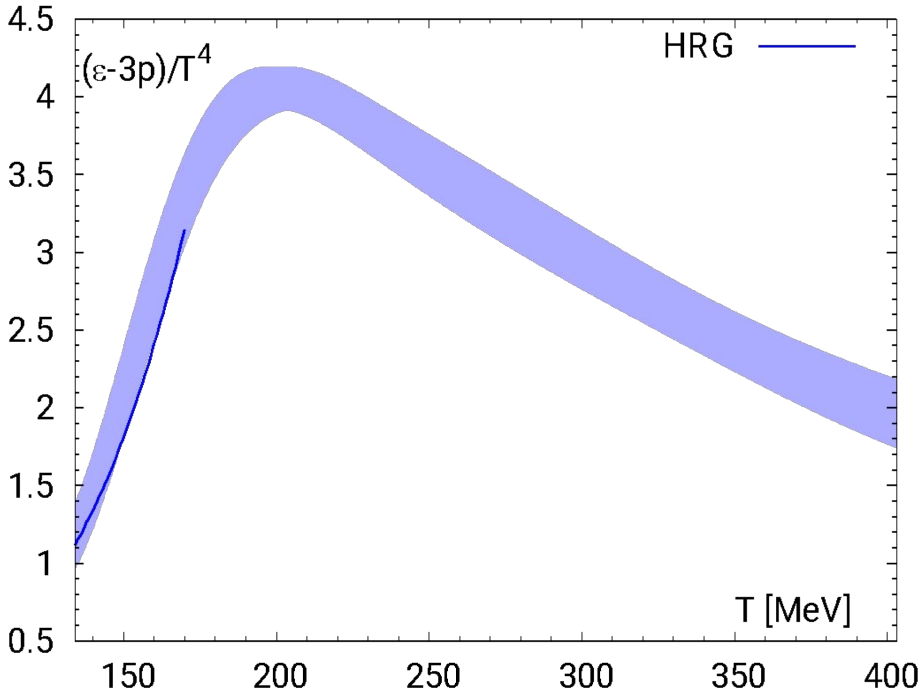
Small cutoff effects for HISQ except for $N_\tau=6$

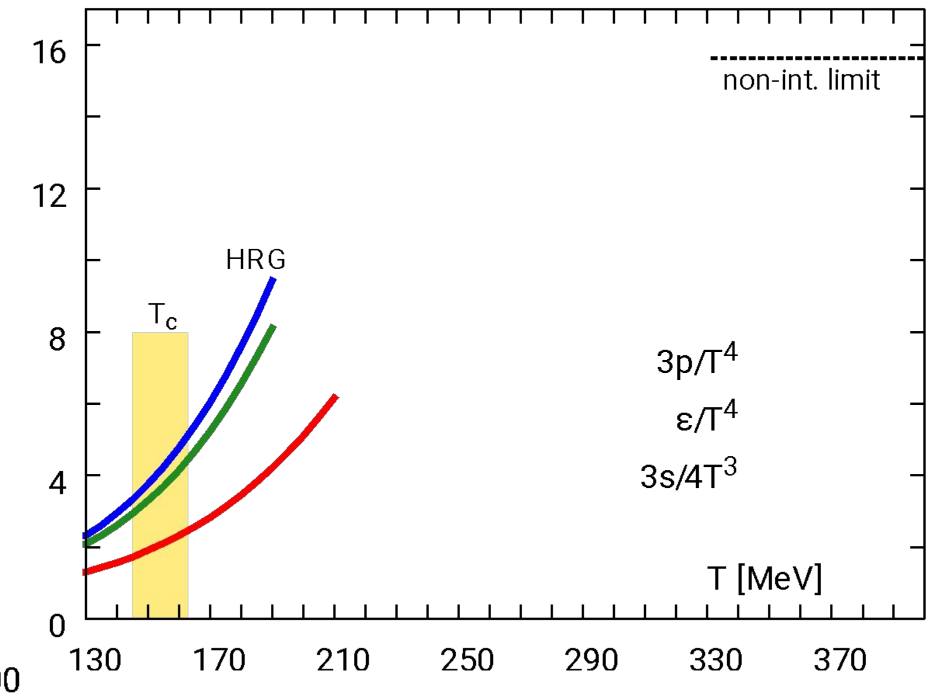
Perform spline interpolation of all the $N_\tau>6$ data with spline coefficients having $a+b/N_\tau^{\,2}$ *form* ,
stabilize the spline demanding that ε-$3p$ is given by HRG at $T=130$ MeV

QCD thermodynamics in the continuum limit

Set the lower integration limit to $T_0=130$ MeV and take $p_0=p^{HRG}(T=130\ \text{MeV})$ ➡ $p(T)$

HotQCD, PRD 90 (2014) 094503



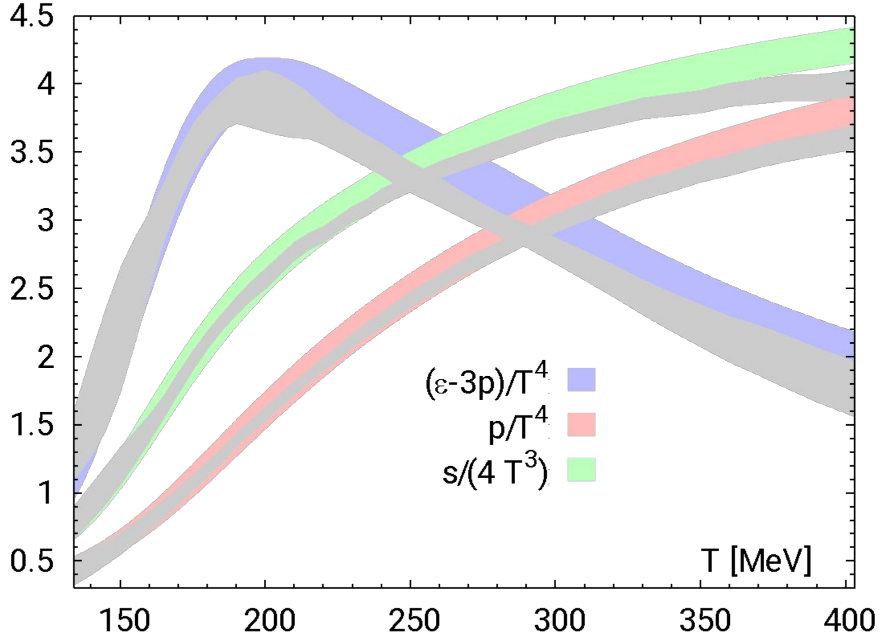


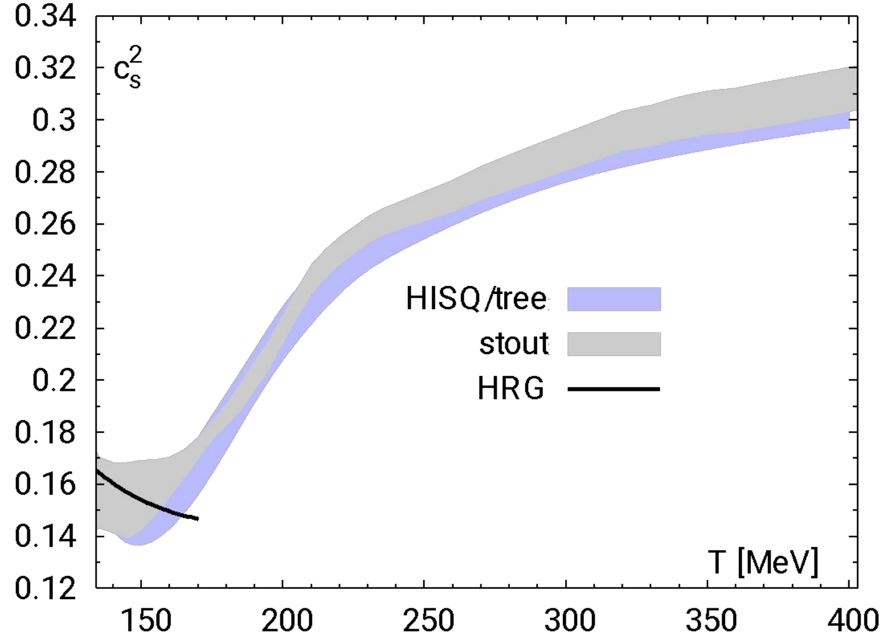
$$T_c = 156 \pm 1.5\ \text{MeV} \qquad \epsilon_{nucl} \simeq 150\text{MeV}/\text{fm}^3$$

$$\epsilon_c = 420(60)\ \text{MeV}/\text{fm}^3 \qquad \epsilon_{proton} \simeq 450\text{MeV}/\text{fm}^3$$

HRG: all resonances from PDG treated as stable (zero width) particles in an ideal gas

Comparison of different continuum limit



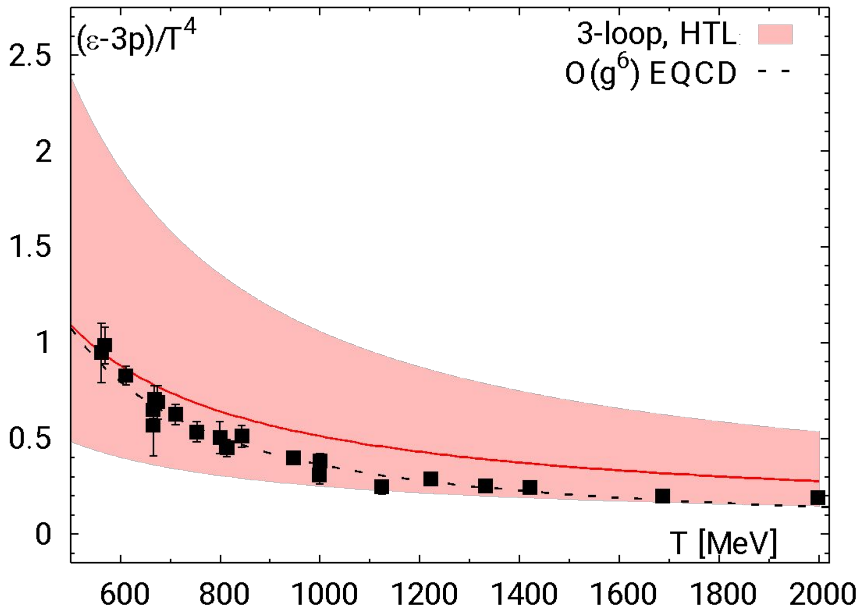


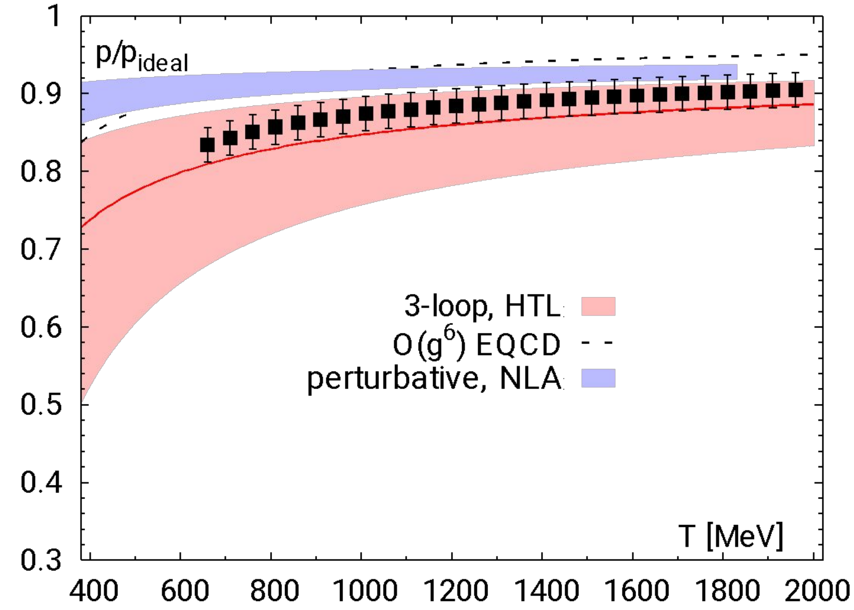
Continuum results obtained with stout and HISQ action agree reasonably well given their errors (some tension for the entropy density)

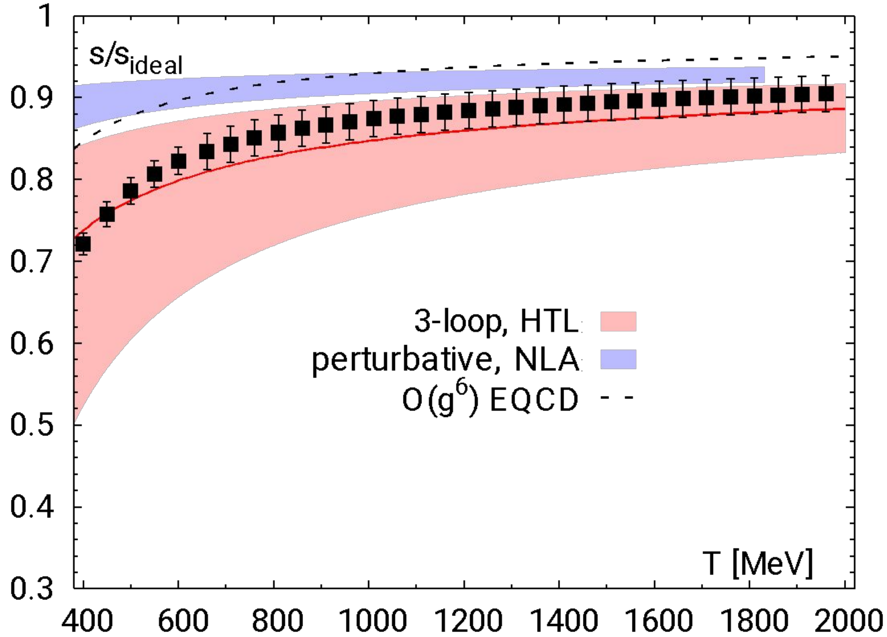
Even in the transition region the speed of sound is not much smaller than the HRG speed of sound (the EoS is never really soft)

HISQ: Bazavov et al, PRD 90 (2014) 094503
stout: Borsányi et al, PLB730 (2014) 99

Comparison of EoS with weak coupling results







Reasonably good agreement
Between the lattice and the weak
coupling calculations
for *T>400* MeV

Bazavov, PP, Weber, PRD97 (2018) 014510

QCD thermodynamics at non-zero chemical potential

Taylor expansion :

$$\frac{p(T,\mu_B,\mu_Q,\mu_S,\mu_C)}{T^4} = \sum_{ijkl} \frac{1}{i!j!k!l!} \chi^{BQSC}_{ijkl} \left(\frac{\mu_B}{T}\right)^i \left(\frac{\mu_Q}{T}\right)^j \left(\frac{\mu_S}{T}\right)^k \left(\frac{\mu_C}{T}\right)^l$$ hadronic

$$\frac{p(T,\mu_u,\mu_d,\mu_s,\mu_c)}{T^4} = \sum_{ijkl} \frac{1}{i!j!k!l!} \chi^{udsc}_{ijkl} \left(\frac{\mu_u}{T}\right)^i \left(\frac{\mu_d}{T}\right)^j \left(\frac{\mu_s}{T}\right)^k \left(\frac{\mu_c}{T}\right)^l$$ quark

$$\chi^{abcd}_{ijkl} = T^{i+j+k+l} \frac{\partial^i}{\partial \mu_b^i} \frac{\partial^j}{\partial \mu_b^j} \frac{\partial^k}{\partial \mu_c^i} \frac{\partial^l}{\partial \mu_d^l} \ln Z(T,V,\mu_a,\mu_b,\mu_c,\mu_d) \Big|_{\mu_a=\mu_b=\mu_c=\mu_d=0}$$

Taylor expansion coefficients give the susceptibilities, i.e. the fluctuations and correlations of conserved charges, e.g.

$$\chi^X_2 = \chi_X = \frac{1}{VT^3}\left(\langle X^2\rangle - \langle X\rangle^2\right) \qquad \chi^{XY}_{11} = \frac{1}{VT^3}\left(\langle XY\rangle - \langle X\rangle\langle Y\rangle\right)$$

information about carriers of the conserved charges (hadrons or quarks)

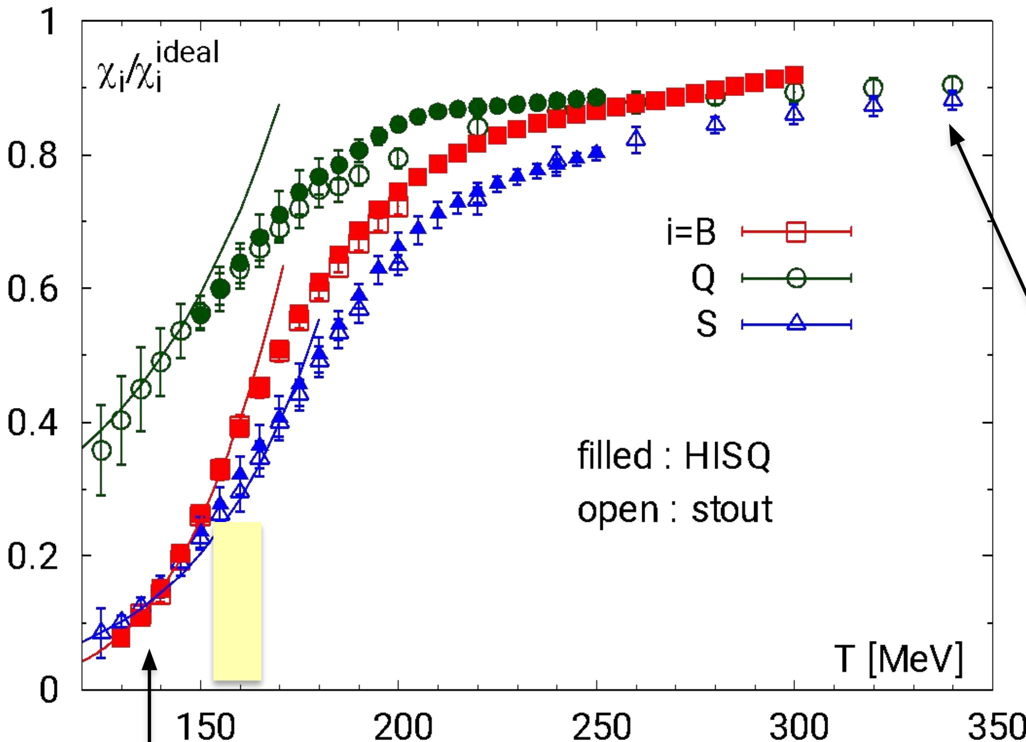
probes of deconfinement

Deconfinement : Fluctuations of Conserved Charges

$$\chi_B = \frac{1}{VT^3}\left(\langle B^2\rangle - \langle B\rangle^2\right)$$ baryon number $$\chi_B^{\mathrm{ideal}} = \frac{1}{3}$$

$$\chi_Q = \frac{1}{VT^3}\left(\langle Q^2\rangle - \langle Q\rangle^2\right)$$ electric charge $$\chi_Q^{\mathrm{ideal}} = \frac{2}{3}$$

$$\chi_S = \frac{1}{VT^3}\left(\langle S^2\rangle - \langle S\rangle^2\right)$$ strangeness $$\chi_S^{\mathrm{ideal}} = 1$$



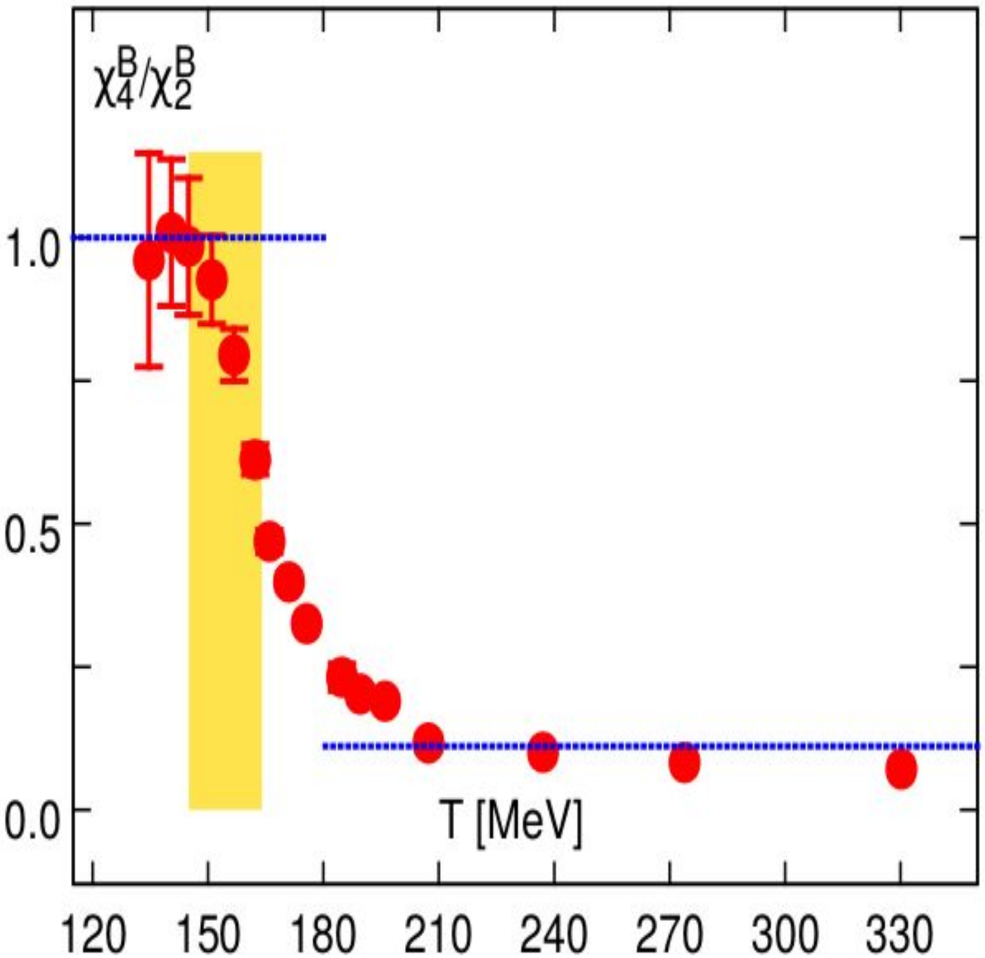
HISQ: Bazavov et al (HotQCD),
PRD86 (2012) 034509
PRD 95 (2017) 054504
stout: Borsányi et al. JHEP 1201 (2012) 138

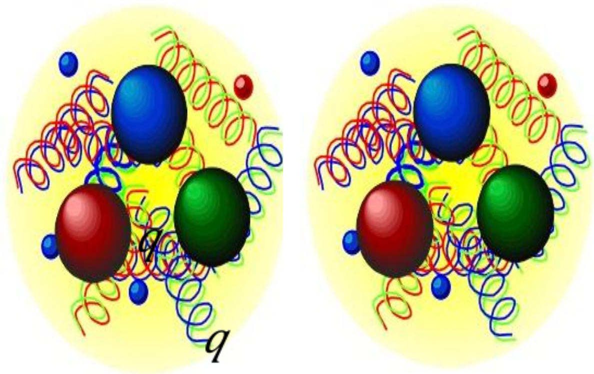
HRG: includes missing states and repulsivemean field,

conserved charges carried by light quarks

conserved charges are carried by massive hadrons

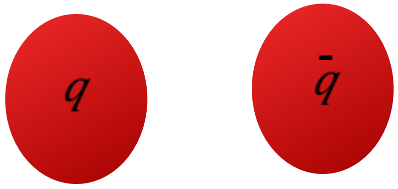
Deconfinement : fluctuations of conserved charges





B=1, -1

$$\chi_2^B = \langle B^2\rangle = 1,\ \chi_4^B = \langle B^4\rangle = 1$$



B=1/3, -1/3

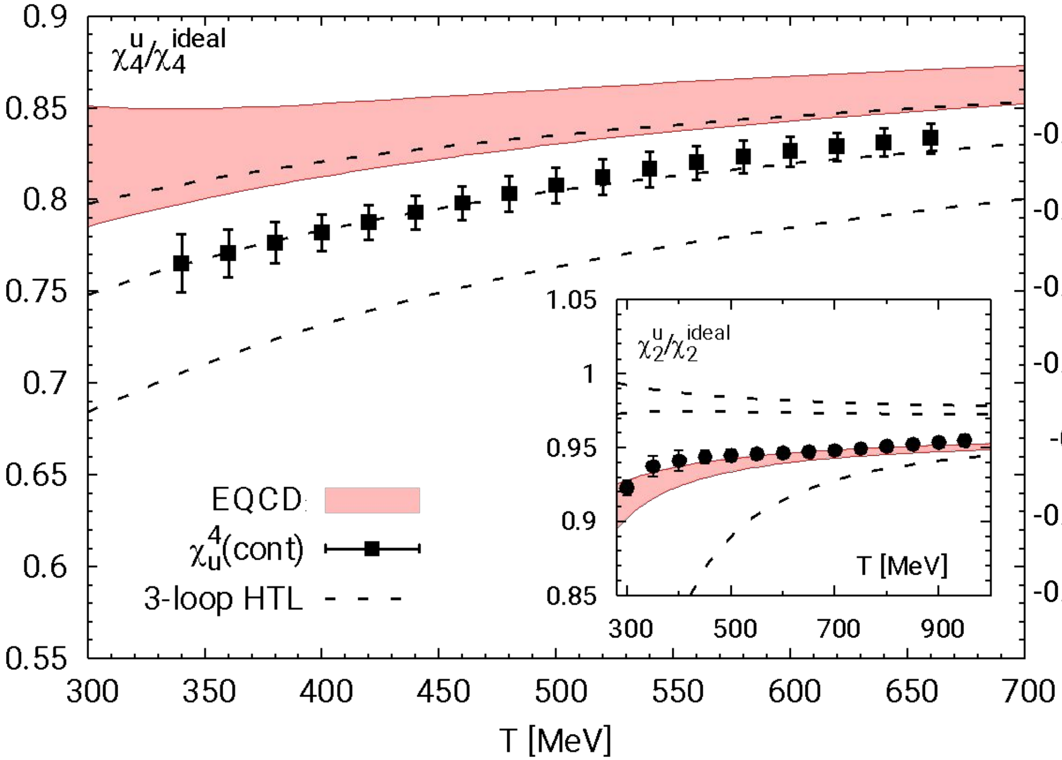
$$\chi_2^B = \langle B^2\rangle = \frac{1}{9},\ \chi_4^B = \langle B^4\rangle = \frac{1}{81}$$

Degrees of freedom for 150 MeV < *T*< 200 MeV ?

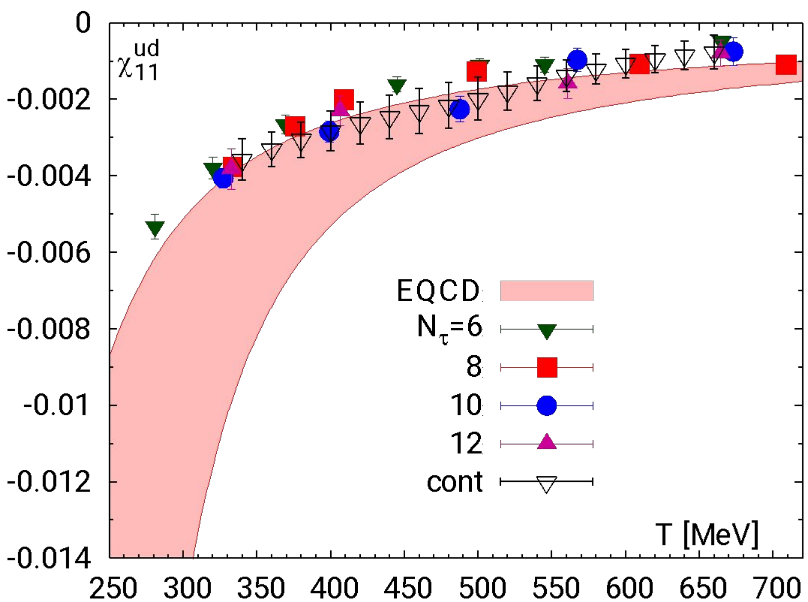
Quark number fluctuations at high T

At high temperatures quark number fluctuations can be described by weak coupling approach due to asymptotic freedom of QCD

quark number fluctuations



quark number correlations



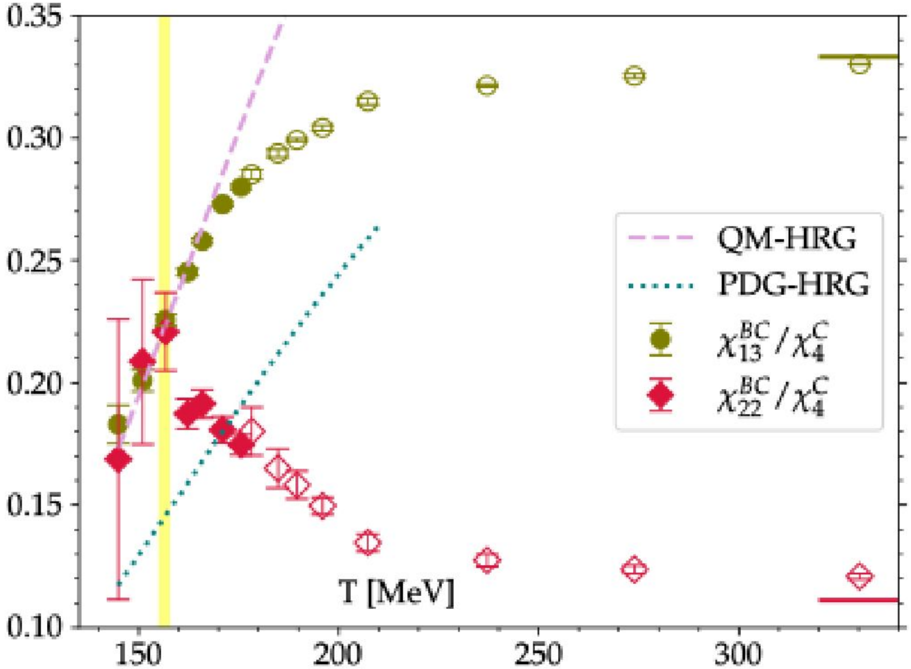
- Good agreement between continuum extrapolated lattice results and the weak coupling approach
- Quark number correlations vanish at any loop order but can be calculated in EQCD and the EQCD calculations agree with the continuum extrapolated lattice results

Bazavov et al, PRD88 (2013) 094021, Ding et at, PRD92 (2015) 074043

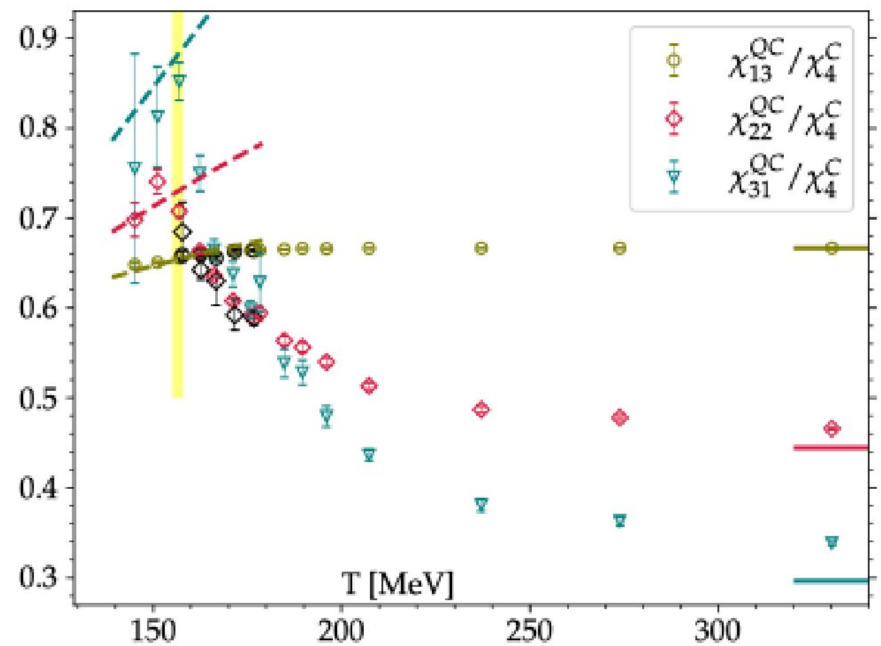
Generalized charm susceptibilities

$$\chi_{nml}^{XYC} = T^{m+n+l} \frac{\partial^{n+m+l} p(T,\mu_X,\mu_Y,\mu_C)/T^4}{\partial\mu_X^n \partial\mu_Y^m \partial\mu_C^l}$$

Bazavov et al, PLB 737 (2014) 210
Bazavov et al, PLB 850 (2024) 138520
Kaczmarek et al, PRD 112 ('25) 034509



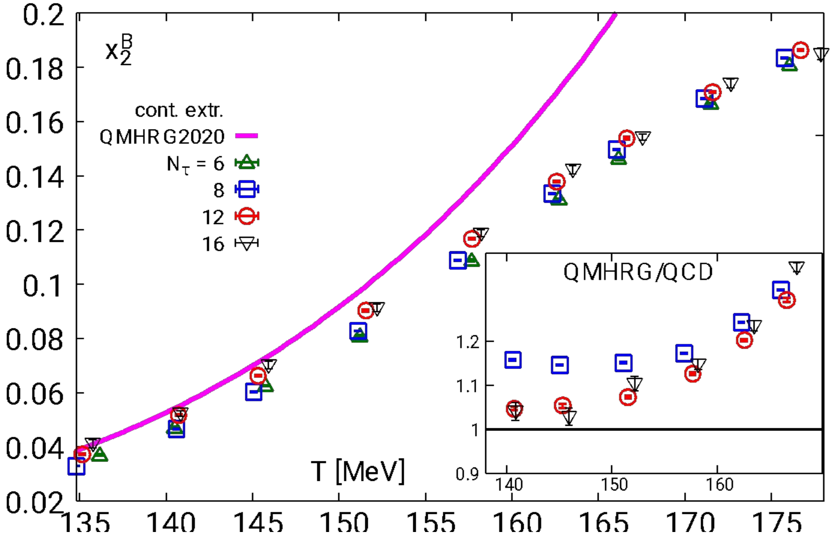
Hadronic description breaks down just above T_c
⇒ open charm deconfines above T_c

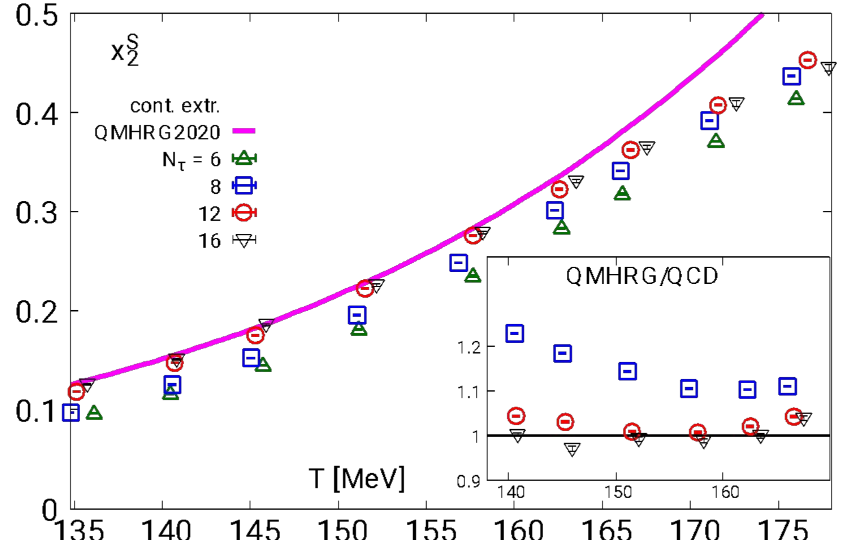


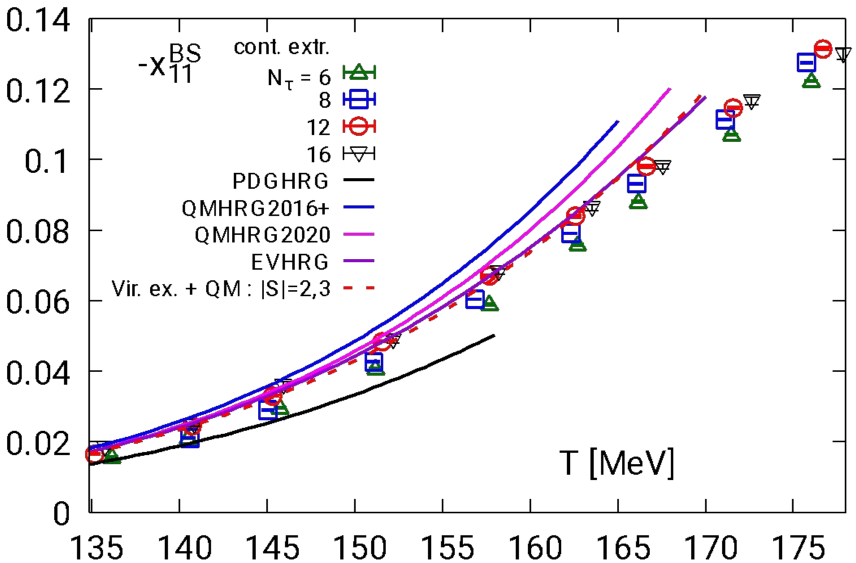
The charm baryon spectrum is not well known (only few states in PDG), HRG works only if the "missing" states are included

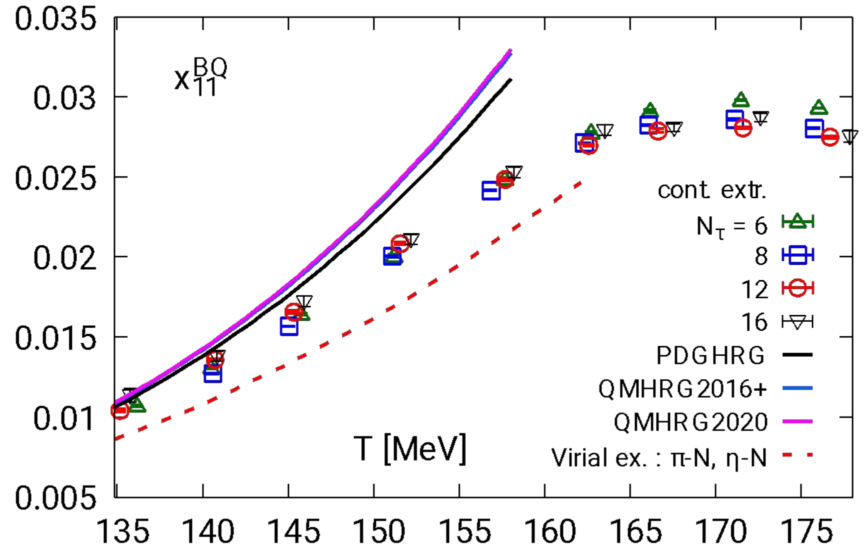
Second order Taylor expansion coefficients and HRG

HISQ, m_π^{phys}, $a = 1/(TN_\tau)$









HRG works up to temperatures ≃145-150 MeV

Charm Hadrons and Quarks across T_c

$M_c >> T$ and Boltzmann approximation holds $\qquad \hat{\mu}_X = \mu_X / T$

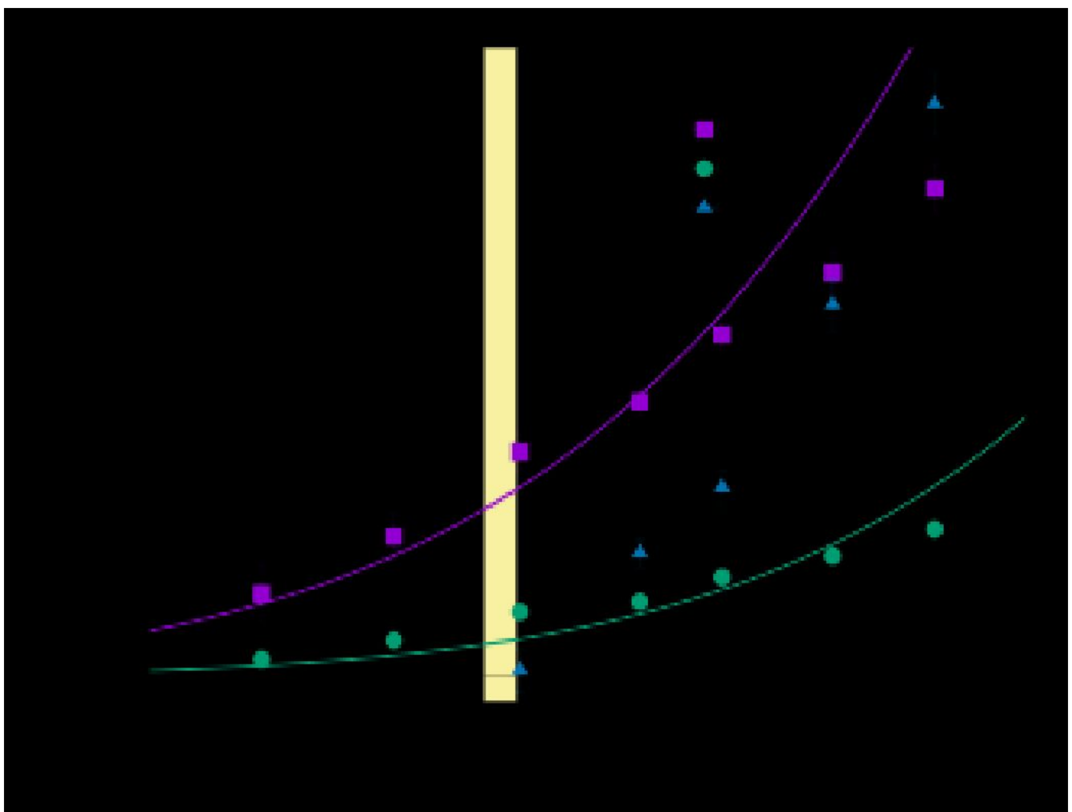
$$p^C(T, \mu_B, \mu_c) = p_q^C(T)\cosh(\hat{\mu}_C + \hat{\mu}_B/3) + p_B^C(T)\cosh(\hat{\mu}_C + \hat{\mu}_B) + p_M^C(T)\cosh(\hat{\mu}_C)$$

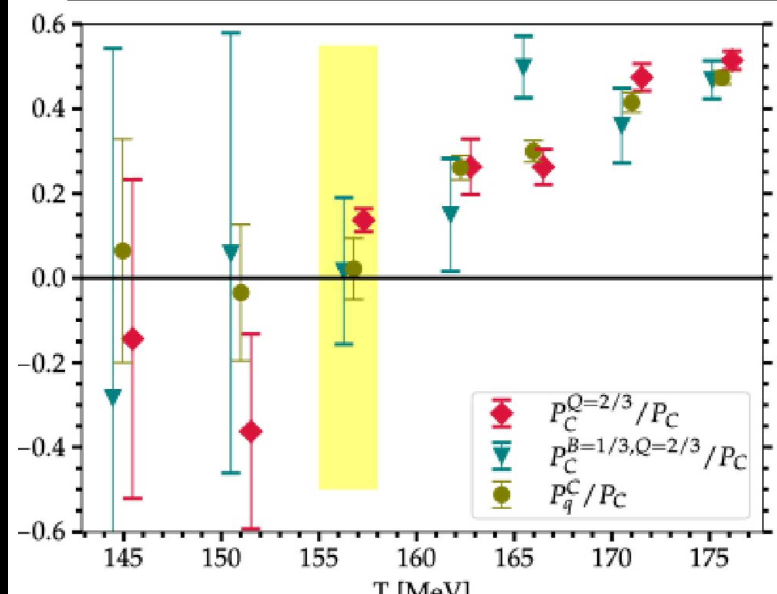
$$\chi_2^C,\ \chi_{13}^{BC},\ \chi_{22}^{BC} \Rightarrow p_q^C(T), p_M^C(T), p_B^C(T)$$

S. Mukherjee, PP, S. Sharma, PRD93 ('16) 014502;
HotQCD, PLB 850 ('24) 138520, PRD 112 ('25) 034509

[redacted]

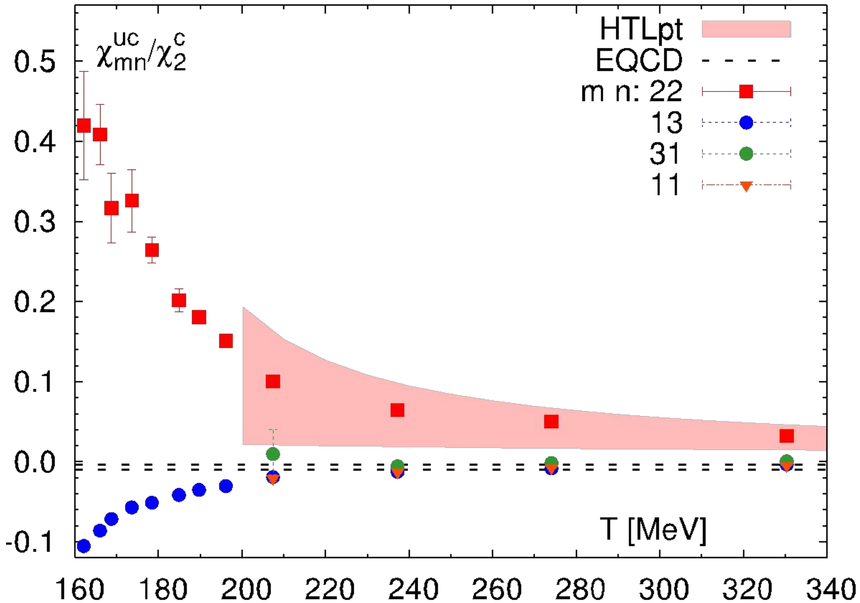
[redacted]



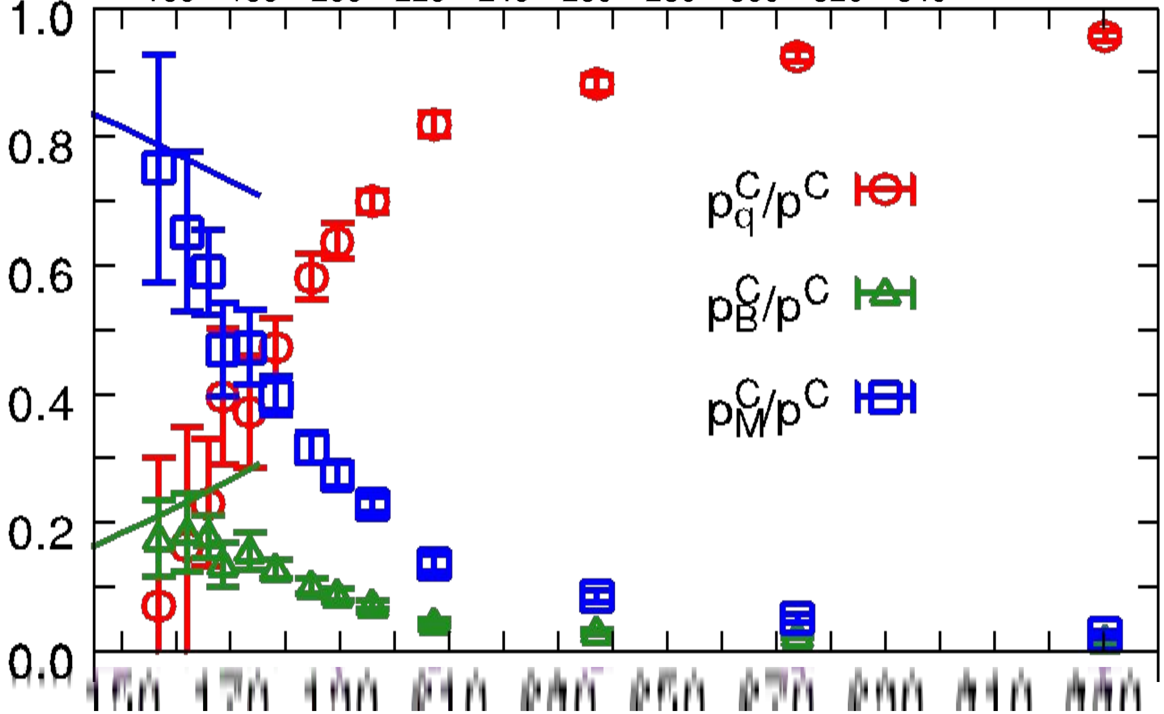


Charm quark appear just above T_c

Quasi-particle model for charm degrees of freedom



Mukherjee, PP, Sharma,
PRD 93 (2016) 014502



Partial pressures
drop because hadronic
excitations become
broad at high temperatures
(bound state peaks merge
with the continuum)

See
Jakovác, PRD88 (2013), 065012
Biró, Jakovác, PRD(2014)065012

Vice versa for quarks

Properties of charm degrees of freedom

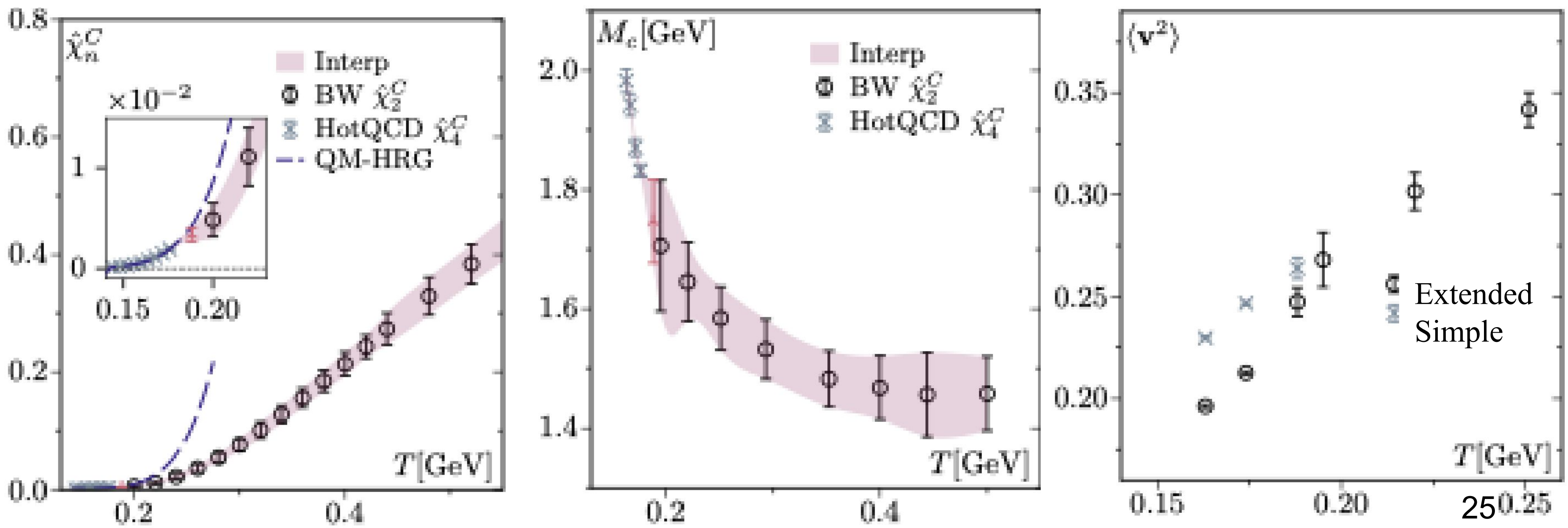
Simple quasi-particle model: [redacted]

Charm
quark
mass

[redacted]

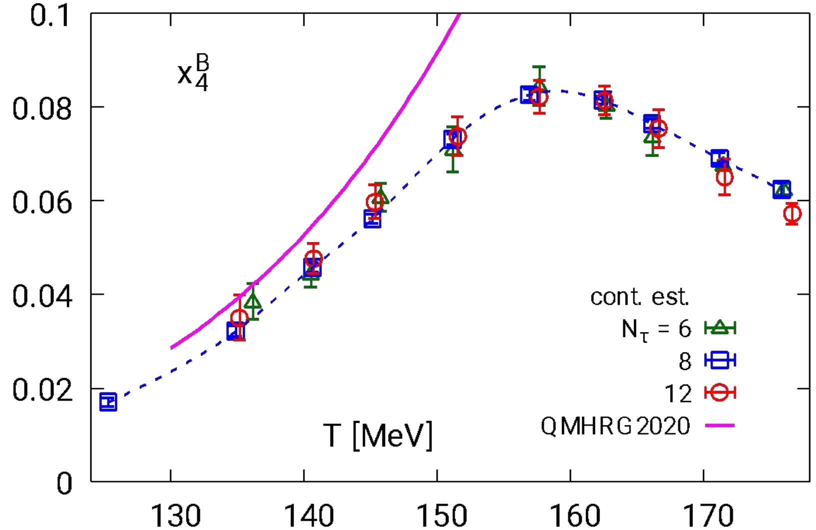
T=0 charmed hadron masses are used plus *T*-dependent charm quark mass

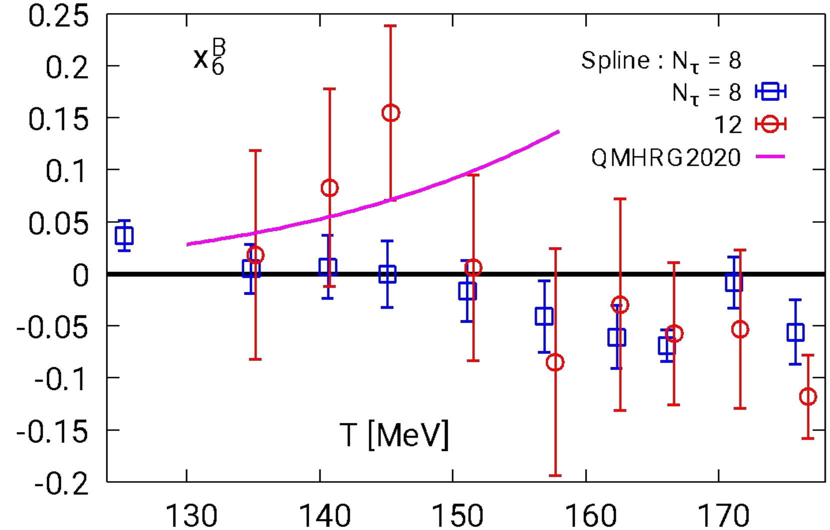
HotQCD, JHEP 09 (2025) 180

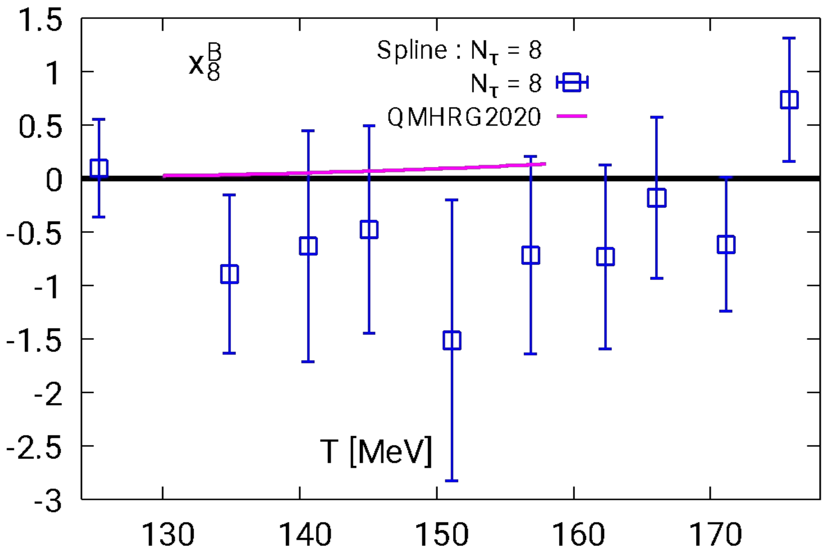


Back-up: Higher order Taylor expansion coefficients and HRG

HISQ, m_π^{phys}, $a = 1/(TN_\tau)$





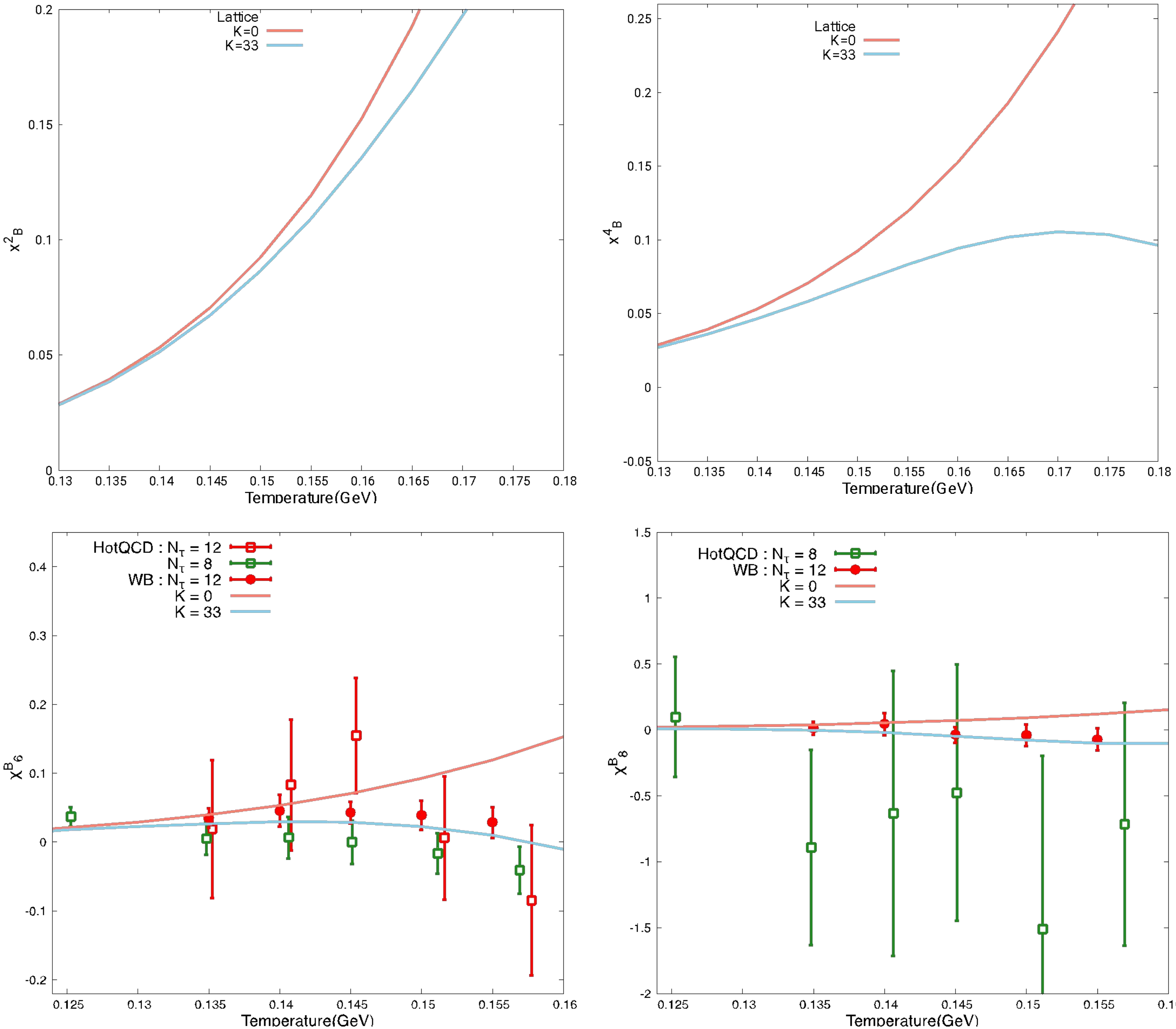


For 4th order expansion coefficient HRG may work only for *T<140* MeV

For 6th and 8th order expansion coefficients turn negative around T_c HRG, only works for *T<135* MeV

Possibly no singularity for real values of baryon chemical potential.

Back-up: HRG with repulsive mean field



D. Biswas, PP,
S. Sharma,
PRC 109 (2024) 055206

Improved agreement between lattice and HRG.

Back-up: QCD phase diagram in extended parameter space

The QCD phase diagram in $T - \mu_B$ plane is related to the nature of the QCD phase transition at zero light quark masses