Lecture 1: General introduction

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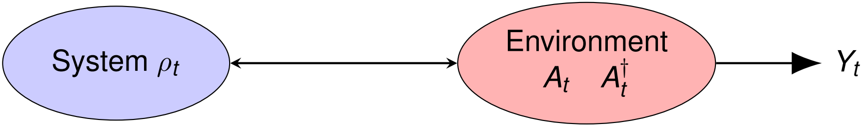
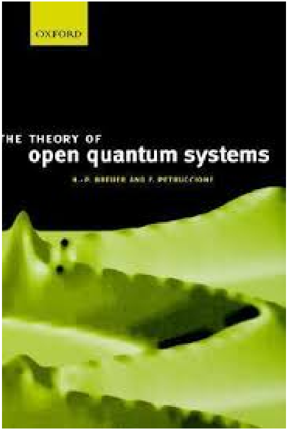
Open quantum systems

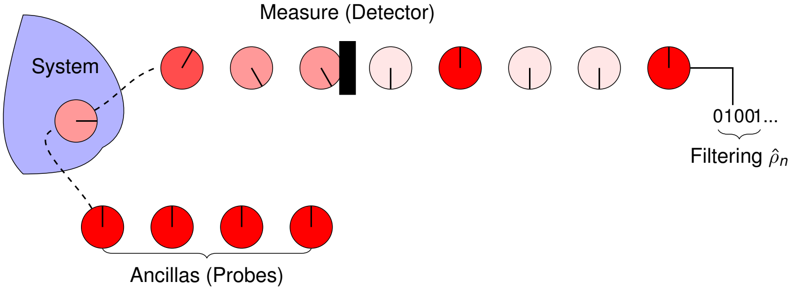
Open quantum systems: a system that interacts with an external quantum system (the environment or a bath)





[Breuer & Petruccione, 2002]

Quantum measurement: indirect measurements



Photon counting and Homodyne detection

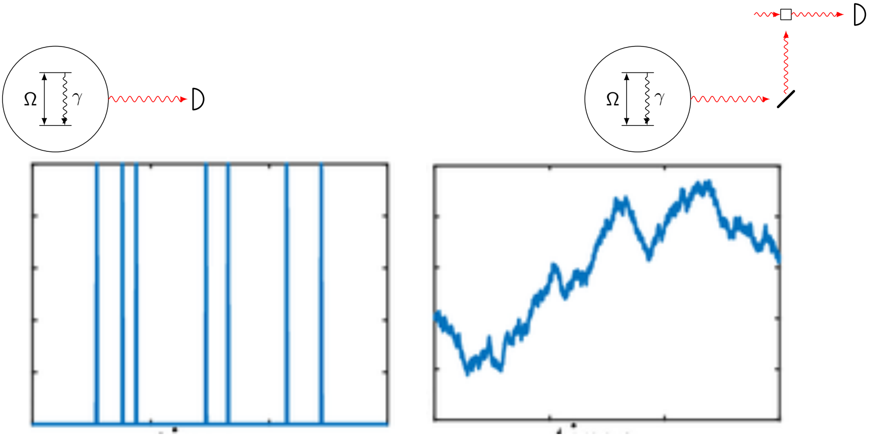


Figure : Two types of detection : Photon counting and Homodyne detection.

Control features

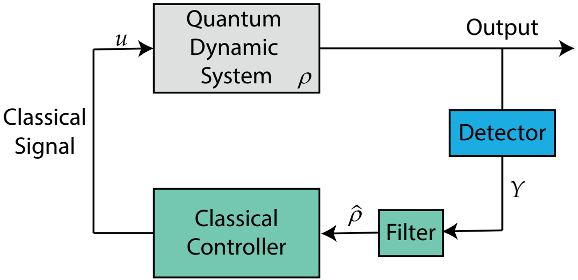
- **Open-loop control:** mostly with Hamiltonian control, implemented by $H = H_0 + \sum_{j=1}^{n} u_j(t) H_j$

  Major goal: Controllability and optimal control

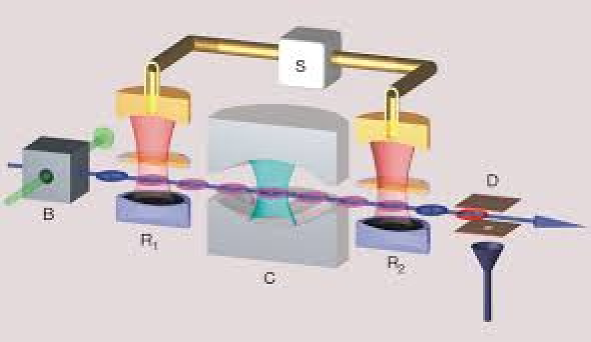
- **Closed-loop control:** measurement-based feedback, coherent feedback

  Major goal: Compensating decoherence induced by the environment

Measurement-based feedback

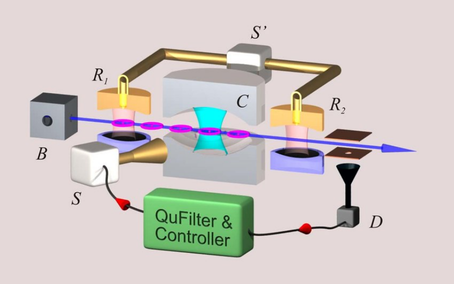


Example: LKB photon box



Here $\rho \to \frac{V_\pm \rho V_\pm^\dagger}{\mathrm{tr}\left(V_\pm \rho V_\pm^\dagger\right)}$ with probability $\mathrm{tr}\left(V_\pm \rho V_\pm^\dagger\right)$.

Feedback stabilizing photon number state

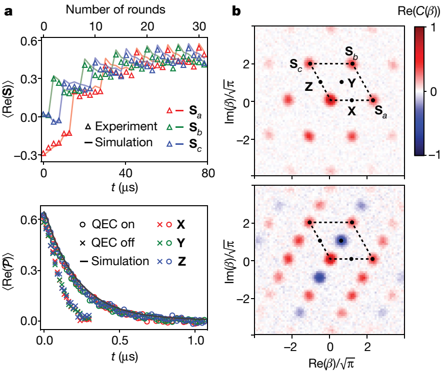


C. Sayrin et al., Nature, 2011

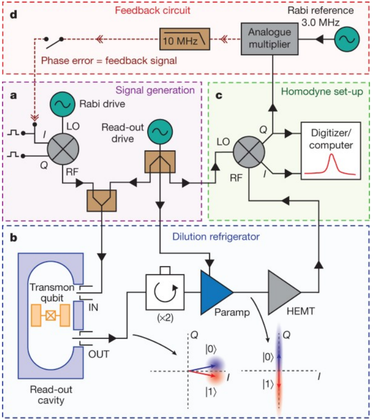
Control input u: application of a unitary $U_u = e^{-\mathrm{i}uH}$:

$$\rho \to U_u \left(\frac{V_\pm \rho V_\pm}{\mathrm{tr}\left(V_\pm \rho V_\pm^\dagger \right)} \right) U_u^\dagger$$

Other examples of measurement-based feedback

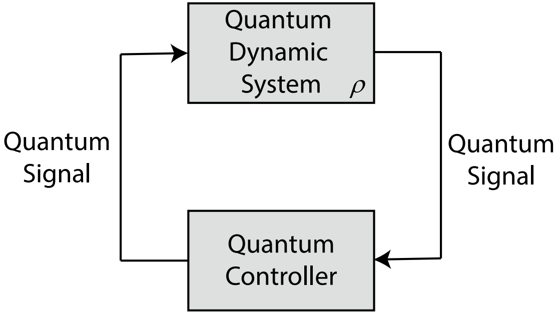


Campagne-Ibarcq et al., Nature, 2020

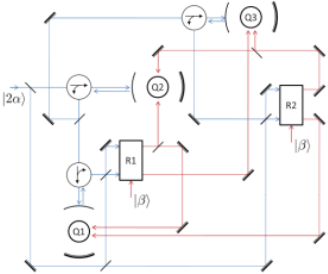


Vijay et al., Nature, 2012

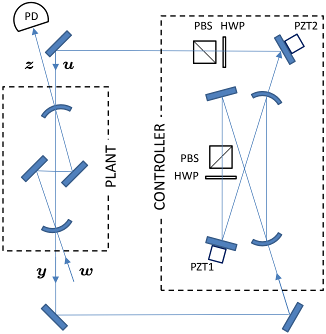
Coherent feedback (without measurement)



Examples of coherent feedback



Photonic QEC network schematic
Kerckhoff et al., PRL, 2010



Coherent feedback with
a dynamic compensator
Mabuchi, PRA, 2008

Some references on quantum trajectories

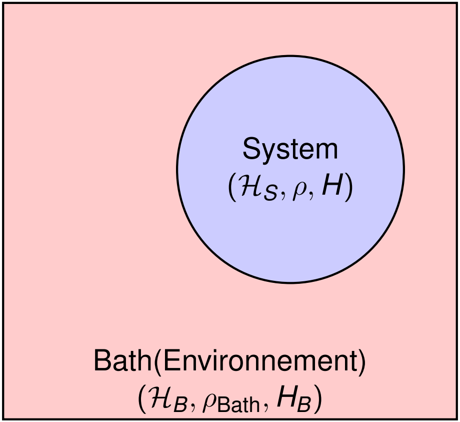
- Davies, Quantum stochastic processes, 1969
- Belavkin, Quantum filtering of Markov signals with white quantum noise, 1980
- Barchielli, Continual measurements for quantum open systems, 1983
- Dalibard, Castin, and Molmer, Wave-function approach to dissipative processes in quantum optics, 1992
- Dum, Zoller, and Ritsch, Monte Carlo simulation of the atomic master equation for spontaneous emission, 1992
- Carmichael, An open systems approach to quantum optics, 1993
- Carmichael, Quantum trajectory theory for cascaded open systems, 1993
- Strunz, Diósi, Gisin, Yu, Quantum trajectories for Brownian motion, 1999

Some references on feedback control

- Belavkin, Theory of the control of observable quantum systems, 1983
- Wiseman, Quantum theory of continuous feedback, 1994
- Doherty and Jacobs, Feedback-control of quantum systems using continuous state-estimation, 1999
- Doherty, Habib, Jacobs, Mabuchi, Tan, Quantum feedback and classical control theory, 2000
- Edwards and Belavkin, Optimal quantum feedback control via quantum dynamic programming, 2005
- Gough, Belavkin, and Smolyanov, Hamilton-Jacobi-Bellman equations for quantum filtering and control, 2005
- van Handel, Stockton, Mabuchi, Feedback control of quantum state reduction
- Mirrahimi, van Handel, Stabilizing feedback controls for quantum systems, 2007
- James, Nurdin, Petersen, H-Infinity Control of Linear Quantum Stochastic Systems, 2008
- Gough, James, The series product and its application to quantum feedforward and feedback networks, 2009

Heuristic derivation of the Lindblad equation [1]

$(\mathcal{H}_{\text{Total}}, \rho_{\text{Total}}, H_{\text{Total}})$

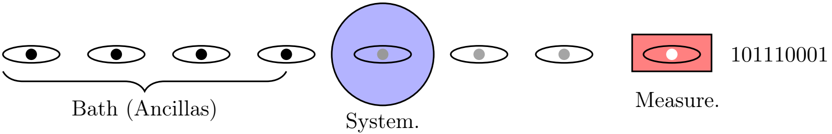


$$\rho_{\text{Total},t} = U_t(\rho_0 \otimes \rho_{\text{Bath}})U_t^\dagger \qquad \rho_{\text{Bath}} = \sum_\nu \lambda_\nu \left|\nu\right\rangle \left\langle\nu\right|$$

[1] Lindblad-Gorini-Kossakawski-Sudarshan, 1976.

Different approaches to derive quantum trajectories

Repeated interaction

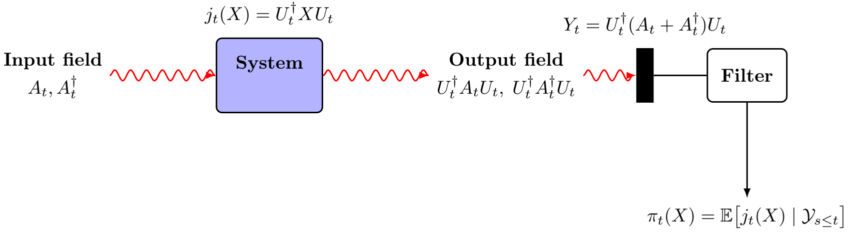


Consider a discrete-time system with interaction time τ at time $n\tau$ it comes in interaction with a fresh ancillary :

$$\mathcal{V}_S = \mathrm{Tr}_a\big(V(\rho \otimes \sigma)V^\dagger\big), \quad V \text{ is unitary acting on } \mathcal{H}_S \otimes \mathcal{H}_B$$

$$\mathrm{d}\hat{\rho}_t = \mathcal{L}(\hat{\rho}_t)\mathrm{d}t + \left(L\hat{\rho}_t + \hat{\rho}_t L^\dagger - \mathrm{Tr}\left((L + L^\dagger)\hat{\rho}_t\right)\hat{\rho}_t\right)\mathrm{d}W_t$$

Quantum filtering



The evolution is described by the following unitary operator: [2]

$$\mathrm{d}U_t = \left\{ L\mathrm{d}A_t^\dagger - L^\dagger \mathrm{d}A_t - \frac{1}{2}L^\dagger L \mathrm{d}t - \mathrm{i}H\mathrm{d}t \right\} U_t, \quad U_0 = I$$

Take $X_t = j_t(X)$ and the observation process $(Y_t)_{t\ge 0}$

$$\mathrm{d}j_t(X) = j_t(\mathcal{L}^\dagger(X))\,\mathrm{d}t + j_t([L^\dagger, X])\mathrm{d}A_t + j_t([X, L])\mathrm{d}A_t^\dagger$$
$$\mathrm{d}Y_t = j_t(L + L^\dagger)\mathrm{d}t + \mathrm{d}A_t + \mathrm{d}A_t^\dagger$$

[2] Hudson and Parthasarathy, 1984.

Quantum filtering

$$[Y_t, Y_r] = 0, \quad t_0 \le r \le t, \quad \text{(self-non-demolition property)}$$

$$[X_t, Y_s] = 0, \quad t_0 \le s \le t, \quad \text{(non-demolition property)}$$

Quantum filter[3]**:** $\hat{X}(t) \equiv \pi_t(X) = \mathbb{E}\left(j_t(X)|\mathcal{Y}_t\right)$

$$\mathrm{d}\pi_t(X) = \pi_t(\mathcal{L}^\dagger(X))\,\mathrm{d}t + \left(\pi_t(XL + L^\dagger X) - \pi_t(L + L^\dagger)\pi_t(X)\right)\mathrm{d}W_t,$$

where W is a Wiener process, $\mathrm{d}W_t = \mathrm{d}Y_t - \pi_t(L + L^\dagger)\,\mathrm{d}t$.

In the Schrödinger picture, $\pi_t(X) = \mathrm{tr}\,(\rho_t X)$,

$$\mathrm{d}\rho_t = \mathcal{L}(\rho_t)\,\mathrm{d}t + \left(L\rho_t + \rho_t L^\dagger - \mathrm{Tr}\left((L + L^\dagger)\rho_t\right)\rho_t\right)\mathrm{d}W_t$$

[3] Belavkin, 1980.

Barchielli's approach

Probability space[4]: $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q}, (W_t)_{t\geq 0}\right)$

Linear Stochastic Schrödinger Equation:

$$\mathrm{d}|\psi_t\rangle = \left(-\mathrm{i}H - \frac{1}{2}L^\dagger L\right)|\psi_t\rangle \mathrm{d}t + L|\psi_t\rangle \mathrm{d}W_t$$

Equation of the Propagator:

$$\mathrm{d}S_t = \left(-\mathrm{i}H - \frac{1}{2}L^\dagger L\right)S_t \mathrm{d}t + LS_t \mathrm{d}W_t$$

[4]Barchielli, 1983.

Barchielli's approach

Probability space: $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q}, (W_t)_{t\geq 0}\right)$

Linear Stochastic Master Equation: denote $\sigma_t = S_t \rho_0 S_t^\dagger$

$$\mathrm{d}\sigma_t = \mathcal{L}(\sigma_t)\,\mathrm{d}t + \left(L\sigma_t + \sigma_t L^\dagger\right)\mathrm{d}W_t$$

where

$$\mathcal{L}(\sigma_t) = -\mathrm{i}\,[H, \sigma_t] + L\sigma_t L^\dagger - \frac{1}{2}\left\{L^\dagger L, \sigma_t\right\}$$

Barchielli's approach

New probability space:

$$\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q}, (W_t)_{t\geq 0}\right) \longrightarrow \left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \tilde{\mathbb{Q}}, (B_t)_{t\geq 0}\right)$$

Stochastic Master Equation: Denote $\rho_t = \sigma_t / \mathrm{Tr}\,(\sigma_t)$

$$\mathrm{d}\rho_t = \mathcal{L}\,(\rho_t)\,\mathrm{d}t + \left(L\rho_t + \rho_t L^\dagger - \mathrm{Tr}\left((L + L^\dagger)\rho_t\right)\rho_t\right)\mathrm{d}B_t$$

where

$$\mathcal{L}\,(\sigma_t) = -\mathrm{i}\,[H, \sigma_t] + L\sigma_t L^\dagger - \frac{1}{2}\left\{L^\dagger L, \sigma_t\right\}$$

Non-Markovian quantum trajectories

- Up to here, environmental correlation times are assumed negligibly short compared to the system's characteristic time scale (Markovian approximation)

- Strunz, Diósi, and Gisin, Open system dynamics with non-Markovian quantum trajectories, PRL, 1999

- Gardiner & Zoller, Quantum Noise: A Handbook of Markovian and Non-Markovian Quantum Stochastic Methods, 2004

- Breuer & Petruccione, The Theory of Open Quantum Systems, 2002

Thank you!