# Finite-dimensional generalized nil-Coxeter and nil-Temperley–Lieb algebras

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# Coxeter groups

The group $S_{n+1}$ of permutations, and $(\mathbb{Z}/2\mathbb{Z}) \wr S_n$ of signed permutations, are examples of *finite Coxeter groups* –
finite groups of orthogonal transformations generated by reflections.

- For $S_{n+1}$, let $s_1 = (1\ 2),\ s_2 = (2\ 3),\ \dots,\ s_n = (n\ n+1)$.

  The permutation group $S_{n+1}$ is generated by $s_1, \dots, s_n$ with the **braid relations**

  $$s_i s_j = s_j s_i \text{ (i.e. } (s_1 s_2)^2 = 1), \quad |i-j|>1, \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},$$

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- The dihedral group $I_2(m)$ (rotation and reflection symmetries of the regular $m$-gon) is generated by $s_1, s_2$ with the braid and Coxeter relations

  $$(s_1 s_2)^m = 1 \text{ (i.e. } s_1 s_2 s_1 \cdots = s_2 s_1 s_2 \cdots), \qquad s_1^2 = s_2^2 = 1.$$

  So e.g. $I_2(3) = \{1,\ s_1,\ s_2,\ s_1 s_2,\ s_2 s_1,\ w_\circ = s_1 s_2 s_1 = s_2 s_1 s_2\} = S_3$.

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The finite real reflection groups were classified in 1934 by Coxeter:

# Coxeter's paper

ANNALS OF MATHEMATICS
Vol. 35, No. 3, July, 1934

## DISCRETE GROUPS GENERATED BY REFLECTIONS

BY H. S. M. COXETER

(Received June 16, 1933)

CONTENTS



### Introduction

It is known that, in ordinary space, the only finite groups generated by reflections are

$[k]$ $(k \geqq 1)$, of order $2k$, with abstract definition

$$R_1^2 = R_2^2 = (R_1R_2)^k = 1,$$

and

$[k_1, k_2]$ $(k_1 \geqq 2,\ k_2 \geqq 2,\ 1/k_1 + 1/k_1 > \frac{1}{2})$, of order $\dfrac{4}{1/k_1 + 1/k_2 - \frac{1}{2}}$, with abstract definition

$$R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^{k_1} = (R_1R_3)^2 = (R_2R_3)^{k_2} = 1.$$

$[1]$ is the group of order 2 generated by a single reflection. Since this is the symmetry group of the one-dimensional polytope[1] { }, we write

$$[1] = [\,].$$

$[k] (k \geqq 3)$ and $[k_1, k_2]$ $(k_1 = 3,\ k_2 = 3, 4, 5)$ are the symmetry groups of the ordinary regular polygons $\{k\}$ and polyhedra $\{k_1, k_2\}$. The rest of the groups can be written as direct products, thus:

$$[2] = [\,] \times [\,],$$

TABLE OF IRREDUCIBLE FINITE GROUPS GENERATED BY REFLECTIONS

| Group | Order[12] | $m$ | $h$ | Central inversion? |
|---|---|---|---|---|
| $[3^n]$ | $(n+2)!$ | $n+1$ | $n+2$ | Only when $n = 0$ |
| $[3^n, 4]$ | $2^{n+2}(n+2)!$ | $n+2$ | $2(n+2)$ | Yes |
| $\begin{bmatrix} 3^n \\ 3 \\ 3 \end{bmatrix}$ | $2^{n+2}(n+3)!$ | $n+3$ | $2(n+2)$ | Only when $n$ is odd |
| $[k]$ | $2k$ | 2 | $k$ | Only when $k$ is even |
| $[3, 5]$ | 120 | 3 | 10 | Yes |
| $[3, 4, 3]$ | 1152 | 4 | 12 | Yes |
| $[3, 3, 5]$ | 14400 | 4 | 30 | Yes |
| $\begin{bmatrix} 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}$ | 51840 | 6 | 12 | No |
| $\begin{bmatrix} 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}$ | 2903040 | 7 | 18 | Yes |
| $\begin{bmatrix} 3, 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}$ | 696729600 | 8 | 30 | Yes |

# Presentation of finite Coxeter group( algebra)s

Every (finite) Coxeter group $W$ is given by:

- A finite set of generators $\{s_i \,|\, i \in I\}$.
- A symmetric Coxeter integer matrix $M = (m_{ij})_{i,j \in I}$, with $2 = m_{ii} \leq m_{ij} \leq \infty$.

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There exist other algebras with the same dimension, which are "deformations" of $\Bbbk W$:

# From braid monoid-algebra...

Given a Coxeter system $I$ and matrix $M \in \mathbb{Z}_{\geq 2}^{I \times I}$,

- The free associative algebra on $I$ is $\Bbbk\langle T_i \,|\, i \in I \rangle$, with basis all words in the generators $T_i$. (Equivalently, the tensor algebra over $\Bbbk$ of $\oplus_i (\Bbbk T_i)$.)
- The corresponding *monoid algebra* is its quotient by a two-sided ideal:

$$\Bbbk\mathcal{B}_M := \Bbbk\langle T_i \,|\, i \in I \rangle \;/\; (T_i T_j T_i \cdots = T_j T_i T_j \cdots \,|\, i \neq j);$$

ANNALS OF MATHEMATICS
Vol. 48, No. 1, January, 1947

**THEORY OF BRAIDS**

BY E. ARTIN

(Received May 20, 1946)

A theory of braids leading to a classification was given in my paper "Theorie der Zöpfe" in vol. 4 of the Hamburger Abhandlungen (quoted as $Z$). Most of the proofs are entirely intuitive. That of the main theorem in §7 is not even convincing. It is possible to correct the proofs. The difficulties that one encounters if one tries to do so come from the fact that projection of the braid, which is an excellent tool for intuitive investigations, is a very clumsy one for rigorous proofs. This has lead me to abandon projections altogether. We shall use the more powerful tool of braid coordinates and obtain thereby farther reaching results of greater generality.

# . . . to generic Hecke algebras

- The associated *generic Hecke algebra* (with parameters $a, b \in \Bbbk$) is:

$$\mathcal{E}_{a,b} := \Bbbk \mathcal{B}_M / (T_i^2 - aT_i - b \,|\, i \in I).$$

- Special case: the *Iwahori–Hecke algebras* $\mathcal{H}_q(W)$ that are prominent in representation theory; here $a = q - 1, \ b = q$.
  (As $q \to 1$, say over $\Bbbk = \mathbb{R}$, the relations go to $\Bbbk W$: "deformation".)

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  (As $q \to 1$, say over $\mathbb{k} = \mathbb{R}$, the relations go to $\mathbb{k}W$: "deformation".)

**Fact:** Each such algebra $\mathcal{E}_{a,b}$ has a "word basis" $\{T_w : w \in W\}$.
So, its dimension is $|W|$.

# Nil-Coxeter algebras

There are three special cases of $(a, b)$ which are interesting from the viewpoint of combinatorics, PBW deformation theory, ...:

1. $a = 0, b = 1$ – group algebra $\Bbbk W$.
2. $a = 1, b = 0$ – $0$-Hecke algebra (Norton, Hivert–Schilling–Thiery, ...).
3. $a = b = 0$ – **nil-Coxeter algebra** $NC_W$. So, $T_i^2 = 0$ – *graded* algebra.

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3. $a = b = 0$ – **nil-Coxeter algebra** $NC_W$. So, $T_i^2 = 0$ – *graded* algebra.

Nil-Coxeter algebras "occur naturally" as differential / divided-difference operators on polynomial rings (and hence in Schubert calculus). E.g. in type $A$, for $1 \leq i \leq n$ the operator $T_i$ is:

$$(T_i f)(x_1, \dots, x_{n+1}) = \frac{f(x_1, \dots, x_i, x_{i+1}, \dots, x_{n+1}) - f(x_1, \dots, x_{i+1}, x_i, \dots, x_{n+1})}{x_{i+1} - x_i}.$$

(The RHS is $\frac{\text{alternating}}{\text{alternating}}$ = symmetric in $\{x_i, x_{i+1}\}$, so $T_i^2 f = 0$.)

# Word basis of $NC_W$

**Example:** The dihedral group $D_n$ (e.g. $S_3$) has elements

$$e, \ s_1, s_2, \ s_1 s_2, s_2 s_1, \ s_1 s_2 s_1, s_2 s_1 s_2, \ \dots,$$

increasing in length all the way to the unique longest element

$$s_1 s_2 s_1 \cdots = w_\circ = s_2 s_1 s_2 \cdots .$$

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Correspondingly, its nil-Coxeter algebra has a "word basis"

$$1 = T_\emptyset,\ T_1, T_2,\ T_1 T_2, T_2 T_1,\ T_1 T_2 T_1, T_2 T_1 T_2,\ \dots,$$

all the way to the unique longest element $T_{w_\circ}$.

Now come to the protagonist of this talk: **"generalized" nil-Coxeter algebras**.

# Generalized Coxeter groups

In 1957, Coxeter studied the "generalized" Coxeter group $W_A(n,d)$, defined as the quotient of the type-$A$ (Artin) braid monoid by $s_i^d = 1$ $\forall i$.
(Such higher orders are typical in *complex* reflection groups.)

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**Theorem (Coxeter, 1957)**

$W_A(n,d)$ *is a finite group if and only if* $\dfrac{1}{n} + \dfrac{1}{d} > \dfrac{1}{2}$, *and then* $|W_A(n,d)| = \left(\frac{1}{n} + \frac{1}{d} - \frac{1}{2}\right)^{1-n} n!/n^{n-1}$.

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Later extended by Koster to cover all generalized Coxeter groups.

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Not necessary in the nil-Coxeter case. Thus:

- Given integers $d_i \geq 2$, **define** the *generalized nil-Coxeter algebra* (over $W$ or $M$) to be

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- **Question** *(a la Coxeter):* For which Coxeter groups $W = (I, M)$ and tuples $\mathbf{d} = (d_i)_i$ is this algebra $NC_M(\mathbf{d})$ finite-dimensional? Does it have a word basis?

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- Clearly, the "usual" nil-Coxeter algebras $NC_M((2, \dots, 2))$ work. Any others?

# Additional motivations / properties of generalized nil-Coxeter algebras

1. *Coxeter combinatorics:* Parallel to Coxeter's question.

2. *Tensor categories:* Generalized nil-Coxeter algebras $NC_M(\mathbf{d})$ possess a coproduct $\Delta(T_i) = T_i \otimes T_i$.
   - This is an algebra map, but cannot have a counit or antipode.
   - Thus, these are not bialgebras; hence their modules do not form a monoidal / tensor category.
   - Yet, the Tannakian formalism applies – to yield *semigroup categories* – no unit object.[1]

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3. *PBW deformations:* Despite no counit or antipode, generalized nil-Coxeter algebras $NC_M(\mathbf{d})$ smash-product polynomial rings admit "PBW deformations".
   (Going beyond the "traditional" PBW program in the literature – Etingof–Ginzburg, Shepler–Witherspoon (and Walton), ...)

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# A novel type-$A$ family

**Back to the question:** Classify the finite-dimensional generalized nil-Coxeter algebras $NC_M(\mathbf{d})$ for $\mathbf{d} \in \mathbb{Z}^I_{\geq 2}$ –
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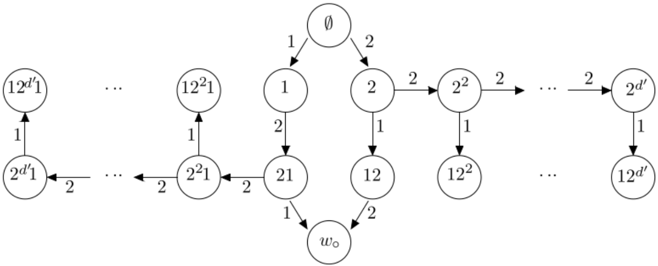
- An obvious positive answer is: $NC_A(1, d) := \Bbbk[T_1]/(T_1^d)$.
- This can be extended to $n = 2$: $NC_A(2, d) := \Bbbk\mathcal{B}_{A_2}/(T_1^2, T_2^d)$.




(In the figure, $d' = d - 1$.)

# The general type-$A$ algebra

**Theorem (K., *Trans. AMS* 2018 + *FPSAC* 2018)**

*For every $n \geq 1$ and $d \geq 2$, the type-$A$ generalized nil-Coxeter algebra*

$$NC_A(n,d) := \Bbbk \mathcal{B}_{A_n} / (T_1^2, \dots, T_{n-1}^2, T_n^d)$$

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- $\Bbbk$-rank: $n!(1+n(d-1))$.
- word basis:
  $\{T_w;\ T_w T_n^k T_{n-1} \cdots T_m \,|\, w \in S_n = W_{A_{n-1}},\ k \in [d-1],\ m \in [n]\}$.
- unique longest word, left/right primitive words, . . .

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- unique longest word, left/right primitive words, ...
- The "usual" length function $\ell$ extends to $NC_A(n,d)$, and its Hilbert–Poincaré series (in $q$) is
  $[n]_q!(1 + [n]_q[d-1]_q)$, where $[n]_q := \frac{q^n-1}{q-1}$, $[n]_q! := \prod_{j=1}^n [j]_q$.

# All finite-dimensional generalized nil-Coxeter algebras

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1. *The algebra $NC_M(\mathbf{d})$ is finite-dimensional (or of finite $\Bbbk$ rank).*
2. *Either $W = W(M)$ is a finite Coxeter group and all $d_i = 2$, or $W$ is of type $A_n$ and $\mathbf{d} = (2, \dots, 2, d)$ (or $(d, 2, \dots, 2)$) – i.e., $NC_M(\mathbf{d}) = NC_A(n, d)$.*

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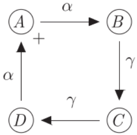
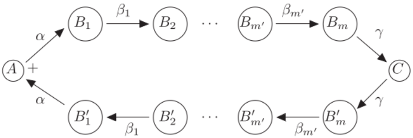
The proof uses a diagrammatic calculus:



# Complex reflection groups and the BMR Freeness Conjecture

The higher nilpotency $T_i^{d_i} = 0$ is reminiscent of *complex* reflection groups $W_{\mathbb{C}}$.

- These groups also have "Coxeter-like" presentations using nodes and edges / generators and relations. The finite groups $W_{\mathbb{C}}$ were classified by Shephard–Todd [*Canadian J. Math.* 1954].
- So, they also have *generic Hecke algebras* $\mathcal{H}_q(W_{\mathbb{C}})\dots$

# Complex reflection groups and the BMR Freeness Conjecture

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**Broué–Malle–Rouquier Freeness Conjecture (*Crelle* 1998)**

*Generic Hecke algebras $\mathcal{H}_q$ over $(W_{\mathbb{C}}, \mathbb{k})$ are free with $\mathbb{k}$-rank $|W_{\mathbb{C}}|$.*

(Proved by Etingof in 2017, in characteristic zero.)

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This was actually **Motivation 4** for us. And we showed (2018):

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*Upshot:* The novel family $NC_A(n, d)$ is strikingly unique, among all real and complex reflection groups!

Does it "occur in nature" (akin to the divided difference operators for $d = 2$)?

# Nil-Temperley–Lieb algebras

What next?

# Nil-Temperley–Lieb algebras

What next?
*Kill long enough braid words!*

The *Temperley–Lieb algebra* in types $A, D, E$ is defined as the quotient of the Iwahori–Hecke algebra by the ideal generated by

$$T_s T_t T_s = \text{lower degree terms}$$

for adjacent nodes $s, t$ in the Coxeter graph.

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## STRUCTURE OF A HECKE ALGEBRA QUOTIENT

C. KENNETH FAN

*Dedicated to my teacher, George Lusztig, on his fiftieth birthday*

### 1. Introduction

Let $W$ be a Coxeter group with Coxeter graph $\Gamma$. Let $\Gamma_g$ be the set of simple generators, which are parametrized by the nodes of $\Gamma$.

Our primary interest in this paper is to understand the case where $\Gamma$ is of type $E$. Therefore, we shall assume that $\Gamma$ is of type $A$, $D$, or $E$, where by $E$, we mean the infinite series $E_n$ which begins $E_5 = D_5$, $E_6$, $E_7$, $E_8$, $E_9 = \hat{E}_8$, etc.

Every $w \in W$ may be written as a product $s_1 s_2 s_3 \cdots s_n$ of generators in $\Gamma_g$. If $n$ is minimal, we call this product "reduced" and define $l(w) = n$. More generally, if $w = w_1 w_2 w_3 \cdots w_n$ satisfies $l(w) = \sum_i l(w_i)$, then we call this product "reduced" as well.

Let $\mathcal{H}$ be the Iwahori-Hecke algebra associated to $W$. This is an algebra over $\mathbb{Q}(q^{1/2})$ (where $q^{1/2}$ is an indeterminate) with generators $T_s$ for each $s \in \Gamma_g$ satisfying the relations $T_s^2 = (q-1)T_s + q$, $T_s T_t = T_t T_s$ if $st = ts$, and $T_s T_t T_s = T_t T_s T_t$ if $sts = tst$, where $s, t \in \Gamma_g$. This algebra has a basis $T_w$, $w \in W$, where we have $T_w = T_{s_1} \cdots T_{s_n}$ whenever $s_1 \cdots s_n$ is a reduced expression for $w$.

Let $\mathcal{I}$ be the two-sided ideal generated by the elements

$$T_{sts} + T_{st} + T_{ts} + T_s + T_t + 1$$

# Nil-Temperley–Lieb algebras

There is a nil-version: given a Coxeter group $W$ with data $(I, M)$, the *nil-Temperley–Lieb algebra* $NTL_W = NTL_M$ is the quotient of $\Bbbk\mathcal{B}_M$ by

- the "braid relations" $T_s T_t T_s = 0$ for adjacent nodes $s \sim t$;
- and the Coxeter relations $T_i^2 = 0$ for all $i \in I$.

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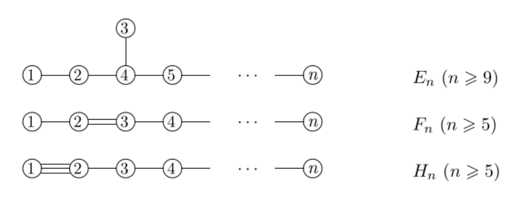
**Theorem (essentially due to Stembridge, C.K. Fan, 1990s)**

*$NTL_W$ has finite $\Bbbk$-rank if and only if $W$ is a finite Coxeter group, or $W$ has one of the following Coxeter graphs:*



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**Question:** In the *generalized* nil-Temperley–Lieb version, with relations $T_s T_t T_s = 0$ and $T_i^{d_i} = 0$,
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*If and only if*

1. *The algebras on the previous slide*;
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*In the second case,*

$$\dim NTL_A(n,d) = (d-1)C_{n+1} - (d-2)C_n + (d-2)\sum_{j=1}^{n-1} jC_{n-j}, \qquad (1.6)$$

*where $C_n$ is the $n$th Catalan number.*

[a] The words in $W$ for which switching between any two reduced expressions uses no non-commutative braid relations.

# Going beyond type $A$

Similarly, one can quotient by all braid words of length $\geq 4$ (but not the braid words $T_s T_t T_s$):

- In the simply-laced types $A, D, E$, this simply yields the "usual" nil-Coxeter algebras (since no "extra" quotienting is needed).

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Similar results if one quotients by the braid words of length $\geq 5$.

- There is exactly *one missing case*: $H_n$ for $n \geq 5$ (equivalently in the length $\geq 4$ case, because there is only one such pair of words: $T_1 T_2 T_1 T_2 T_1 = T_2 T_1 T_2 T_1 T_2$ – see the Figure on Slide 18).

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If instead one kills all braid words of length $\geq 6$, then since there were no such in any non-dihedral finite Coxeter group, we just get back the usual nil-Coxeter algebra (or the algebras $NC_A(n,d)$) – hence of finite rank.

# Table of findings (from the preprint)

Table of all finite-dimensional generalized nil-Temperley–Lieb algebras.
In it, $J_{<k}$ means we quotient by all braid words of length $\geq k$.

| $W$ | $J_{<3}$ | $J_{<4}$ | $J_{<5}$ | $J_{<k}$ $(6 \leqslant k \leqslant \infty)$ |
|---|---|---|---|---|
| $A_n, D_n, E_6, E_7, E_8$ | $\|W_{\rm fc}\|$ | $\|W\|$ | $\|W\|$ | $\|W\|$ |
| $B_n$ | $\|W_{\rm fc}\|$ | $\sum_{k=0}^{n} \binom{n}{k}^2 k!$ | $\|W\|$ | $\|W\|$ |
| $F_4$ | $\|W_{\rm fc}\|$ | 304 | $\|W\|$ | $\|W\|$ |
| $H_3$ | $\|W_{\rm fc}\|$ | 76 | 76 | $\|W\|$ |
| $H_4$ | $\|W_{\rm fc}\|$ | 1460 | 1460 | $\|W\|$ |
| $I_2(m)$ | $\|W\|$ if $m<3$, else $\|W_{\rm fc}\| = \|W\|-1$ | $\|W\|$ if $m<4$, else $\|W\|-1$ | $\|W\|$ if $m<5$, else $\|W\|-1$ | $\|W\|$ if $m<k$, else $\|W\|-1$ |
| $E_n$ $(n \geqslant 9)$ | $\|W_{\rm fc}\|$ | – | – | – |
| $F_n$ $(n \geqslant 5)$ | $\|W_{\rm fc}\|$ | – | – | – |
| $H_n$ $(n \geqslant 5)$ | $\|W_{\rm fc}\|$ | ? | ? | – |
| $A_n$, $\mathbf{d} = (d, 2, \dots, 2)$, $d>2$ | (see (1.6)) | $n!(1+n(d-1))$ | $n!(1+n(d-1))$ | $n!(1+n(d-1))$ |

# Open questions

1. Do the algebras $NC_A(n,d)$ "occur in nature"? How about the generalized $XYX$-algebras $NTL_A(n,d)$ with $d \geq 3$?

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3. For which $n \geq 5$ does the $H_n$ nil-Temperley–Lieb algebra become of finite $\mathbb{k}$-rank, when one quotients by braid words of length $\geq 4$ (equivalently, length $\geq 5$ – that is, $T_1T_2T_1T_2T_1$ and $T_2T_1T_2T_1T_2$) ?

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