Geometric interpretation of timelike entanglement entropy

Michal P. Heller

based on 2408.15752 + work in progress with Fabio Ori and Alex Serantes

Introduction

Top down sharp geometric probes of the bulk

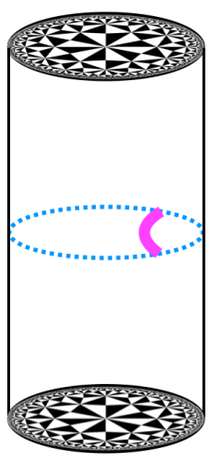
hep-th/9906226 by Balasubramanian and Ross, hep-th/0603001 / 0705.0016 by RT / HRT, …

Holography encodes gravity in the language of boundary CFTs

What we know about the bulk ultimately stems from $Z_{\text{bulk}}[J] = Z_{\text{CFT}}[J]$

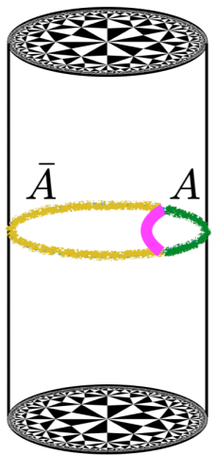
This gives us direct access to correlation functions / thermodynamics / entropies

While all these qties are geometric, they are only sometimes sharp in the bulk



$$\langle O_\Delta(x_1) O_\Delta(x_2) \rangle \approx e^{-m_\Delta \,(\text{proper length})}$$

boundary anchored
spacelike geodesics

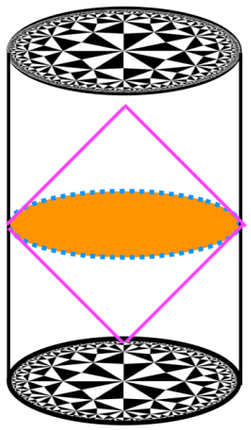


$$S = \frac{\text{proper area}}{4G_N}$$

holographic
entanglement entropy

What they are good for in the bulk?

They are also bottom up sharp geometric object, like holographic complexity, but these are in the vast majority of cases at best only qualitatively understood



$\mathcal{C}_V \sim$ volume of $\begin{matrix}\text{max (Lorentzian)}\\ \text{min (Euclidean)}\end{matrix}$ volume time slice

$\mathcal{C}_A \sim$ bulk action in the Wheeler - de Witt patch

$\mathcal{C}_{V\,2.0} \sim$ bulk volume of the Wheeler - de Witt patch

$\mathcal{C}_{anything} \sim$ covariantly defined bulk volumes using a whole class of functionals

...

1402.5674 by Susskind, 1509.07876 by Brown et al., 1610.02038 by Couch et al., ... , 2111.02429 by Belin et al.

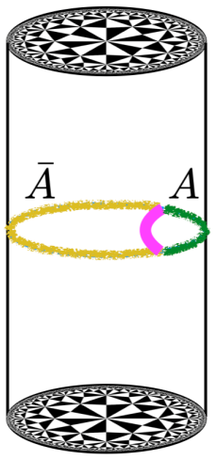
We like such sharp quantities because due to their localized nature they allow to directly probe black hole regions of interests (horizon, interior, singularity)

And if they are top down and we could calculate them independently on the boundary, the match with the bulk representation indicates bulk geometry works

Timelike entanglement entropy

2210.09457, 2302.11695 by Doi, Harper, Mollabashi, Takayanagi, Taki

Bulk:



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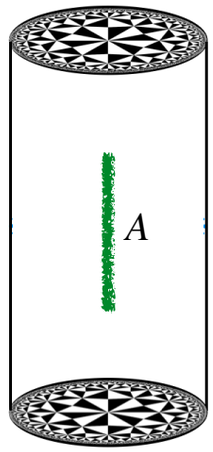
entanglement entropy

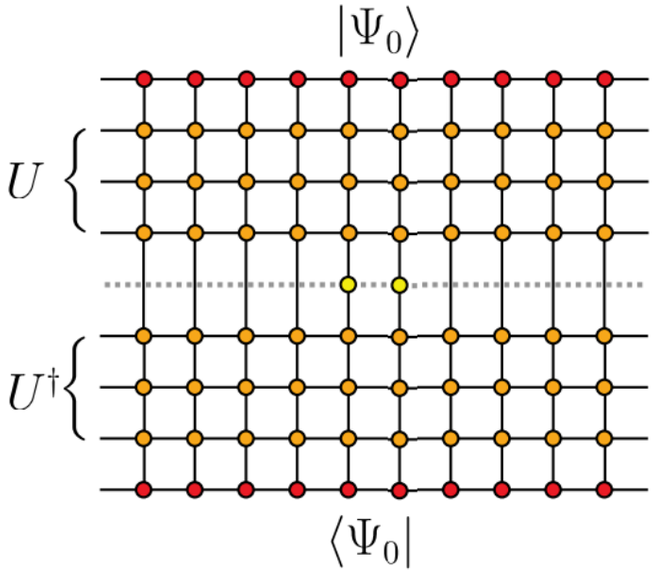


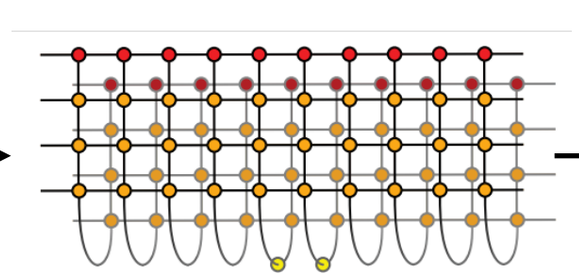
???

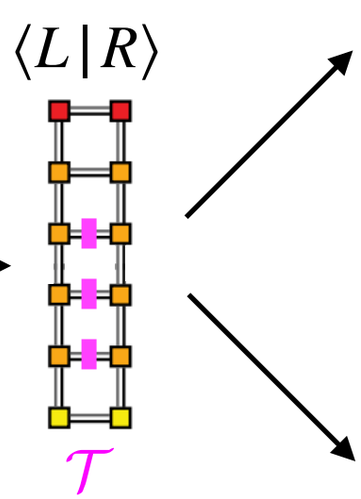
(holographic) timelike entanglement entropy

Tensor network connection: temporal entanglement

0904.1926 by Bañuls, Hastings & Verstraete, 1411.7950 by Hastings & Mahajan, 2307.11649 by Carignano, Marimón & Tagliacozzo*, ….







entropies:

$$S[\langle L|L\rangle^{-1}\,\mathrm{tr}_{\bar{\mathcal{T}}}|L\rangle\langle L|]$$

or

$$S[\langle R|R\rangle^{-1}\,\mathrm{tr}_{\bar{\mathcal{T}}}|R\rangle\langle R|]$$

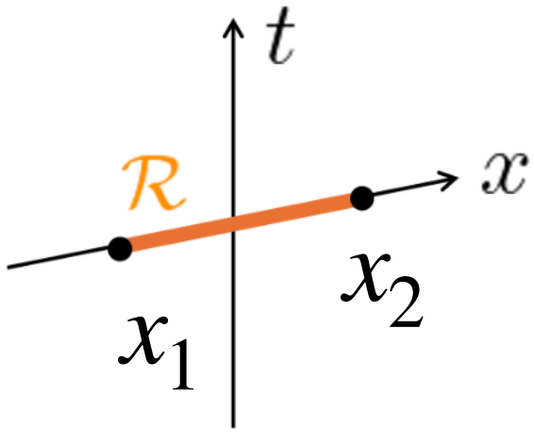
pseudoentropy:

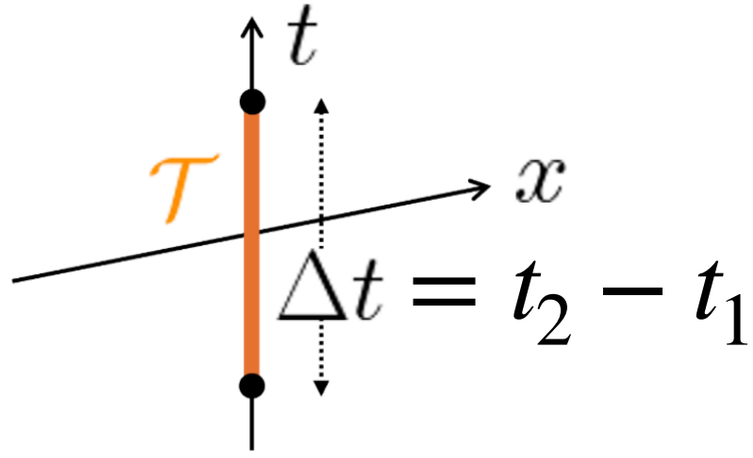
$$S[\langle L|R\rangle^{-1}\,\mathrm{tr}_{\bar{\mathcal{T}}}|R\rangle\langle L|]$$

* pictures adopted from this work

TEE in QFTs via analytic continuation: example

2210.09457, 2302.11695 by Doi, Harper, Mollabashi, Takayanagi, Taki





CFT$_{1+1}$: $S = \frac{c}{3} \log \frac{\sqrt{(x_1 - x_2)^2}}{\epsilon}$

$$S = \frac{c}{3} \log \frac{\sqrt{-(t_1 - t_2)^2}}{\epsilon}$$
$$= \frac{c}{3} \left(\log \frac{\sqrt{(t_1 - t_2)^2}}{\epsilon} + i\frac{\pi}{2} \right)$$

Why this talk?

Tempting bulk picture based on a comparison with CFT_{1+1} analytic continuations:

2210.09457, 2302.11695

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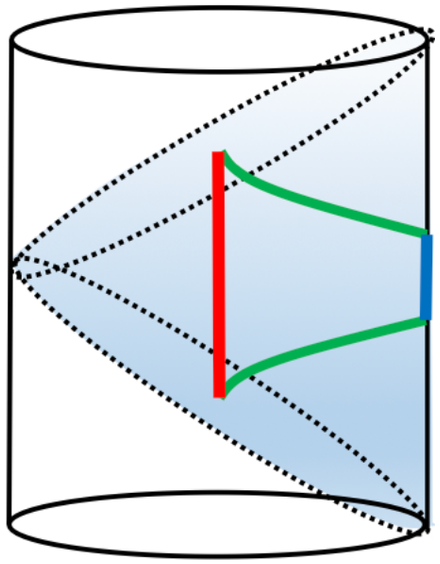
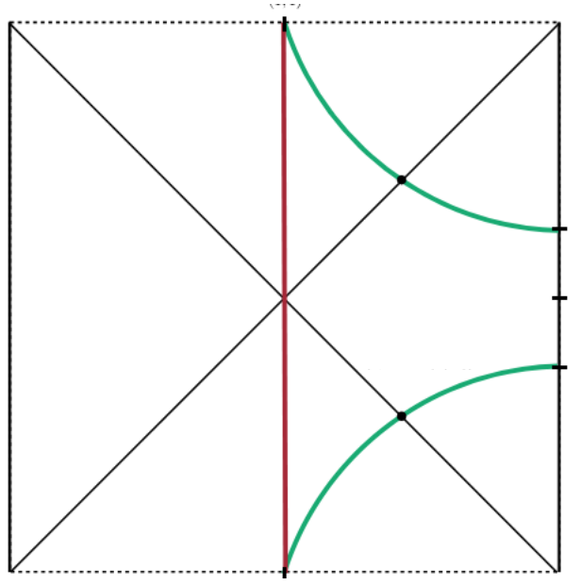




$$\frac{c}{3} log \frac{\sqrt{(t_1 - t_2)^2}}{\epsilon} + \frac{c}{3}\left(i\frac{\pi}{2}\right)$$

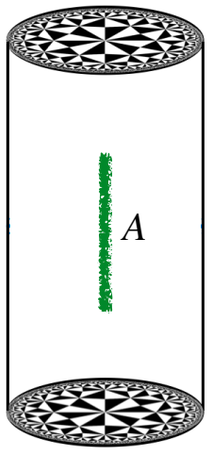
Goal: working out what the prescription is and applying it beyond AdS_3/CFT_2

2408.15752 with Fabio Ori and Alex Serantes

Our proposal

Our proposal v1

2408.15752 with Fabio Ori and Alex Serantes



holographic timelike entanglement entropy

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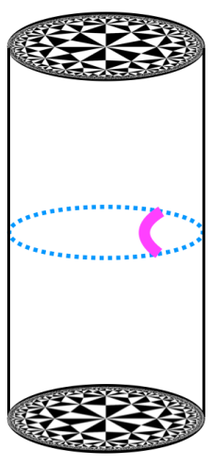

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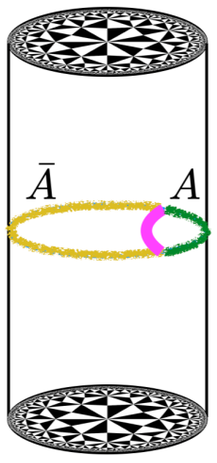

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holographic
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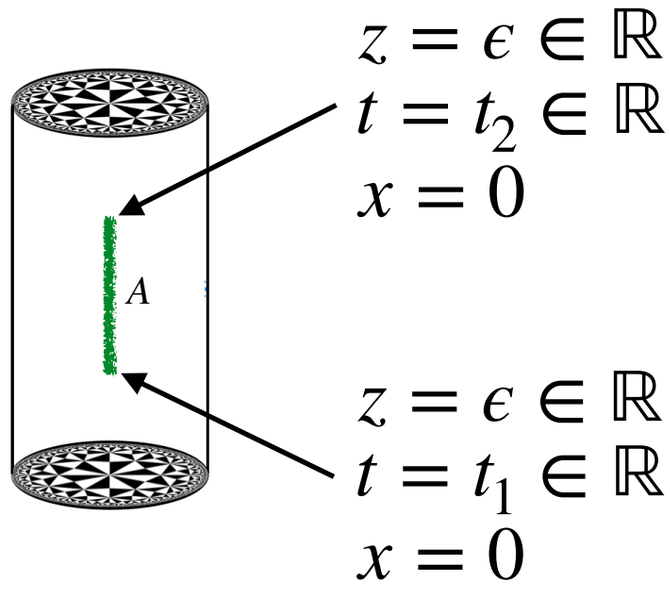
Its key features

Extremal surface means we extremize the surface function no matter what

Complex means the surface lives in the bulk metric $g_{ab}(x^c)$ with x^c complexified

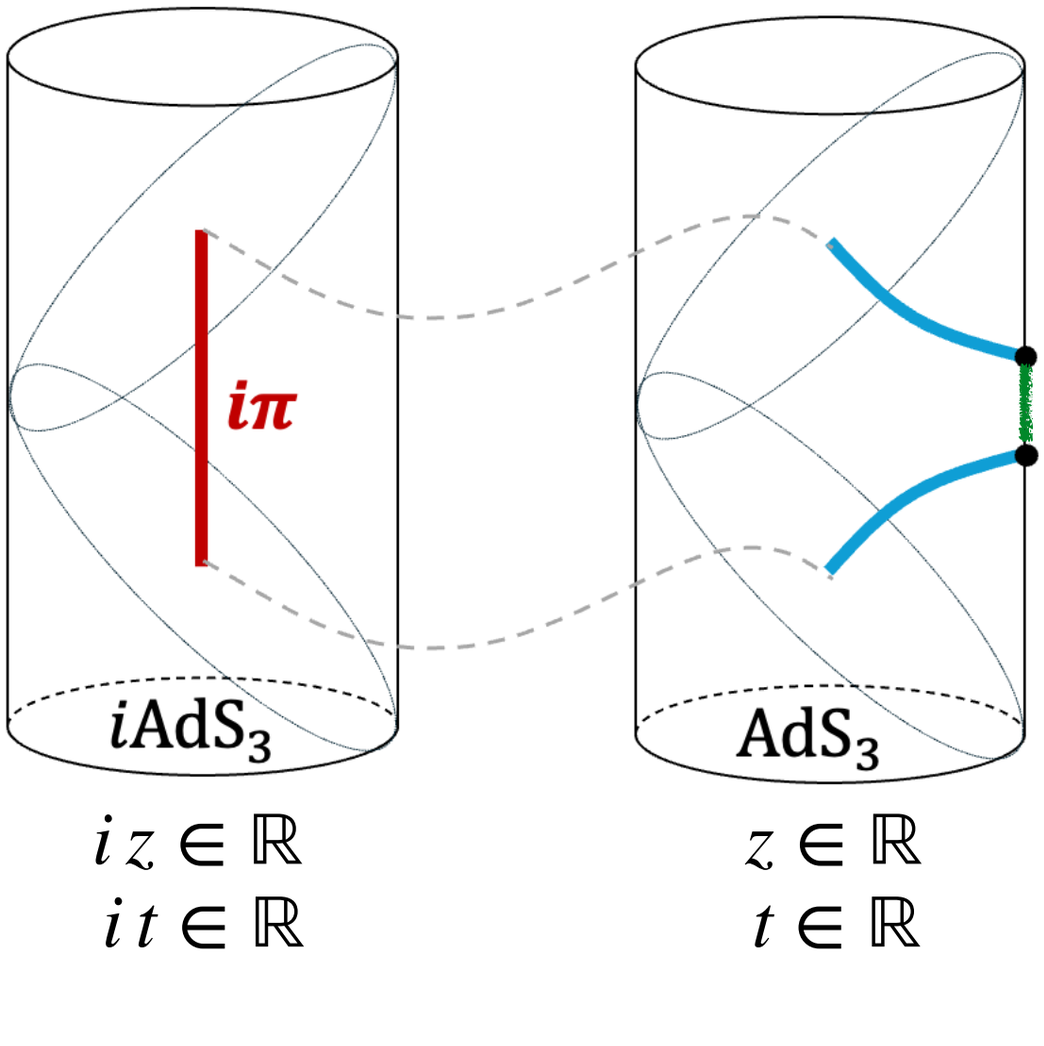
Boundary anchored means that the surface satisfies real boundary conditions, e.g.

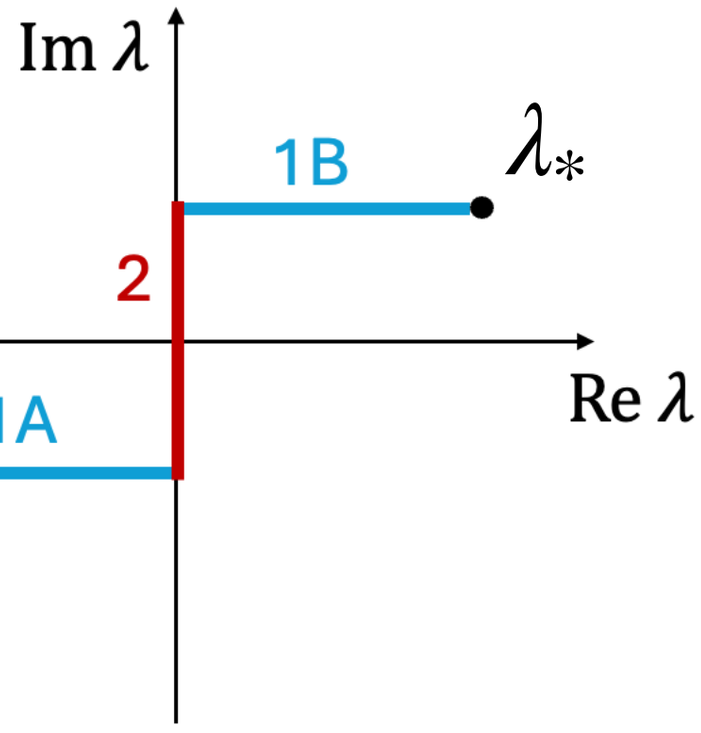


Crosscheck: timelike interval + empty AdS_3

Vacuum in CFT_{1+1}

Complex geodesic reproduces analytic continuation of the CFT_{1+1} result





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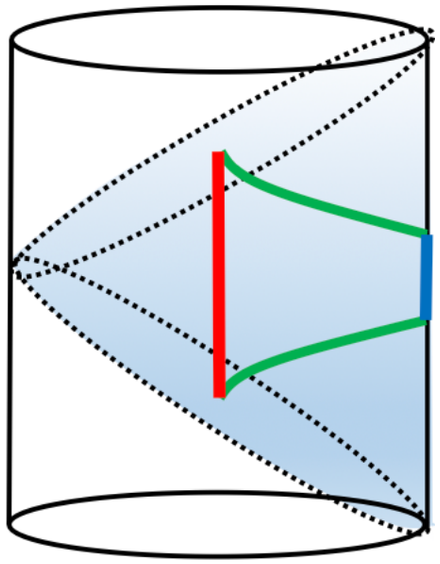

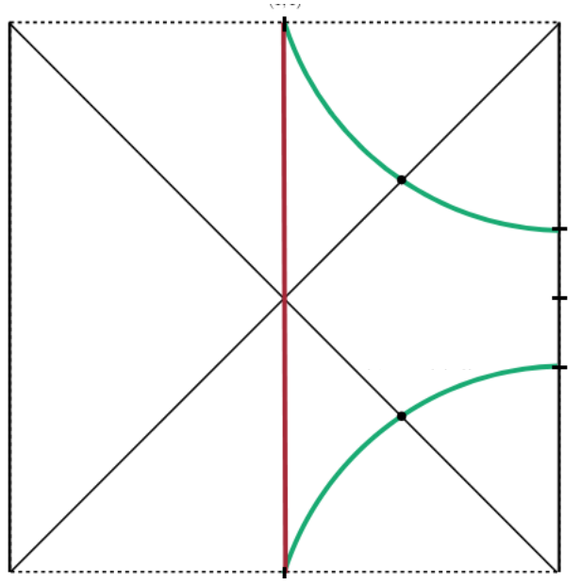

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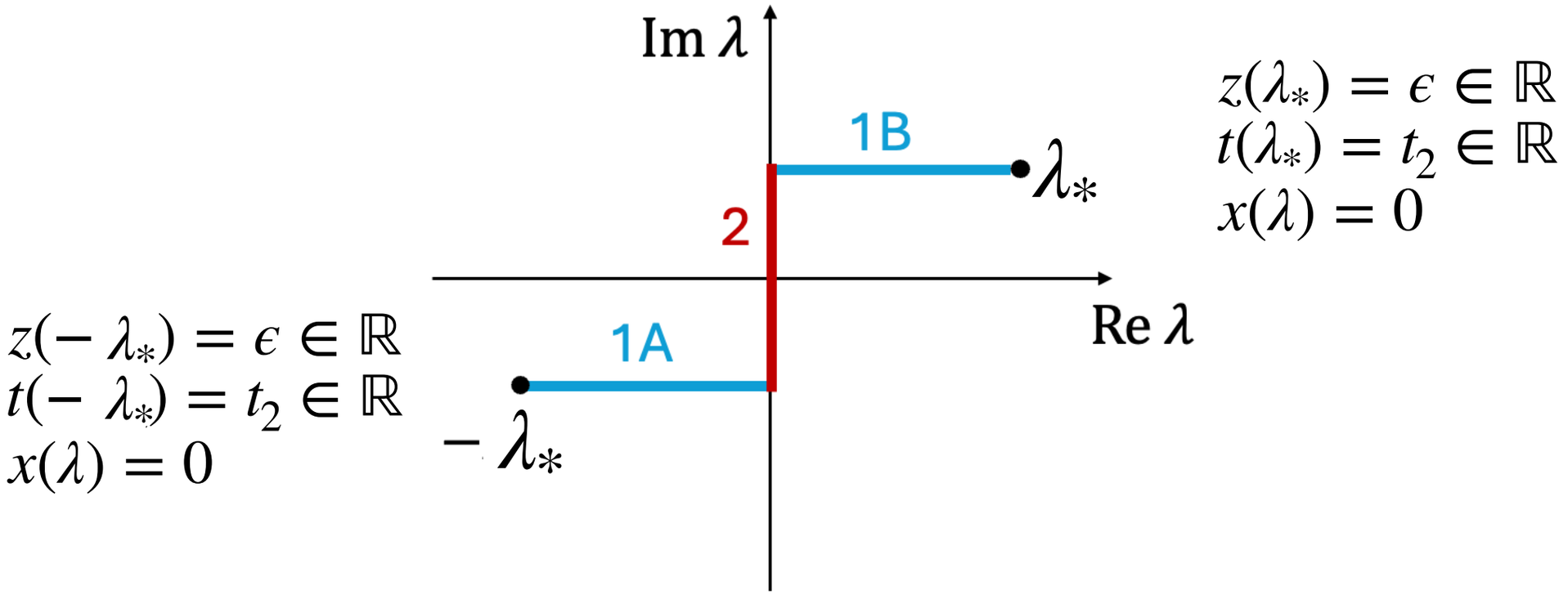
2408.15752 with Fabio Ori and Alex Serantes

A necessary feature

Different paths in the complex affine parameter plane possible (proper area = $2\lambda_*$)

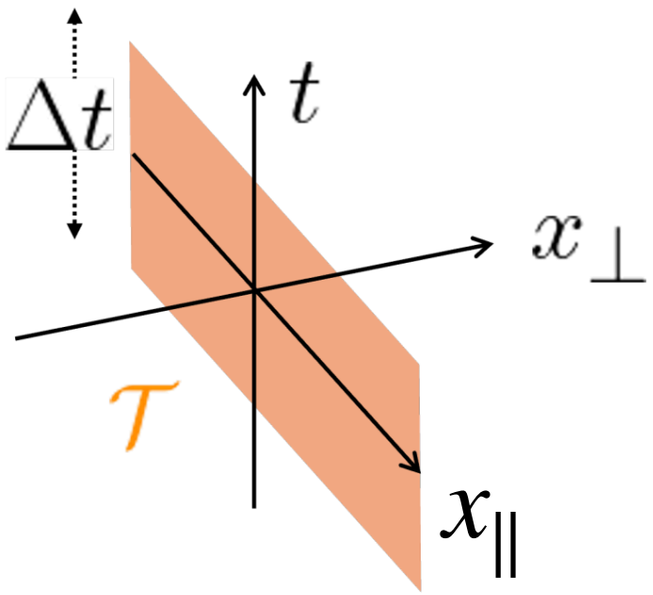


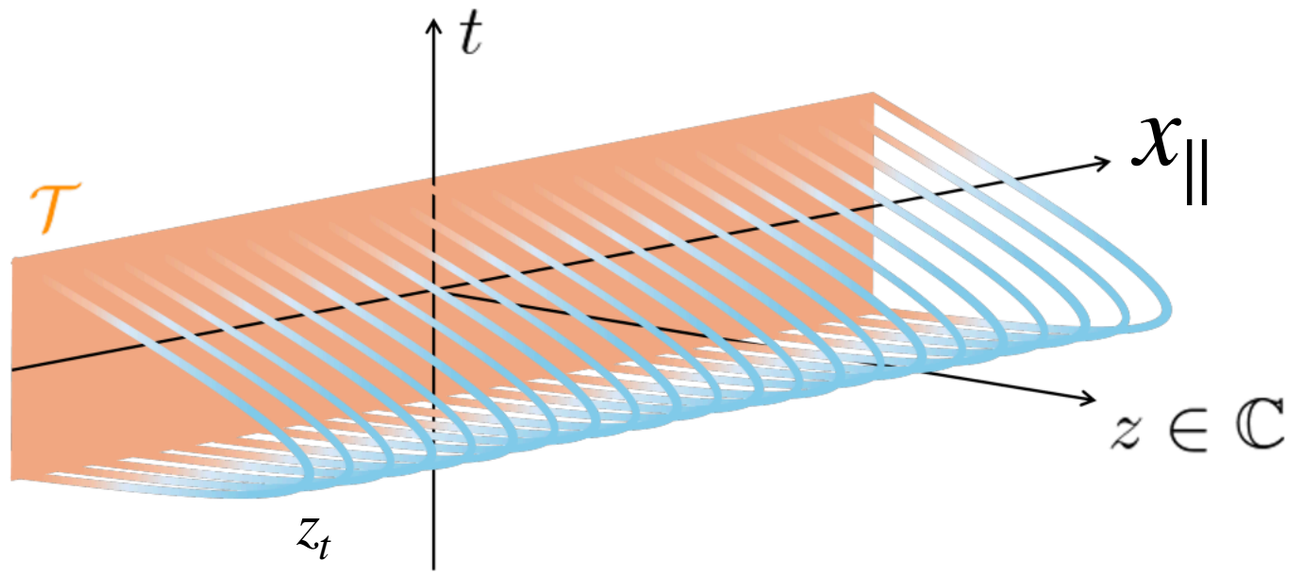
Here only a few can be interpreted in terms of paths in some real geometries and we believe this feature only follows from extreme simplicity of this setup

However, for our proposal it does not matter, any equivalent path is good

Prediction: timelike strip + AdS_4 black brane

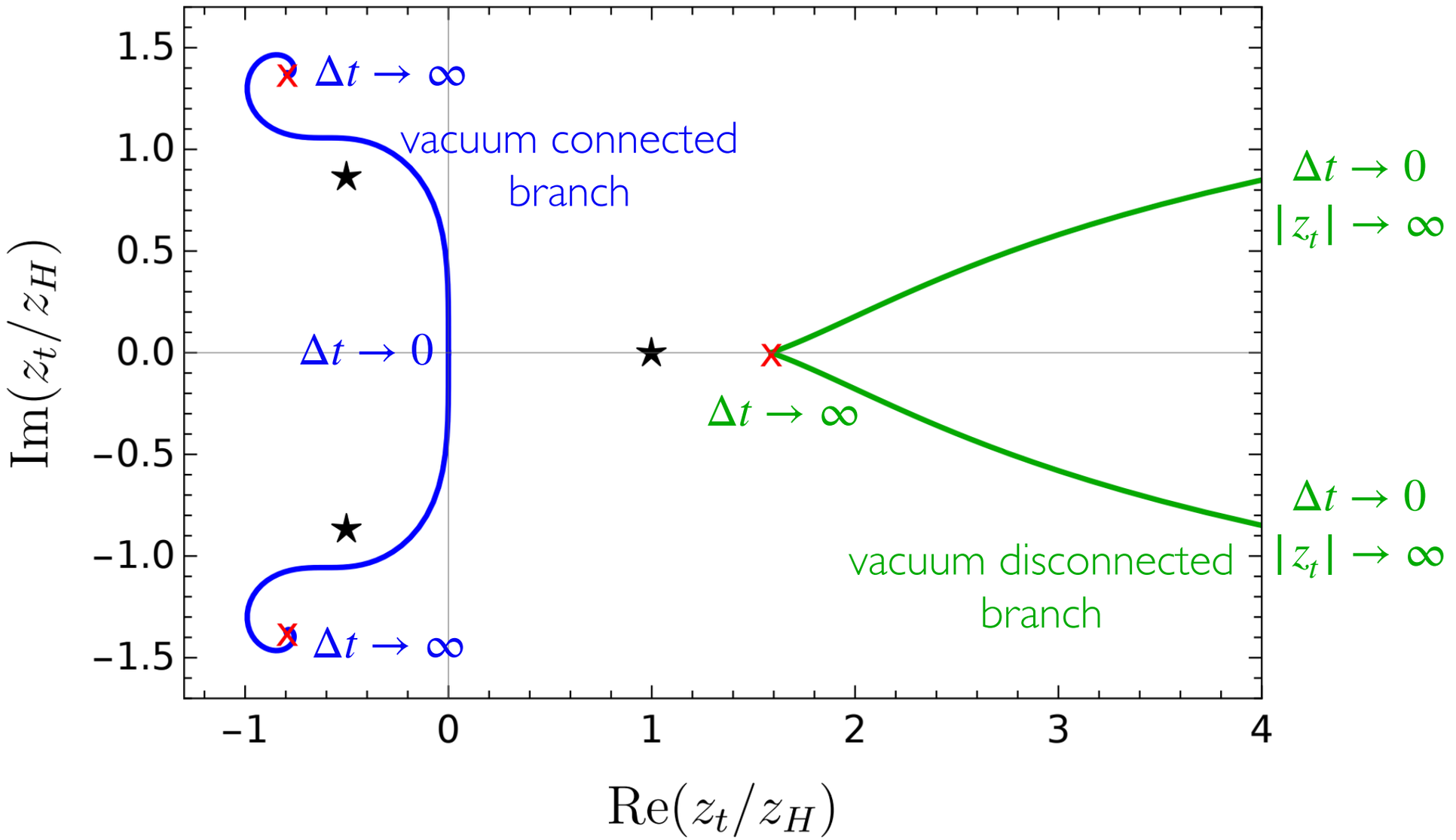
Setup





$$ds^2 = \frac{1}{z^2}\left(-f(z)dt^2 + \frac{dz^2}{f(z)} + dx_\parallel^2 + dx_\perp^2\right) \qquad \text{with} \qquad f(z) = 1 - \frac{z^3}{z_H^3}$$

Novelty: multiple possible surfaces!



★ : solutions of $f(z) = 0$

× : critical solutions having $z = \text{const}$

Which ones to pick?

Our proposal v2a

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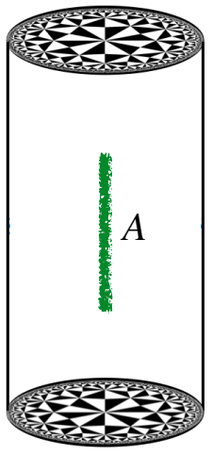


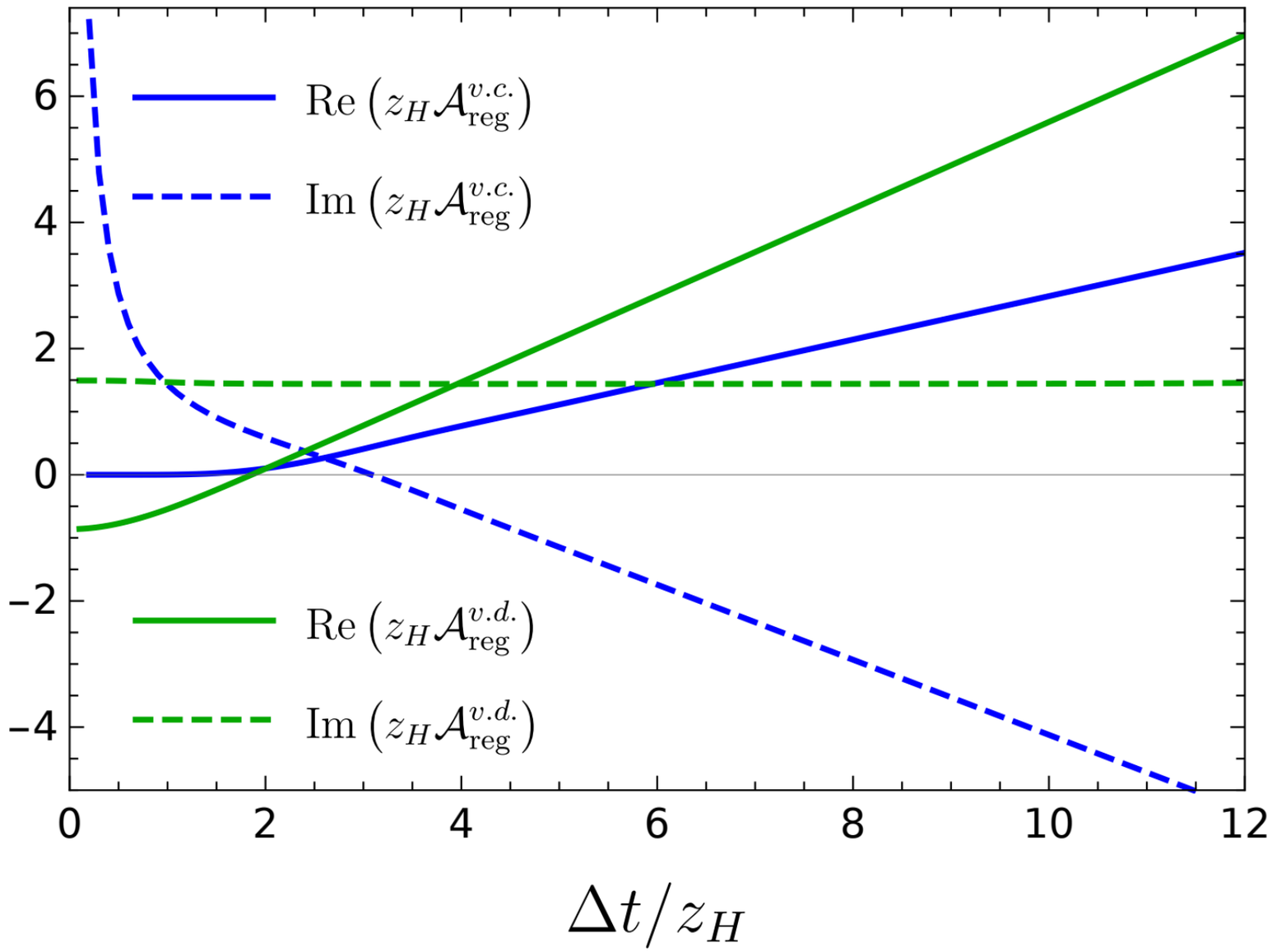
holographic timelike entanglement entropy

=

$$S = \frac{\text{proper area}}{4G_N}$$ of a complex boundary anchored extremal surface of codimension two with minimal Re(proper area)

Conundrum

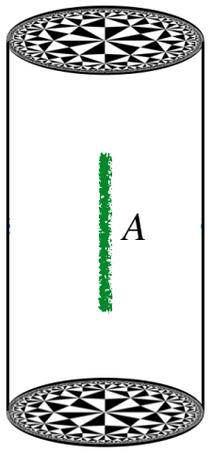
$$\mathcal{A}_{\rm reg} = \lim_{\epsilon\to 0}\left(\mathcal{A} - \frac{2}{\epsilon}\right):$$



If one minimizes Re(proper area), then at short Δt dominate the ones probing (complexified) black brane singularity, which violates UV/IR correspondence!

Our proposal v2b

2408.15752 with Fabio Ori and Alex Serantes



holographic timelike
entanglement entropy

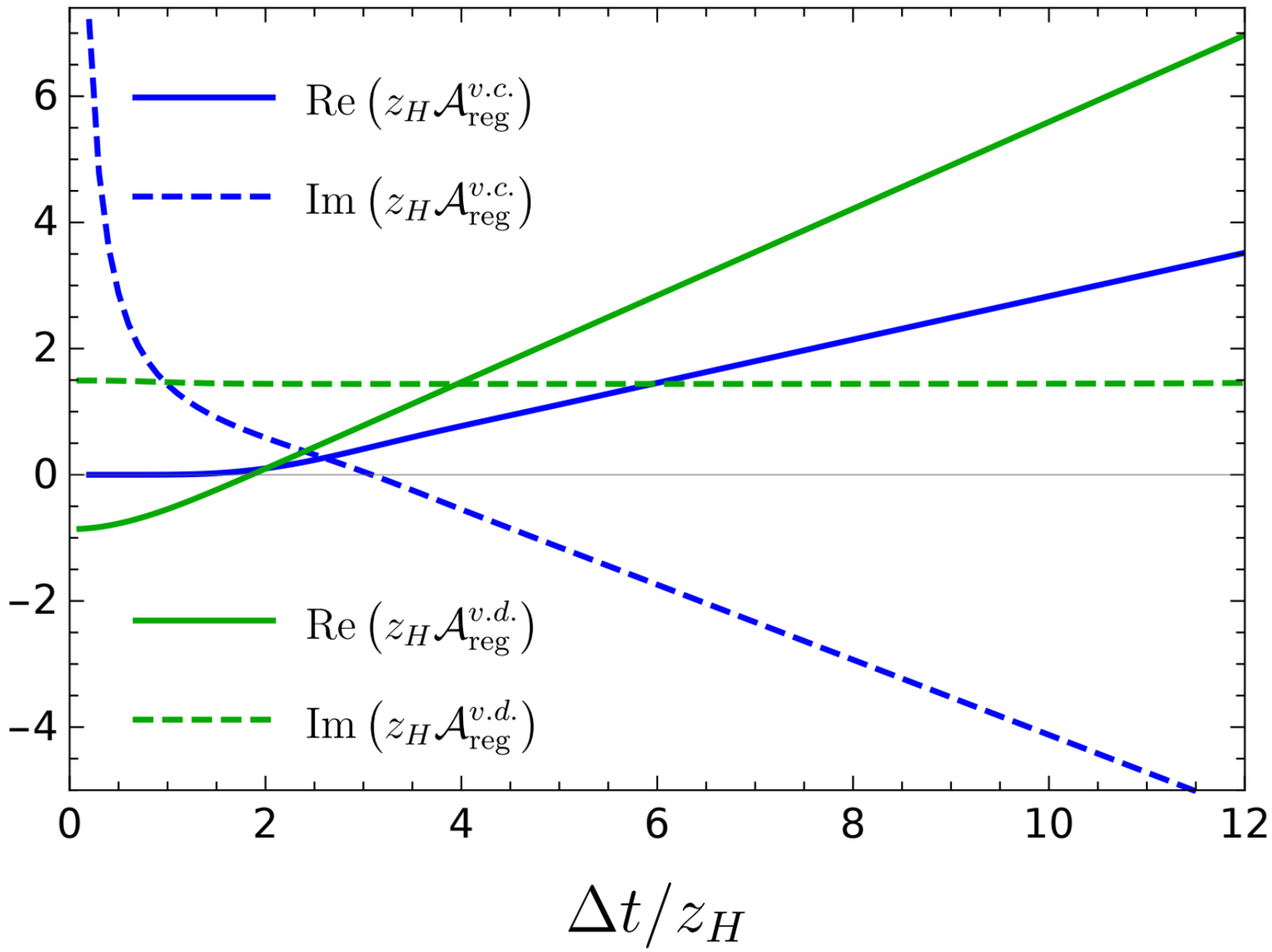
=

$$S = \frac{\text{proper area}}{4G_N}$$ of a complex

boundary anchored extremal
surface of codimension two
that reduces to entanglement entropy
upon analytic continuation

Possible resolution

$$\mathcal{A}_{\rm reg} = \lim_{\epsilon\to 0}\left(\mathcal{A} - \frac{2}{\epsilon}\right):$$



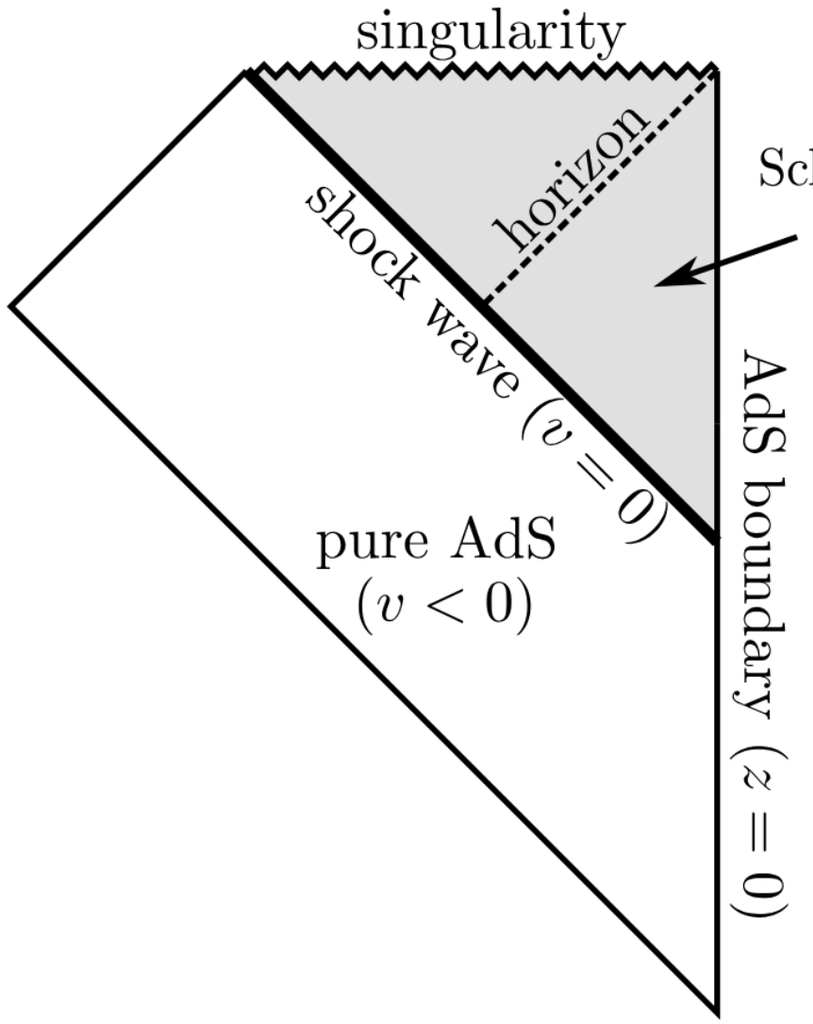
In this case, the vacuum connected surfaces would dominate the short distances and the singularity probing ones would be subdominant or non-contributing saddles

Vaidya quenches

work in progress with Fabio Ori and Alex Serantes

Quenches work in progress with Fabio Ori and Alex Serantes

The prescription allows to study also holographic quenches in CFT_d



$$ds^2 = -\frac{L^2}{z^2}\left(-(1 - M(v)z^d)\,dv^2 - 2dvdz + d\vec{x}^2\right)$$

e.g. with $M(v) = \frac{M_f}{2}\left(1 + \tanh \gamma v\right)$

here $\gamma \to \infty$

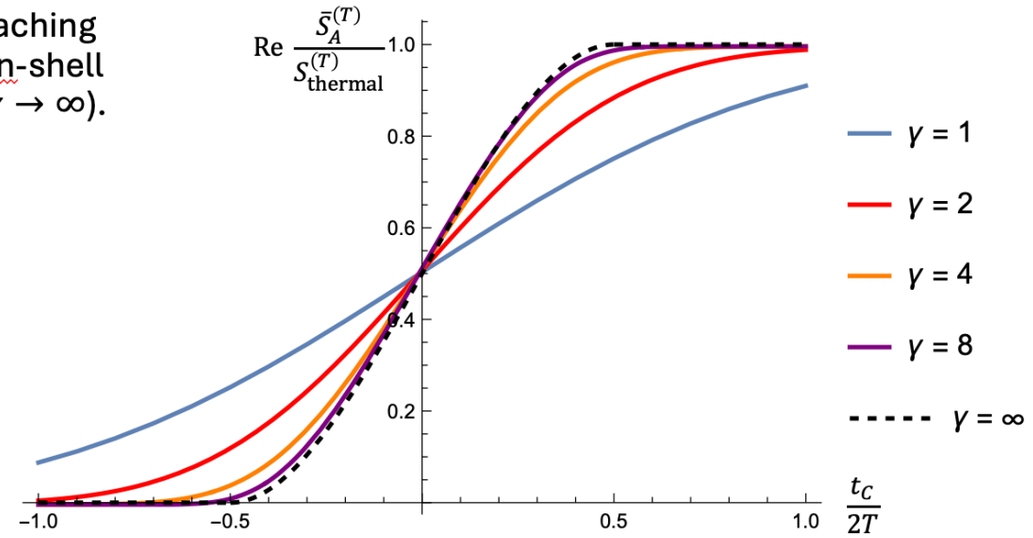
Timelike entanglement entropy in quenches

work in progress with Fabio Ori and Alex Serantes

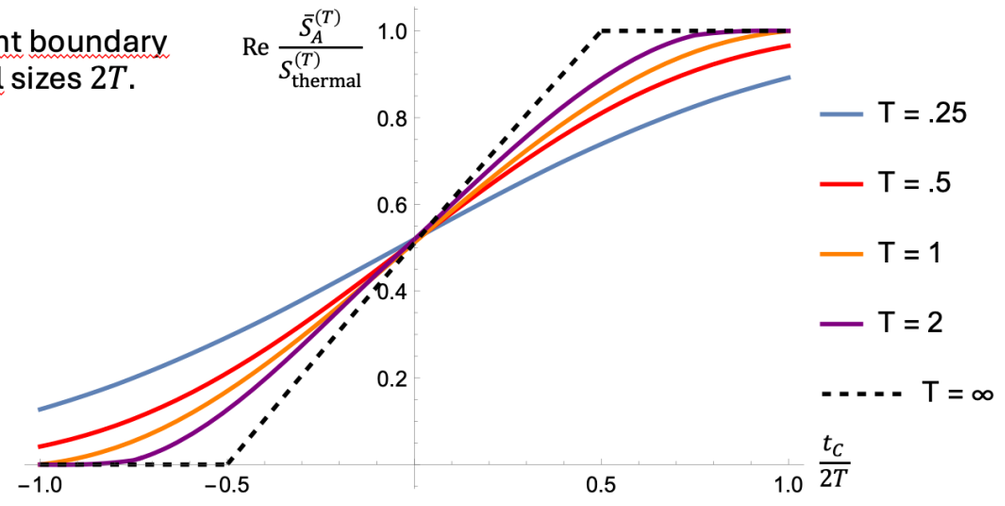
CFT$_{1+1}$ and $\gamma \to \infty$: two ways of doing things:

- matching exact solutions at v = 0
  1212.6066 by Balasubramanial et al.
- fully fledged numerics

Approaching the thin-shell limit ($\gamma \to \infty$).



Different boundary interval sizes $2T$.



Otherwise, it is numerics in complexified Vaidya, pilot results in 4 bulk dimensions

Summary

Summary

Sharp and well defined observables are rare and precious (bulk tomography)

Brilliant idea in 2022/2123: analytic continuation of holographic entanglement entropy to timelike subregions might as well be such a quantity

2210.09457, 2302.11695 by Doi, Harper, Mollabashi, Takayanagi, Taki

Us: providing a bulk prescription and studying geometric interpretation

2408.15752 with Fabio Ori and Alex Serantes

Take home: holographic timelike entanglement entropy is necessarily given by complex extremal codimension-2 hypersurfaces

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Its key features

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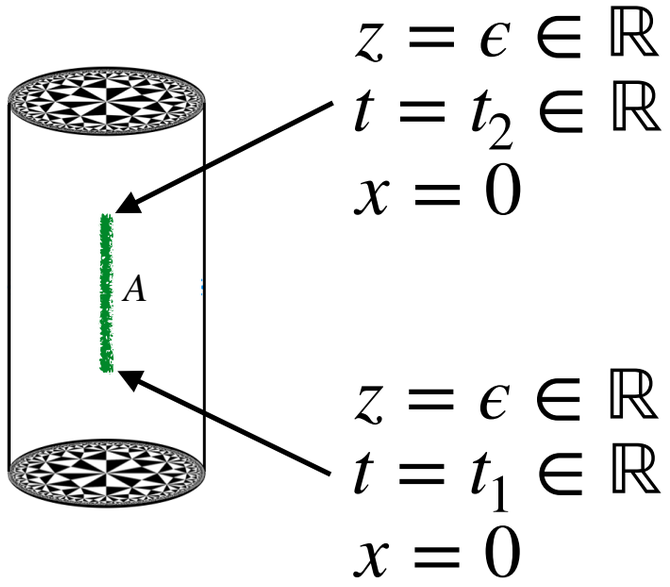


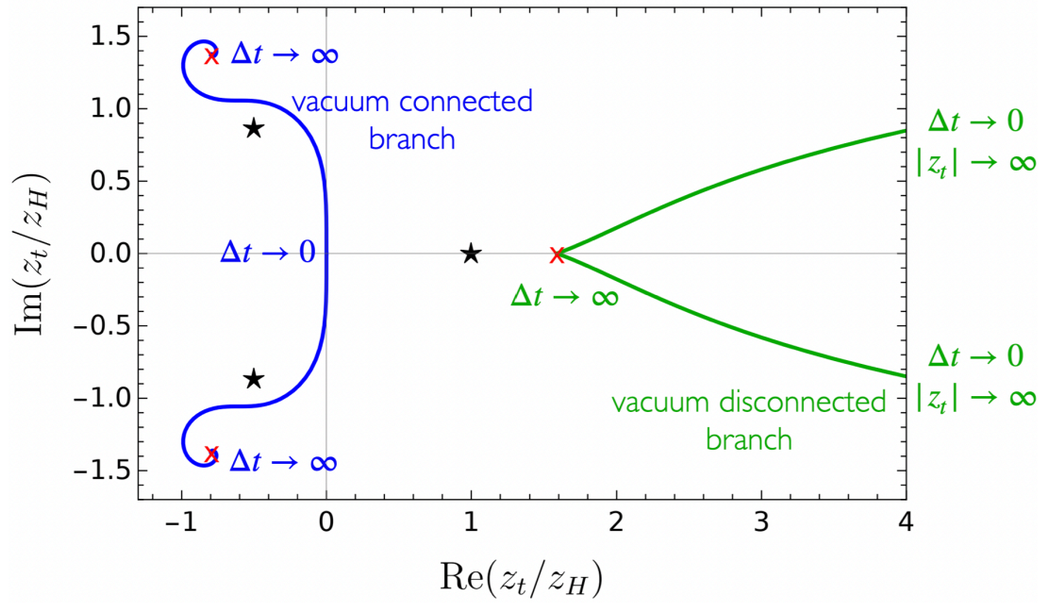
Outlook

Outlook

Which complex extremal surface to pick (min Re or reducing to HEE)?

Replica trick derivation?

Holographic timelike entanglement entropy as probe of black hole singularity?



Robust physical properties to match with CFT / tensor networks?

Tip of the iceberg of novel bottom up holographic geometric probes?