Minimal surfaces & Labourie's Conjecture

Lecture 1: Equivariant harmonic maps

Lecture 2: Minimal surfaces + Labourie Conjecture

Lecture 3: Counterexamples to Labourie Conjecture

Equivariant harmonic maps

(M,μ), (N,ν) Riemannian manifolds.

$f: M \xrightarrow{C^2} N$, $df: TM \to TN$ interpreted as a section $df \in \Gamma(T^*M \otimes f^*TN)$.

μ, ν give rise to norm $|\cdot|_{\mu,\nu}$ and connection $\nabla = \nabla^{\mu,\nu}$ on $T^*M \otimes f^*TN$.

$|df|^2_{\mu,\nu} = tr_\mu f^*\nu$.

Defn: f is harmonic if $tr_\mu \nabla df = 0$.

M, N compact, f harmonic iff it's a critical point for the energy $\mathcal{E} = \frac{1}{2}\int_M |df|^2_{\mu,\nu} dV_\mu$

Basic examples:

- harmonic functions $\Omega^{\subset \mathbb{R}^n} \to \mathbb{R}^m$

- geodesics $[0,T] \to (N,\nu)$, more generally, totally geodesic maps
- holomorphic maps between Kähler manifolds
- Hopf fibrations $S^3 \to S^2$, $S^7 \to S^4$, etc

Thm. (Eells–Sampson, 1964) M, N compact, $\underbrace{K_N}_{\text{sectional curvature}} \le 0$. In every homotopy class of maps $M \to N$ $\exists$ a harmonic map $f: (M,\mu) \to (N,\nu)$

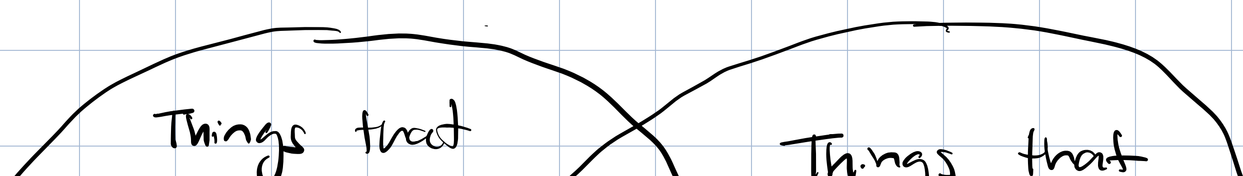
Proof by heat flow $\quad F = F(x,t)$

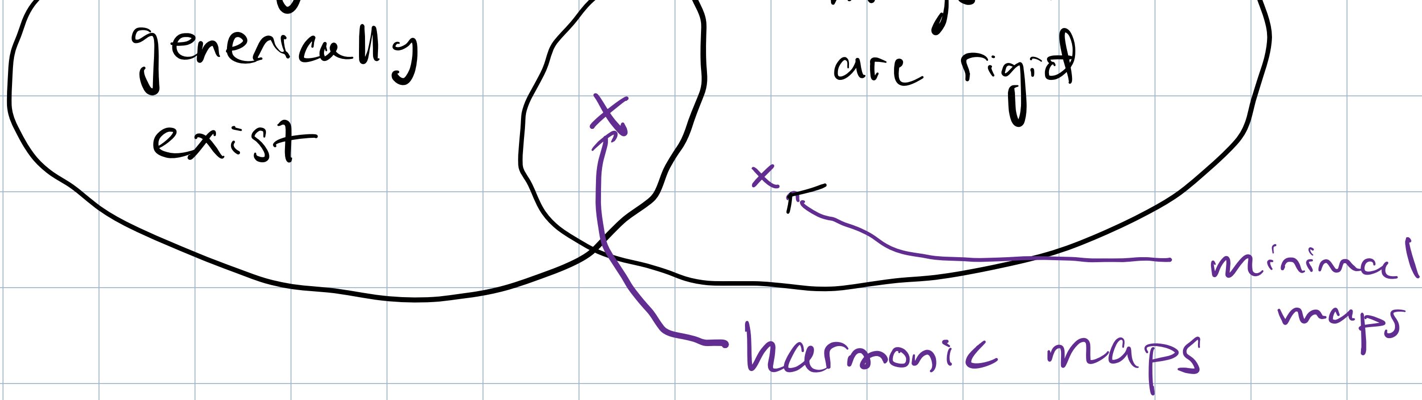
$$\frac{\partial F}{\partial t} = \mathrm{tr}_\mu \nabla dF.$$

Thm. (Hartman, Sampson 1967, 1968)

Above, if $K_N < 0$, then f is unique unless $f(M)$ is contained in a geodesic. Equivalently, $f_*(\pi_1 M)$ is abelian

$K_N \le 0$





Thm. (Siu, 1980) (M,μ), (N,ν) closed Kähler manifolds, $\dim_{\mathbb{C}} \geq 2$. Assume N complex hyperbolic. Then any degree 1 harmonic map $(M,\mu) \to (N,\nu)$ is a biholomorphism.

Corollary (Siu, 1980) (M,μ), (N,ν) as above. If $\pi_1 M$ is isomorphic to $\pi_1 N$ then M and N are biholomorphic or anti-biholomorphic.

Equivariant harmonic maps

$\rho : \pi_1 M \to \mathrm{Isom}(N,\nu)$, $\tilde{M}$ = universal cover of M, $\pi_1 M \curvearrowright \tilde{M}$ by Deck transformations, $\tilde{M}/\pi_1 M = M$.

Defn: $f : \tilde{M} \to (N,\nu)$ is ρ-equivariant if $\forall \gamma \in \pi_1 M$, $f \circ \gamma = \rho(\gamma) \circ f$.

<u>Ex.</u> N_0 manifold with universal cover N

$$\begin{tikzcd} \tilde{f}: \tilde{M} \arrow[r] \arrow[d] & (\tilde{N}, v) \arrow[d] \\ f: M \arrow[r] & (N_0, v) \end{tikzcd}$$

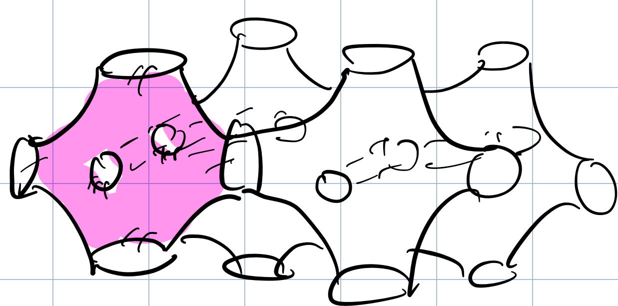
$\rho = f_* : \pi_1 M \to \pi_1 N_0 < \mathrm{Isom}(N, v)$

Σ_3

$$\begin{tikzcd} \Sigma_3 \arrow[r] & (S^1)^3 = \mathbb{R}^3/\mathbb{Z}^3 \\ \mathbb{H}^2 \arrow[u] \arrow[r] & \mathbb{R}^3 \arrow[u] \end{tikzcd}$$

$= \{ z \in \mathbb{C} : \mathrm{Re}\, z > 0 \}$

Schwarz P-surface 1880s



<u>Exercise</u>: f ρ-equivariant, $|df|^2_{\mu,v}$ is $\pi_1 M$-invariant and hence descends to M.

Thus, can define energy $\mathcal{E}(f) = \int_M |df|^2_{\mu,v}\, dV_\mu$

Now, assume $K_N \leq 0$, (N, v) complete and simply connected ($C.H.$ thm $\Rightarrow N \simeq_{top} \mathbb{R}^n$)

<u>Ex.</u> $(N, v) = (\mathbb{H}^n, x_n^{-2} \sum_i dx_i^2)$, $K_N = -1$.

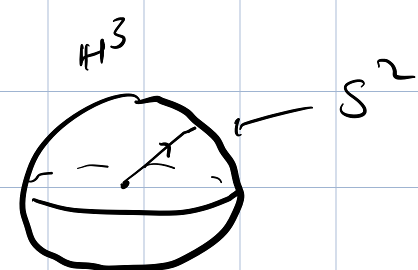
$\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$

<u>Def'n</u>: Fix $O \in N$. Geodesic rays

$\gamma_1, \gamma_2 : [0, \infty) \to (N, v)$, $\gamma_i(0) = O$, are equivalent if $\forall t$, $d(\gamma_1(t), \gamma_2(t)) \leq K$ (some $K > 0$). An equivalence class is called an endpoint.

The Gromov boundary is the set of endpoints of geodesics rays.

<u>Ex.</u> $\partial_\infty \mathbb{H}^n = S^{n-1}$



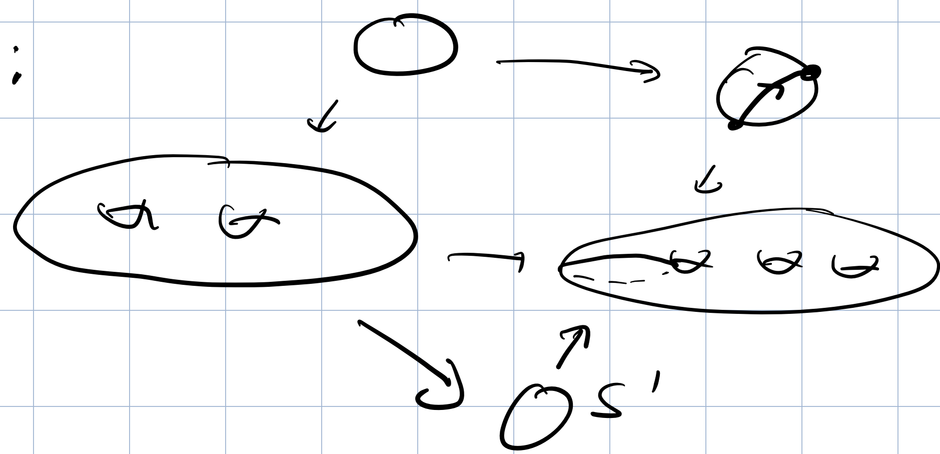
Any isometry of (N, v) extends to a bijection of $\partial_\infty N$.

Def'n: $\rho : \pi_1 M \to \mathrm{Isom}(N, v)$ is irreducible if $\forall \xi \in \partial_\infty N$ $\exists \gamma \in \pi_1 M$ s.t. $\rho(\gamma)\xi \neq \xi$.

Ex. $K_{N_0} < 0$

$M \to N_0$
closed manifolds

Non-example:



M compact

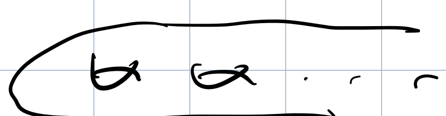
Thm. (Donaldson, Corlette, Labourie 1986, 1988, 1991)
(N, v) as above, ρ irreducible. Then
$\exists !$ ρ-equiv. harmonic map $(\tilde{M}, m) \to (N, v)$.

Proof by heat flow.

Can generalize to non-compact situations.

Non-compact surfaces: Wolf, Simpson, Jost, Gupta, S. - 2019, Gupta - ?

Totally open: equivariant harmonic maps for infinite type surfaces



See Schoen Conjecture, Markovic, Benoist – Hulin,

Associated bundles: out of ρ we have

$$N_\rho = \tilde{M} \times_\rho N = \{(p,x) \in \tilde{M} \times N\} \big/ _{(p,x) \sim (\gamma \cdot p, \rho(\gamma)x)}$$

$$\downarrow$$

$$M$$

Comes with a flat connection D by taking exterior derivative in each triv.

$$\begin{tikzcd} \pi^* N_\rho \arrow[r] \arrow[d] & N_\rho \arrow[d] \\ \tilde{M} \arrow[r, "\pi"] \arrow[u, bend left, "\pi^* s"] & M \arrow[u, bend right, "s"'] \end{tikzcd}$$

$\tilde{M}$ contractible,

$$\pi^* N_\rho \simeq \tilde{M} \times N$$

$$\pi^* s(p,x) = (p, f(x))$$

for some ρ-equiv. $\tilde{M} \to N$.

Harmonic maps from Riemann surfaces

Metrics μ, μ' on M are conformally equivalent if $\exists\, v : M \to \mathbb{R}$ s.t. $\mu' = e^v \mu$.
From now on, M is a closed surface, genus $g \geq 2$, Σ_g. A conformal class of metrics on Σ_g is equivalent to

a Riemann surface structure S on Σ_g.

(By Beltrami eqn, can find cdt z s.t. $\mu = \mu_0(z)|dz|^2$)

Exercise: $f:(\Sigma_g, \mu) \to (N, \nu)$, $v: \Sigma_g \to \mathbb{R}$, $\mathcal{E}(f)$ is the same if we replace μ with $e^v \mu$.

$\Longrightarrow$ Harmonic maps depend only on the conformal class of μ, or equivalently the Riemann surface structure.

Henceforth, we just specify R.S. S.

Complex geometry:

Warm-up: harmonic functions

$$\left\{ \begin{array}{c} \text{harmonic} \\ f: \mathbb{C} \to \mathbb{R} \end{array} \right\} \Big/ \text{trans.} \longleftrightarrow \left\{ \begin{array}{c} \text{holomorphic} \\ \phi: \mathbb{C} \to \mathbb{C} \end{array} \right\}$$

$$\frac{\partial^2 f}{\partial \bar{z} \partial z} = 0 \qquad f \longmapsto \partial f / \partial z \qquad \frac{\partial \phi}{\partial \bar{z}} = 0$$

$$\phi \longmapsto f(z) = \int^z \operatorname{Re} \phi(\zeta)\, d\zeta$$

$\int_{z_0} \mathrm{Re}\,\varphi(\zeta)\,d\zeta$.

Riemann surfaces : $S \to (N, v)$

$$T^*S^{\mathbb{C}} = \underset{dz}{(T^*S)^{1,0}} \oplus \underset{d\bar z}{(T^*S)^{0,1}}$$

Split $df = \underset{f_z dz}{\partial f} + \underset{f_{\bar z} d\bar z}{\bar\partial f}$, $\nabla = \nabla^{1,0} + \nabla^{0,1}$

<u>Exercise</u>: f is harmonic iff $\nabla^{0,1}\partial f = 0$

$$\partial f \in (T^*S)^{1,0} \otimes_{\mathbb{C}} f^*TN^{\mathbb{C}}$$

<u>Thm.</u> (Koszul–Malgrange) Given a complex v. bundle E over a complex manifold M, with an operator $\bar\partial_E : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ satisfying the $\bar\partial$-Liebniz rule, if $\bar\partial_E^2 = 0$, then $\exists$ holomorphic v. bundle structure on E s.t. $\bar\partial_E$ is the del-bar operator.

Del-bar operator: $E \to M$ hol. v. bundle, $s_1, \dots, s_n$ local frame of hol. sections,

$\partial_F(\sum_i f_i s_i) = \sum_i \partial f_i \otimes s_i)$

Upshot: $\nabla^{0,1}$ induces hol. structure on $f^*TN^{\mathbb{C}}$ in which ∂f is a hol-$f^*TN^{\mathbb{C}}$-valued 1-form.

<u>Harmonic maps from surfaces to symmetric spaces</u>

$$X_n^{\mathbb{C}} = SL(n,\mathbb{C})/SU(n) = \{A \in SL(n,\mathbb{C}) : A = \bar{A}^T, A > 0\}$$

$$= \{\text{Hermitian metrics on } \mathbb{C}^n \text{ inducing } 1 \text{ on } \Lambda^n\mathbb{C}^n\}$$

$$X_n \subset X_n^{\mathbb{C}}, \quad X_n = SL(n,\mathbb{R})/SO(n,\mathbb{R})$$

$$= \{A \in SL(n,\mathbb{R}) : A = A^T, A > 0\}$$

$$= \{\text{Inner products on } \mathbb{R}^n \text{ inducing } 1 \text{ on } \Lambda^n\mathbb{R}^n\}$$

$$T_H X_n^{\mathbb{C}} = \{A \in M(n,\mathbb{C}) : A = \bar{A}^T, H^{-1}A \text{ traceless}\}$$

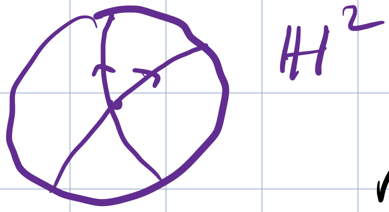
Metric v on $X_n^{\mathbb{C}}$: $v_I(A,B) = \frac{n}{2}\operatorname{tr}(AB)$

$SL(n,\mathbb{C})$-invariant: $v_H(A,B) = \frac{n}{2}\operatorname{tr}(H^{-1}AH^{-1}B)$.

For $n=2$, $X_n = \mathbb{H}^2$, $X_n^{\mathbb{C}} = \mathbb{H}^3$

- $\forall n$, $K_{X_n^{\mathbb{C}}} \leq 0$
- $X_n^{\mathbb{C}}$ complete, simply connected
- through each point in $X_n^{\mathbb{C}}$ $\exists$

$(n-1)$-dimensional flat subspaces



Ex. At Id, can take real diagonal matrices.

Flatness: $R(X,Y)Z = -[[X,Y],Z]$.

$\rho : \pi_1 \Sigma_g \to SL(n,\mathbb{C}) \curvearrowright X_n^{\mathbb{C}}$ by isometries

irreducible iff composition of ρ w/ $ad : SL(n,\mathbb{C}) \to \mathfrak{sl}(n,\mathbb{C})$ totally reducible with finite centralizer.

$\tilde{S} \longrightarrow X_n^{\mathbb{C}}$ ρ-equiv., ρ irreducible

1) $E_\rho = \tilde{S} \times_\rho \mathbb{C}^n$ with flat connection D.

2) $(X_n^{\mathbb{C}})_\rho = \tilde{S} \times_\rho X_n^{\mathbb{C}} = \mathrm{Met}_1(E) =$ Hermitian metrics on E_ρ inducing 1 on $\Lambda^n E_\rho$.

An equivariant map $f : \tilde{S} \to X_n^{\mathbb{C}}$ is equivalent to a Hermitian metric

H on E.

3) $\mathfrak{sl}(n,\mathbb{C})$-valued 1-form $\omega = -\frac{1}{2} H^{-1} dH$ induces an iso. between $f^* T X_n^{\mathbb{C}} \to S$ and the space $\mathrm{End}_0^H(E)$ of H-self adjoint traceless endomorphisms of E.

$T = T^{*_H} = H^{-1} \bar{T}^{T} H$.

Derivative of f on H is $\Psi_H \in \mathrm{End}_0^H(E)$.

Note $\mathrm{End}_0^H(E)^{\mathbb{C}} = \mathrm{End}_0(E) =$ traceless endomorphisms.

Define connection on E by

$$\nabla_H = D - \Psi_H, \text{ extends to } \mathrm{End}_0^H(E).$$

<u>Exercise</u>: ∇_H on $\mathrm{End}_0^H(E)$ is the pullback of the L.C. connection on $X_n^{\mathbb{C}}$.

Decompose $T^*S^{\mathbb{C}} = (T^*S)^{1,0} \oplus (T^*S)^{0,1}$,

$\Psi_H = \Psi_H^{1,0} + \Psi_H^{0,1}$. <u>Rmk.</u> $\Psi_H^{0,1} = (\Psi_H^{1,0})^{*_H}$

f harmonic iff $\nabla^{0,1} \partial f = 0$

iff $\nabla_H^{0,1}\psi_H^{1,0}=0$.

KM thm $\Rightarrow$ $\nabla_H^{0,1}$ induces complex structure on E.

deg $E = 0$

<u>Defn</u>: A $SL(n,\mathbb{C})$-Higgs bundle $(E,\bar\partial_E,\phi)$ on S is a hol. v. bundle $(E,\bar\partial_E)\to S$ with $\phi\in\Omega^{1,0}(\mathrm{End}\,E)$ s.t. $\bar\partial_E\phi=0$ called the Higgs field.

Equivariant harmonic map $\tilde{S}\to X_n^{\mathbb{C}}$ gives rise to a Higgs bundle on S, $(E_\rho,\nabla_H^{0,1},\psi_H^{1,0})$

Flatness of D + holomorphicity of $\psi_H^{1,0}$ is expressed via Hitchin's self-duality eqns $F(\nabla_H)+[\psi_H^{1,0},(\psi_H^{1,0})^{*_H}]=0$.

<u>Higgs bundles</u> $(E,\bar\partial_E,\phi)$, when does it come from a harmonic map.

Given $(E,\bar\partial_E)$, Hermitian metric

H on E, $\exists!$ connection ∇_H, Chern connection, s.t. $\nabla_H H = 0$, $\nabla_H^{0,1} = \bar{\partial}_E$.

We want to find H s.t.

$$F(\nabla_H) + [\phi, \phi^{*H}] = 0 . \qquad (*)$$

$\Rightarrow$ $D = \nabla_H + \phi + \phi^{*_H}$ is flat, get holonomy rep ρ, for which H induces a ρ-equivariant map

<u>Def'n</u>: $(E, \bar{\partial}_E, \phi)$ is stable if for any ϕ-inv. hol. proper subbundle $F \subset E$, $\deg F < 0$.

<u>Thm.</u> (Hitchin 1986, Simpson 1988)
$(E, \bar{\partial}_E, \phi)$ is stable and has no non-trivial automorphisms (simple) iff one can find unique H solving S.D. eqn's $(*)$.

Non-abelian Hodge correspondence

$$\begin{tikzcd} \left\{ \rho : \pi_1 \Sigma_g \to SL(n,\mathbb{C}) \text{ irreducible} \right\} /\!/ \text{conj.} \arrow[r, leftrightarrow] \arrow[r, bend right, "\text{harmonic map D.C.L.}"'] & \left\{ \text{stable } SL(n,\mathbb{C})\text{-Higgs bundles} \right\} /\!/ \text{iso.} \arrow[l, bend right, "\text{S.D. Hitchin, Simpson}"'] \end{tikzcd}$$

Rmk. Higgs bundle perspective on harmonic maps to compact Lie groups/symmetric spaces.
Hitchin: harmonic maps from a 2-torus to the 3-sphere, chapter 1

Lecture 2: Minimal surfaces and Labourie's Conjecture.

Teichmuller space Σ_g closed oriented surface, $g \geq 2$.

Def'n: Teichmüller space T_g is the space of equivalence classes of pairs $[S, f]$, S = Riemann surface on Σ_g, $f : \Sigma_g \to S$ o.p. diffeomorphism.

$[S_1, f_1] \sim [S_2, f_2)$ if $f_1 \circ f_2^{-1}$ is a biholomorphism isotopic to id.

Uniformization thm. Every simply connected R.S. is $\mathbb{CP}^1, \mathbb{C}, \mathbb{H}^2$.

$S \simeq_{top} \Sigma_g$, then $\tilde{S} = \mathbb{H}^2$ since

$\mathrm{Aut}(\mathbb{H}^2) = PSL(2, \mathbb{R})$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \dfrac{az+b}{cz+d}$

$\exists$ discrete subgroup $\Gamma < PSL(2, \mathbb{R})$ s.t.

$S \simeq \mathbb{H}^2 / \Gamma$.

Two descriptions of T_g:

- $$\begin{array}{ccc} \tilde{f} : \tilde{\Sigma}_g & \longrightarrow & \tilde{S} \\ \downarrow & & \downarrow \\ f : \Sigma_g & \longrightarrow & S = \mathbb{H}^2/\Gamma \end{array}$$ $\tilde{f}$ is f_*-equivariant

  $[S, f]$ is equiv. to a discrete + faithful rep (Fuchsian rep) $\rho : \pi_1 \Sigma_g \to PSL(2,\mathbb{R})$

up to conjugation.

- Since $\mathbb{H}^2$ carries a $PSL(2,\mathbb{R})$-inv. hyp. metric, T_g is also $[\mu, f]$, where μ is a hyp. metric on Σ_g, $f : \Sigma_g \to (\Sigma_g, \mu)$, equiv. relation analogous to above.

$$MCG(\Sigma_g) = \mathrm{Diff}^+(\Sigma_g) / \mathrm{Diff}_0(\Sigma_g) \curvearrowright T_g$$

$[\varphi] \cdot [S, f] = [S, f \circ \varphi^{-1}]$, properly discontinuous, $T_g / MCG = \mathcal{M}_g$ moduli space

Harmonic maps and T_g

Fix R.S. S on Σ_g, with conformal metric $\mu = \mu_0(z) |dz|^2$.

Let (N, ν) be Riemannian manifold.

$f^* \nu$: extend to $TS^{\mathbb{C}}$, decompose into $(1,1)$, $(2,0)$, $(0,2)$ components.

$|dz|^2 \quad dz^2 \quad d\bar{z}^2$

$$f^*v = v(\partial f, \bar\partial f) + v(\partial f, \partial f) + v(\bar\partial f, \bar\partial f)$$
$$= |df|^2_{\mu, v}\, \mu + q(f) + \bar q(f)$$

f harmonic $\Rightarrow \bar\partial q(f) = \bar\partial v(\partial f, \partial f)$
$$= 2v(\nabla^{0,1}\partial f, \partial f) = 0$$

K canonical bundle of S, $K^i = K^{\otimes i}$,

$H^0(S, K^i)$ = holomorphic sections.

$q(f) \in H^0(S, K^2)$.

<u>Defn</u>: $q(f)$ is the Hopf differential of f.

<u>Rmk.</u> For $N = X_n^{\mathbb{C}} = SL(n, \mathbb{C})/SU(n)$, get Higgs bundle $(E, \bar\partial_E, \phi)$ out of f.
$$\frac{n}{2}\operatorname{tr}(\phi^2) = q(f).$$

Now, take $(N, v) = (\Sigma_g, v)$, v hyp.
E.S. or D.C.L. $\exists!$ harmonic $f_v : S \to (\Sigma_g, v)$ homotopic to id. Get $q(f_v)$.

$[v, Id] \mapsto q(f_v)$ descends to a

map $\quad \mathcal{T}_g \longrightarrow H^0(S, K^2)$

hyperbolic model

<u>Thm.</u> (Wolf 1984/1989, Hitchin 1986, Wan)

The map above is a homeomorphism.

Hitchin's proof uses Higgs bundles.

<u>Hitchin representations</u> $\forall n$, $\mathrm{Rep}(\Sigma_g, SL(n,\mathbb{R}))$

$= \mathrm{Hom}(\pi_1\Sigma_g, SL(n,\mathbb{R}))/SL(n,\mathbb{R})$.

$\mathcal{T}_g$ in rep. model is a connected

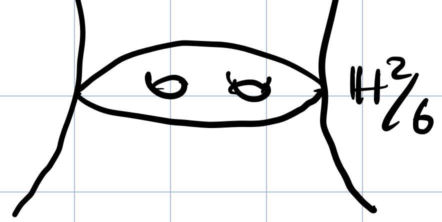
component of $\mathrm{Rep}(\Sigma_g, SL(2,\mathbb{R}))$.

$\exists$ irrep $\iota_n : SL(2,\mathbb{R}) \hookrightarrow SL(n,\mathbb{R})$.

$\sigma: \pi_1\Sigma_g \to SL(2,\mathbb{R})$ Fuchsian,

consider $\iota_n \circ \sigma : \pi_1\Sigma_g \to SL(n,\mathbb{R})$

$) \qquad (\; X_n/$



$\iota_n \circ \sigma$

Def'n: Hitchin component

$Hit(\Sigma_g, n)$ is a connected component of $Rep(\Sigma_g, SL(n,\mathbb{R}))$ containing $\iota_n \circ \sigma$.

Consists of classes of Hitchin representations

$SL(n,\mathbb{C})$-Higgs bundles: $(E, \bar\partial_E, \phi)$,

$(E, \bar\partial_E) \to S$ rank n deg 0 hol. vector bundle, $\phi \in \Omega^{1,0}(\mathrm{End}\, E)$, $\bar\partial_E \phi = 0$.

$$\mathcal{M}_H(n) = \left\{ \begin{matrix} \text{rank } n \\ \text{stable, simple} \\ (E, \bar\partial_E, \phi) \end{matrix} \right\} \Big/ \text{iso.}$$

moduli space of Higgs bundles

$$\downarrow$$

$$\bigoplus_{i=2}^{n} H^0(S, K^i)$$

$q \in H^0(S, K^i)$
$q = q(z)\, dz^i$

Informally, compute characteristic polynomial of ϕ

$\det(\lambda I - \phi) = \lambda^n + \lambda^{n-2} q_2 + \cdots + \lambda q_{n-1} + q_n$

$q_i \in H^0(S, K^i)$. No q_1 since last time we imposed $\operatorname{tr} \phi = 0$.

Rmk. q_2 is a scalar multiple of $\operatorname{tr} \phi^2$.

Thm. (Hitchin, 1990) $\forall n$ $\exists$ section $s: \bigoplus_{i=2}^{n} H^0(S, K^i) \to \mathcal{M}_H(n)$ whose image under NAH is $\mathrm{Hit}(\Sigma_g, n)$.

$T_g \hookrightarrow \mathrm{Hit}(\Sigma_g, n)$ is described by

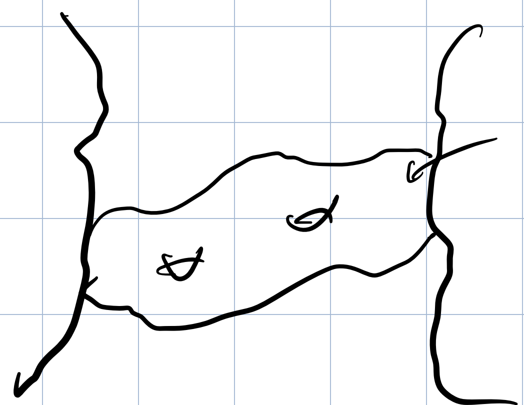
$$\{ s(q_2, 0, \ldots, 0) : q_2 \in H^0(S, K^2) \}.$$

Goldman + Choi-Goldman (late 80s / early 90s)

$\mathrm{Hit}(\Sigma_g, 3)$ parametrizes convex $\mathbb{RP}^2$-structures on Σ_g.

Thm. (Labourie 2006) Hitchin reps are Anosov (hence discrete + faithful)

ρ Hitchin $\quad X_n/\rho$



More developments: Labourie, Guichard-Wienhard, etc.

Question: ρ Hitchin, $f: \tilde{S} \to X_n$ ρ-equiv. harmonic map. Is f is an immersion?

Labourie Conjecture

Thm. (Labourie 2006/2008) $MCG \curvearrowright Hit(\Sigma_g, n)$ prop. discontinuously.

However, no good MCG-action on $\bigoplus_{i=2}^{n} H^0(S, K^i)$

Proposal: use minimal maps.

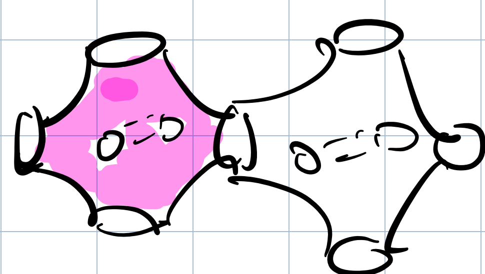
$\rho : \pi_1 \Sigma_g \longrightarrow \mathrm{Isom}(N, \nu)$.

$f : \tilde{\Sigma}_g \to N$ ρ-equiv.,

$$A(f) = \int_{\Sigma_g} \det f^* \nu.$$

area of image of fundamental domain for $\tilde{\Sigma}_g \to \Sigma_g$.

<u>Ex.</u> $\mathbb{H}^2 \longrightarrow \mathbb{R}^3$ Schwarz - P



$A(P)$ = area of pink

<u>Def'n</u>: The image of f is a minimal surface if f is a critical point for A.

Well-known facts

(1) $f : \tilde{S} \to (N, \nu)$ harmonic, then image of f is minimal iff f is (weakly) conformal iff $f^*\nu = |df|^2_{\mu,\nu}\,\mu$,

i.e., $q(f) = 0$.

In this case, call f a minimal map.

Conversely, every minimal surface is the image of a minimal map.

<u>Ex.</u> Schwarz P-surface is minimal.

Set of minimal maps for Hitchin reps w/ underlying R.S. S is parametrized by $(0, q_3, \ldots, q_n) \in \bigoplus_{i=2}^{n} H^0(S, K^i)$.

$$\mathcal{M}_n(\Sigma_g) \longrightarrow \mathcal{T}_g, \quad \text{hol. v. bundle}$$

$$\mathcal{M}_n(\Sigma_g)\big|_{[S,f]} = \bigoplus_{i=3}^{n} H^0(S, K^i)$$

$= \left\{ \begin{array}{l} \text{Space of minimal} \\ \text{surfaces for Hitchin reps} \end{array} \right\}$

Holonomy map

$$L_n : M_n(\Sigma_g) \longrightarrow \mathrm{Hit}(\Sigma_g, n)$$

$MCG(\Sigma_g)$ - equiv.

<u>Labourie Conjecture</u>: ρ Hitchin rep.

$\exists!$ ρ-invariant minimal surface in X_n.

<u>Thm.</u> (Labourie 2006/2008) Existence always holds.

$\Rightarrow$ L_n is surjective.

Conjecture is about uniqueness. When true for given n, L_n is bijection.

<u>Thm.</u> (Labourie 2007) Uniqueness for $n=3$.

See also Loftin ($\sim$2000)

$$\mathcal{M}_H(n) \longrightarrow \bigoplus^{n} H^0(S, K^i)$$

$i=2$

Conjectures related to spectral data:

Katzarkov - Noll - Pandit - Simpson

Labourie's Existence thm.

(N, ν) complete s.c., $K_N \leq 0$,

$\rho : \pi_1 \Sigma_g \to \mathrm{Isom}(N, \nu)$ s.t. $\forall$ R.S.

S on Σ_g $\exists!$ harmonic map f_S^ρ.

Energy function $E_\rho : \mathcal{T}_g \to (0, \infty)$

$$E_\rho = \mathcal{E}(f_S^\rho) = \frac{1}{2} \int_S |df_S^\rho|^2_{\mu, \nu} \, dV_\mu$$

Computation of dE_ρ shows

$[S, id]$ is a critical point iff

$q(f_S^\rho) = 0$ iff f_S^ρ minimal.

(Douglas? Wentworth 2007)

N_0 = closed manifold, $\tilde{N_0} = N$.

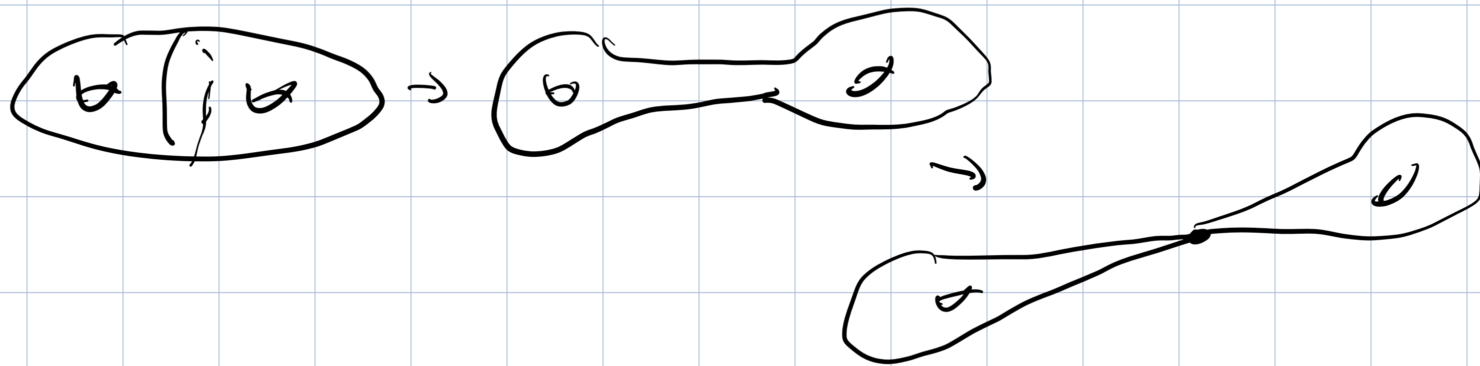
$f : \Sigma_g \to N_0$ is incompressible if $\forall$ simple closed curves $\gamma \in \pi_1 \Sigma_g$, $f_*([\gamma]) \neq 0$.

<u>Thm.</u> (Schoen–Yau 1979). $f : \Sigma_g \to (N_0, \nu)$ incompressible, then E_{f_*} is proper.

Proper: $\forall K > 0$, $E_{f_*}^{-1}([0, K]) \subset T_g$ is compact. Hence we can minimize E_{f_*}.

<u>Proof idea</u>

Going to ∞ in T_g via pinching curve



<u>Keen's Collar Lemma</u>: γ s.c.c. on hyp. surface $(\Sigma_g, \ )$,

$\ell_\mu(\gamma)$ = length of geodesic homotopic to γ, $\exists$ collar C_γ around γ conformally equivalent to $[0, L_{\mu,\gamma}] \times [0,1]$, $L_{\mu,\gamma} \gtrsim \ell_\mu(\gamma)^{-1}$.

<u>Lemma</u>: $f : \Sigma_g \to (N_0, \nu)$,

$$\int_{C_\gamma} |df|^2_{\mu,\nu}\, dV_\mu \geq L_{\mu,\gamma}\, \ell_\nu(f(\gamma))^2$$

geodesic length in (N, r)

<u>Proof</u>: Conformal invariance of energy,

$$\int_{C_\gamma} |df|^2_{\mu,\nu}\, dV_\mu = \int_0^{L_{\mu,\gamma}} \int_0^1 |df(\partial_x)|_\nu^2 + |df(\partial_y)|_\nu^2\, dx\, dy$$

$$\geq \int_0^{L_{\mu,\gamma}} \int_0^1 |df(\partial_y)|_\nu^2\, dx\, dy$$

$$\geq L_{\mu,\gamma} (\ell_\nu(f(\gamma)))^2 \text{ by C.S.}$$

By compactness $\exists\, \varepsilon > 0$ s.t. $\forall$ s.c.c. γ, $\ell_\nu(f(\gamma)) \geq \varepsilon$, If $E_\rho(S) \leq K$

$$K \geq \int_{C_\gamma} |df|^2_{u,v} \, dV_u \geq L_{u,\gamma} \varepsilon^2$$

$\Rightarrow$ $L_{u,\gamma} \leq K \varepsilon^{-2} \Rightarrow \ell_u(\gamma) \geq C.$

<u>Mumford compactness</u>: in a fundamental domain for $MCG \curvearrowright T_g$, pinching curves is the only way to go to ∞.

Finish proof: deal w/ MCG-action.

Back to Labourie's existence for Hitchin reps.

(N, ν) complete s.c., $K_N \leq 0$, $\rho : \pi_1 \Sigma_g \to \mathrm{Isom}(N, \nu)$.

$$\ell(\rho\gamma) = \inf_{x \in N} d_\nu(x, \rho(\gamma) x)$$

<u>Defn</u>: ρ is well-displacing if for any hyp. metric ν on Σ_g, $\exists$ $A, B > 0$ s.t. $\forall \gamma \in \pi_1 \Sigma_g$,

$$\ell(\rho(\gamma)) \geq A \, \ell_\nu(\gamma) - B$$

<u>Thm.</u> If ρ is well-displacing, E_ρ is proper.

<u>Thm.</u> Hitchin reps. are well-displacing

Proof is an adaptation of SY proof.

See also my notes on webpage on Labourie's conjecture. Gave new and easier proof of properness.

<u>Uniqueness in rank 2</u>

Hitchin reps are defined $\forall$ split real simple Lie groups

<u>Labourie</u>: Uniqueness $\forall$ Hitchin reps in rank 2.

$SL(3,\mathbb{R})$, $Sp(4,\mathbb{R})$, G_2'

<u>Idea</u>: Special symmetry in Higgs bundles cyclic Higgs bundles

$\Rightarrow$ minimal surfaces lift to curves in some bundle over symmetric space that are "J-holomorphic."

Shows such curves have no infinitesimal variations.

<u>Question/Problem</u>: Understood the proof.

<u>Question</u>: Are minimal surfaces

for Hitchin reps w/ cyclic Higgs bundles stable?