ICTS Bengaluru: Quantum Trajectories January 2025

Slides based on Lecture Notes with Mazyar Mirrahimi and on
A tutorial introduction to quantum stochastic master equations based on the qubit/photon system,
Annual Reviews in Control 54, 252-261 (2022)

pierre.rouchon@minesparis.psl.eu
Quantic research team
Laboratoire de Physique de l'Ecole Normale Supérieure,
Mines Paris-PSL, Inria, ENS-PSL, Université PSL, CNRS.

Outline

1. Schrödinger ($\hbar = 1$): wave funct. $|\psi\rangle \in \mathcal{H}$, density op. $\rho \sim |\psi\rangle\langle\psi|$

$$\frac{d}{dt}|\psi\rangle = -i\mathrm{H}|\psi\rangle, \quad \mathrm{H} = \mathrm{H}_0 + u\mathrm{H}_1 = \mathrm{H}^\dagger, \quad \frac{d}{dt}\rho = -i[\mathrm{H}, \rho].$$

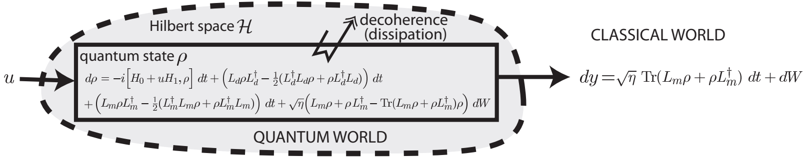
2. Origin of dissipation: collapse of the wave packet induced by the measurement of $\mathrm{O} = \mathrm{O}^\dagger$ with spectral decomp. $\sum_y \lambda_y \mathrm{P}_y$:
   - measurement outcome y with proba. $\mathbb{P}_y = \langle\psi|\mathrm{P}_y|\psi\rangle = \mathrm{Tr}\left(\rho \mathrm{P}_y\right)$ depending on $|\psi\rangle$, ρ just before the measurement
   - measurement back-action if outcome y:

$$|\psi\rangle \mapsto |\psi\rangle_+ = \frac{\mathrm{P}_y|\psi\rangle}{\sqrt{\langle\psi|\mathrm{P}_y|\psi\rangle}}, \quad \rho \mapsto \rho_+ = \frac{\mathrm{P}_y \rho \mathrm{P}_y}{\mathrm{Tr}\left(\rho \mathrm{P}_y\right)}$$

3. Tensor product for the description of composite systems (S, C):
   - Hilbert space $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_c$
   - Hamiltonian $\mathrm{H} = \mathrm{H}_s \otimes \mathrm{I}_c + \mathrm{H}_{sc} + \mathrm{I}_s \otimes \mathrm{H}_c$
   - observable on sub-system C only: $\mathrm{O} = \mathrm{I}_s \otimes \mathrm{O}_c$.

[1] S. Haroche and J.M. Raimond (2006). *Exploring the Quantum: Atoms, Cavities and Photons.* Oxford Graduate Texts.

Diffusive stochastic master equation[2]



$t \mapsto \rho_t$ continuous time function (not differentiable), solution of

$$d\rho_t = -i\left[H_0 + uH_1, \rho_t\right]\,dt + \left(\sum_{\nu=d,m} L_\nu\rho_t L_\nu^\dagger - \tfrac{1}{2}(L_\nu^\dagger L_\nu\rho_t + \rho_t L_\nu^\dagger L_\nu)\right)\,dt + \dots$$

$$\dots + \sqrt{\eta}\left(L_m\rho_t + \rho_t L_m^\dagger - \mathrm{Tr}(L_m\rho_t + \rho_t L_m^\dagger)\rho_t\right)\,dW_t,$$

where $\eta \in [0,1]$ and the same Wiener process W_t is shared by the state dynamics and the output map

$$dy_t = \sqrt{\eta}\ \mathrm{Tr}(L_m\rho_t + \rho_t L_m^\dagger)\ dt + dW_t.$$

---

[2] A. Barchielli and M. Gregoratti. *Quantum Trajectories and Measurements in Continuous Time: the Diffusive Case*. Springer Verlag, 2009.

$t \mapsto \rho_t$ piece wise smooth time function, solution of

$$d\rho_t = \left(-i[\mathrm{H},\rho_t] + \mathrm{V}\rho_t\mathrm{V}^\dagger - \tfrac{1}{2}(\mathrm{V}^\dagger\mathrm{V}\rho_t + \rho_t\mathrm{V}^\dagger\mathrm{V})\right)\, dt$$
$$+ \left(\frac{\bar\theta\rho_t + \bar\eta\mathrm{V}\rho_t\mathrm{V}^\dagger}{\bar\theta + \bar\eta\ \mathrm{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)} - \rho_t\right)\left(dy_t - \left(\bar\theta + \bar\eta\ \mathrm{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)\right) dt\right)$$

where $\bar\theta \geq 0$ (dark count rate) and $\bar\eta \in [0,1]$ (detection efficiency) and where the counting detector outcome $dy_t \in \{0,1\}$ with

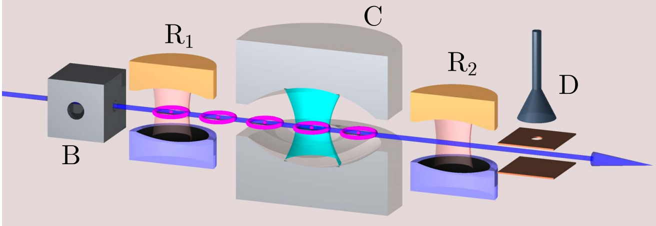
- $dy_t = 0$ with probability $1 - \left(\bar\theta + \bar\eta\ \mathrm{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)\right) dt$ and then

$$\rho_{t+dt} = \rho_t + \Big(- i[\mathrm{H},\rho_t] + \mathrm{V}\rho_t\mathrm{V}^\dagger - \tfrac{1}{2}(\mathrm{V}^\dagger\mathrm{V}\rho_t + \rho_t\mathrm{V}^\dagger\mathrm{V})$$
$$+ \bar\eta\Big(\ \mathrm{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)\rho_t - \mathrm{V}\rho_t\mathrm{V}^\dagger\Big)\Big)\, dt$$

- $dy_t = 1$ with probability $\left(\bar\theta + \bar\eta\ \mathrm{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)\right) dt$, and then

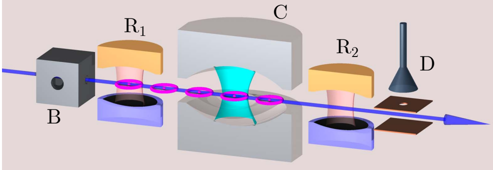
$$\rho_{t+dt} = \frac{\bar\theta\rho_t + \bar\eta\mathrm{V}\rho_t\mathrm{V}^\dagger}{\bar\theta + \bar\eta\ \mathrm{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)}.$$

[3] see, e.g., J. Dalibard, Y. Castin, and K. Mølmer. Wave-function approach to dissipative processes in quantum optics. *Phys. Rev. Lett.*, 68(5):580–583, February 1992.



- Dispersive qubit/photon interaction: $\mathbf{H}_{int} = -\chi\big(|e\rangle\langle e| - |g\rangle\langle g|\big) \otimes \mathbf{n}$ (with χ a constant parameter) yields $e^{-iT\mathbf{H}_{int}}$, the Schrödinger propagator during the time $T > 0$, given with $\theta = \chi T$ by
$$\mathbf{U}_\theta = |g\rangle\langle g| \otimes e^{-i\theta\mathbf{n}} + |e\rangle\langle e| \otimes e^{i\theta\mathbf{n}}.$$
- resonant qubit/photon interaction: $\mathbf{H}_{int} = i\frac{\omega}{2}\Big(|g\rangle\langle e| \otimes \mathbf{a}^\dagger - |e\rangle\langle g| \otimes \mathbf{a}\Big)$ (with ω a constant parameter) yields $e^{-iT\mathbf{H}_{int}}$, the Schrödinger propagator during the time $T > 0$, given with $\theta = \omega T/2$ by
$$\mathbf{U}_\theta = |g\rangle\langle g| \otimes \cos(\theta\sqrt{\mathbf{n}}) + |e\rangle\langle e| \otimes \cos(\theta\sqrt{\mathbf{n}+\mathbf{I}}) + |g\rangle\langle e| \otimes \frac{\sin(\theta\sqrt{\mathbf{n}})}{\sqrt{\mathbf{n}}}\mathbf{a}^\dagger - |e\rangle\langle g| \otimes \mathbf{a}\frac{\sin(\theta\sqrt{\mathbf{n}})}{\sqrt{\mathbf{n}}}.$$

[4] LKB for Laboratoire Kastler Brossel.



$$U = \left(\left(\left(\frac{|g\rangle - |e\rangle}{\sqrt{2}}\right)\langle g| + \left(\frac{|g\rangle + |e\rangle}{\sqrt{2}}\right)\langle e|\right) \otimes I\right)$$
$$\left(|g\rangle\langle g| \otimes e^{-i\theta \mathbf{n}} + |e\rangle\langle e| \otimes e^{i\theta \mathbf{n}}\right)$$
$$\left(\left(\left(\frac{|g\rangle + |e\rangle}{\sqrt{2}}\right)\langle g| + \left(\frac{-|g\rangle + |e\rangle}{\sqrt{2}}\right)\langle e|\right) \otimes I\right)$$

applied on $|\Psi\rangle = |g\rangle \otimes |\psi\rangle$ yields

$$U\ (|g\rangle|\psi\rangle) = |g\rangle\ \cos(\theta \mathbf{n})|\psi\rangle + |e\rangle\ i\sin(\theta \mathbf{n})|\psi\rangle.$$

Markov process induced by the passage of qubit number k:

$$|\psi_{k+1}\rangle = \begin{cases} \frac{\cos(\theta \mathbf{n})|\psi_k\rangle}{\sqrt{\langle\psi_k|\cos^2(\theta \mathbf{n})|\psi_k\rangle}} & \text{if } y_k = g \text{ with probability } \langle\psi_k|\cos^2(\theta \mathbf{n})|\psi_k\rangle\ ; \\ \frac{i\sin(\theta \mathbf{n})|\psi_k\rangle}{\sqrt{\langle\psi_k|\sin^2(\theta \mathbf{n})|\psi_k\rangle}} & \text{if } y_k = e \text{ with probability } \langle\psi_k|\sin^2(\theta \mathbf{n})|\psi_k\rangle\ ; \end{cases}$$

where $y_k \in \{g, e\}$ classical signal produced by measurement of qubit k.

The density operator formulation ($\rho \equiv |\psi\rangle\langle\psi|$):

$$\rho_{k+1} = \begin{cases} \frac{M_g \rho_k M_g^\dagger}{\mathrm{Tr}\left(M_g \rho_k M_g^\dagger\right)} & \text{if } y_k = g \text{ with probability } \mathrm{Tr}\left(M_g \rho_k M_g^\dagger\right) ; \\ \frac{M_e \rho_k M_e^\dagger}{\mathrm{Tr}\left(M_e \rho_k M_e^\dagger\right)} & \text{if } y_k = e \text{ with probability } \mathrm{Tr}\left(M_e \rho_k M_e^\dagger\right) ; \end{cases}$$

with measurement Kraus operators $M_g = \cos(\theta N)$ and $M_e = \sin(\theta N)$. Notice that $M_g^\dagger M_g + M_e^\dagger M_e = I$.
For θ/π irrational, almost sure convergence towards a Fock state $|\bar{n}\rangle\langle\bar{n}|$ for some $\bar{n}$ based on the Lyapunov function (super-martingale)

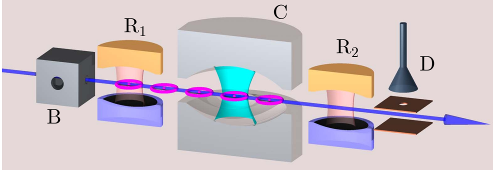
$$V(\rho) = \sum_{0 \le n_1 < n_2} \sqrt{\langle n_1|\rho|n_1\rangle\langle n_2|\rho|n_2\rangle}$$

that converges in average towards 0 since

$$\mathbb{E}\left(V(\rho_{k+1}) \,\middle|\, \rho_k\right) \le \left(\max_{0 \le n_1 < n_2} |\cos(\theta(n_1 \pm n_2))|\right) V(\rho_k).$$

Probability that a realisation converges towards $|\bar{n}\rangle\langle\bar{n}|$ given by its initial population $\langle\bar{n}|\rho_0|\bar{n}\rangle$

Photons measured by resonant qubits (1)



Wave function $|\Psi\rangle$ of the composite qubit/photon system just before D:

$$\Bigg(|g\rangle\langle g|\cos(\theta\sqrt{\mathbf{n}}) + |e\rangle\langle e|\cos(\theta\sqrt{\mathbf{n}+\mathbf{I}})$$

$$+ |g\rangle\langle e|\frac{\sin(\theta\sqrt{\mathbf{n}})}{\sqrt{\mathbf{n}}}\mathbf{a}^\dagger - |e\rangle\langle g|\mathbf{a}\frac{\sin(\theta\sqrt{\mathbf{n}})}{\sqrt{\mathbf{n}}}\Bigg)|g\rangle|\psi\rangle$$

$$= |g\rangle\ \cos(\theta\sqrt{\mathbf{n}})|\psi\rangle - |e\rangle\ \mathbf{a}\frac{\sin(\theta\sqrt{\mathbf{n}})}{\sqrt{\mathbf{n}}}|\psi\rangle$$

Resulting Markov process associated to the measurement of the observable $\sigma_\mathbf{z} = |e\rangle\langle e| - |g\rangle\langle g|$ with classical output signal $y \in \{g, e\}$:

$$|\psi_{k+1}\rangle = \begin{cases} \dfrac{\cos(\theta\sqrt{\mathbf{n}})|\psi_k\rangle}{\sqrt{\langle\psi_k|\cos^2(\theta\sqrt{\mathbf{n}})|\psi_k\rangle}} & \text{if } y_k = g \text{ with probability } \langle\psi_k|\cos^2(\theta\sqrt{\mathbf{n}})|\psi_k\rangle\ ; \\ -\dfrac{\mathbf{a}\frac{\sin(\theta\sqrt{\mathbf{n}})}{\sqrt{\mathbf{n}}}|\psi_k\rangle}{\sqrt{\langle\psi_k|\sin^2(\theta\sqrt{\mathbf{n}})|\psi_k\rangle}} & \text{if } y_k = e \text{ with probability } \langle\psi_k|\sin^2(\theta\sqrt{\mathbf{n}})|\psi_k\rangle\ ; \end{cases}$$

Density operator formulation;

$$\rho_{k+1} = \begin{cases} \frac{M_g \rho_k M_g^\dagger}{\mathrm{Tr}\left(M_g \rho_k M_g^\dagger\right)} & \text{if } y_k = g \text{ with probability } \mathrm{Tr}\left(M_g \rho_k M_g^\dagger\right); \\ \frac{M_e \rho_k M_e^\dagger}{\mathrm{Tr}\left(M_e \rho_k M_e^\dagger\right)} & \text{if } y_k = e \text{ with probability } \mathrm{Tr}\left(M_e \rho_k M_e^\dagger\right); \end{cases}$$

with measurement Kraus operators $M_g = \cos(\theta\sqrt{n})$ and $M_e = a\frac{\sin(\theta\sqrt{n})}{\sqrt{n}}$. Notice that, once again, $M_g^\dagger M_g + M_e^\dagger M_e = I$.

For $\theta\sqrt{n}/\pi$ irrational for all n, almost surely towards vacuum state $|0\rangle\langle 0|$. Results from the following the Lyapunov function (super-martingale)

$$V(\rho) = \mathrm{Tr}\left(n\rho\right)$$

since

$$\mathbb{E}\left(V(\rho_{k+1}) \mid \rho_k\right) = V(\rho_k) - \mathrm{Tr}\left(\sin^2(\theta\sqrt{n})\rho_k\right).$$

With measurement imperfections, use Bayes rule by taking as quantum state, the expectation value of ρ_{k+1} knowing ρ_k and the information provides by the imperfect measurement outcome.
Assume detector D broken. From

$$\rho_{k+1} = \begin{cases} \frac{\mathsf{M}_g \rho_k \mathsf{M}_g^\dagger}{\mathrm{Tr}\left(\mathsf{M}_g \rho_k \mathsf{M}_g^\dagger\right)} & \text{if } y_k = g \text{ with probability } \mathrm{Tr}\left(\mathsf{M}_g \rho_k \mathsf{M}_g^\dagger\right) ; \\ \frac{\mathsf{M}_e \rho_k \mathsf{M}_e^\dagger}{\mathrm{Tr}\left(\mathsf{M}_e \rho_k \mathsf{M}_e^\dagger\right)} & \text{if } y_k = e \text{ with probability } \mathrm{Tr}\left(\mathsf{M}_e \rho_k \mathsf{M}_e^\dagger\right) ; \end{cases}$$

we get the quantum channel:

$$\rho_{k+1} = \mathcal{K}(\rho_k) \triangleq \mathbb{E}\left(\rho_{k+1} \;\middle|\; \rho_k\right) = \mathsf{M}_g \rho_k \mathsf{M}_g^\dagger + \mathsf{M}_e \rho_k \mathsf{M}_e^\dagger.$$

When the qubit detector D, producing the classical measurement signal $y_k \in \{g, e\}$, has errors characterized by the error rate $\eta_e \in (0,1)$ (resp. $\eta_g \in (0,1)$) the probability of detector outcome g (resp. e) knowing that the perfect outcome is e (resp. g), Bayes law gives directly

$$\rho_{k+1} = \begin{cases} \mathbb{E}\left(\rho_{k+1} \;\middle|\; y_k = g, \rho_k\right) = \frac{(1-\eta_g)\mathsf{M}_g\rho_k\mathsf{M}_g^\dagger + \eta_e\mathsf{M}_e\rho_k\mathsf{M}_e^\dagger}{\mathrm{Tr}\left((1-\eta_g)\mathsf{M}_g\rho_k\mathsf{M}_g^\dagger + \eta_e\mathsf{M}_e\rho_k\mathsf{M}_e^\dagger\right)} \\ \qquad \text{with probability } \mathbb{P}(y_k = g|\rho_k) = \mathrm{Tr}\left((1-\eta_g)\mathsf{M}_g\rho_k\mathsf{M}_g^\dagger + \eta_e\mathsf{M}_e\rho_k\mathsf{M}_e^\dagger\right), \\ \\ \mathbb{E}\left(\rho_{k+1} \;\middle|\; y_k = e, \rho_k\right) = \frac{\eta_g\mathsf{M}_g\rho_k\mathsf{M}_g^\dagger + (1-\eta_e)\mathsf{M}_e\rho_k\mathsf{M}_e^\dagger}{\mathrm{Tr}\left(\eta_g\mathsf{M}_g\rho_k\mathsf{M}_g^\dagger + (1-\eta_e)\mathsf{M}_e\rho_k\mathsf{M}_e^\dagger\right)} \\ \qquad \text{with probability } \mathbb{P}(y_k = e|\rho_k) = \mathrm{Tr}\left(\eta_g\mathsf{M}_g\rho_k\mathsf{M}_g^\dagger + (1-\eta_e)\mathsf{M}_e\rho_k\mathsf{M}_e^\dagger\right) \end{cases}$$

Notice that a broken detector corresponds to $\eta_e = \eta_g = 1/2$ and one recovers the above quantum channel.

Outline

MINES PARIS | PSL

General structure of discrete-time SME based on a quantum channel with the following Kraus decomposition (which is not unique)

$$\mathcal{K}(\rho) = \sum_{\mu} \mathsf{M}_\mu \rho \mathsf{M}_\mu^\dagger \quad \text{where} \sum_{\mu} \mathsf{M}_\mu^\dagger \mathsf{M}_\mu = \mathsf{I}$$

and a left stochastic matrix $(\eta_{y,\mu})$ where y corresponds to the different imperfect measurement outcomes. With $\mathcal{K}_y(\rho) = \sum_\mu \eta_{y,\mu} \mathsf{M}_\mu \rho \mathsf{M}_\mu^\dagger$, ones gets the following SME:

$$\rho_{k+1} = \frac{\mathcal{K}_{y_k}(\rho_k)}{\mathrm{Tr}\left(\mathcal{K}_{y_k}(\rho_k)\right)} \quad \text{where } y_k = y \text{ with probability } \mathrm{Tr}\left(\mathcal{K}_y(\rho_k)\right)$$

Notice that $\mathcal{K} = \sum_y \mathcal{K}_y$ since η is left stochastic.

Here the Hilbert space $\mathcal{H}$ is arbitrary and can be of infinite dimension, the Kraus operator M_μ are bounded operator on $\mathcal{H}$ and ρ is a density operator on $\mathcal{H}$ (Hermitian, trace-class with trace one, non-negative). When the index y or μ are continuous, discrete sums are replaced by integrals and probabilities by probability densities.

MINES PARIS | PSL

Probe photon in the coherent state $|i\frac{\alpha}{\sqrt{2}}\rangle$ with $\alpha > 0$. Just before D the composite qubit/photon wave function $|\Psi\rangle$ reads:

$$\Big(|g\rangle\langle g|e^{-i\theta\mathbf{n}} + |e\rangle\langle e|e^{i\theta\mathbf{n}}\Big)|\psi\rangle|i\tfrac{\alpha}{\sqrt{2}}\rangle = \langle g|\psi\rangle\,|g\rangle\;|ie^{-i\theta}\tfrac{\alpha}{\sqrt{2}}\rangle + \langle e|\psi\rangle\,|e\rangle\;|ie^{i\theta}\tfrac{\alpha}{\sqrt{2}}\rangle.$$

Measurement outcome $y \in \mathbb{R}$ corresponding to observable

$$\mathrm{Q} = \frac{\mathrm{a}+\mathrm{a}^\dagger}{\sqrt{2}} \equiv \int_{-\infty}^{+\infty} q|q\rangle\langle q|dq \text{ where } \langle q|\,q'\rangle = \delta(q-q').$$

Since $|ie^{\pm i\theta}\frac{\alpha}{\sqrt{2}}\rangle = \frac{1}{\pi^{1/4}}\int_{-\infty}^{+\infty} e^{iq\alpha\cos\theta} e^{-\frac{(q\pm\alpha\sin\theta)^2}{2}}|q\rangle dq$, we have

$$\langle g|\psi\rangle\,|g\rangle\;|ie^{-i\theta}\tfrac{\alpha}{\sqrt{2}}\rangle + \langle e|\psi\rangle\,|e\rangle\;|ie^{i\theta}\tfrac{\alpha}{\sqrt{2}}\rangle$$

$$= \frac{1}{\pi^{1/4}}\int_{-\infty}^{+\infty} e^{iq\alpha\cos\theta}\left(e^{-\frac{(q-\alpha\sin\theta)^2}{2}}\langle g|\psi\rangle\,|g\rangle + e^{-\frac{(q+\alpha\sin\theta)^2}{2}}\langle e|\psi\rangle\,|e\rangle\right)|q\rangle dq.$$

Thus

$$|\psi_{k+1}\rangle = e^{iy_k\alpha\cos\theta}\,\frac{e^{-\frac{(y_k-\alpha\sin\theta)^2}{2}}\langle g|\psi_k\rangle\,|g\rangle + e^{-\frac{(y_k+\alpha\sin\theta)^2}{2}}\langle e|\psi_k\rangle\,|e\rangle}{\sqrt{e^{-(y_k-\alpha\sin\theta)^2}|\langle g|\psi_k\rangle|^2 + e^{-(y_k+\alpha\sin\theta)^2}|\langle e|\psi_k\rangle|^2}}$$

where $y_k \in [y, y+dy]$ with prob. $\frac{e^{-(y-\alpha\sin\theta)^2}|\langle g|\psi_k\rangle|^2 + e^{-(y+\alpha\sin\theta)^2}|\langle e|\psi_k\rangle|^2}{\sqrt{\pi}}dy.$

Density operator formulation

$$\rho_{k+1} = \frac{M_{y_k} \rho_k M_{y_k}^\dagger}{\mathrm{Tr}\left(M_{y_k} \rho_k M_{y_k}^\dagger\right)} \quad \text{where } y_k \in [y, y+dy] \text{ with probability } \mathrm{Tr}\left(M_y \rho_k M_y^\dagger\right) dy$$

and measurement Kraus operators

$$M_y = \frac{1}{\pi^{1/4}} e^{-\frac{(y-\alpha \sin\theta)^2}{2}} |g\rangle\langle g| + \frac{1}{\pi^{1/4}} e^{-\frac{(y+\alpha \sin\theta)^2}{2}} |e\rangle\langle e|.$$

Notice that

$$\mathrm{Tr}\left(M_y \rho M_y^\dagger\right) = \frac{1}{\sqrt{\pi}} e^{-(y-\alpha \sin\theta)^2} \langle g|\rho|g\rangle + \frac{1}{\sqrt{\pi}} e^{-(y+\alpha \sin\theta)^2} \langle e|\rho|e\rangle$$

and $\int_{-\infty}^{+\infty} M_y^\dagger M_y \, dy = |g\rangle\langle g| + |e\rangle\langle e| = I$.
For $\alpha \neq 0$, almost sure convergence towards $|g\rangle$ or $|e\rangle$ deduced from Lyapunov function

$$V(\rho) = \sqrt{\langle g|\rho|g\rangle\langle e|\rho|e\rangle} \text{ with } \mathbb{E}\left(V(\rho_{k+1}) \;\middle|\; \rho_k\right) = e^{-\alpha^2 \sin^2\theta} \; V(\rho_k).$$

Detection imperfections: probability density of y knowing perfect detection q is a Gaussian given by $\frac{1}{\sqrt{\pi\sigma}} e^{-\frac{(y-q)^2}{\sigma}}$ for some error parameter $\sigma > 0$. Then the above Markov process becomes

$$\rho_{k+1} = \frac{\mathcal{K}_{y_k}(\rho_k)}{\mathrm{Tr}\left(\mathcal{K}_{y_k}(\rho_k)\right)}$$

where

$$\mathcal{K}_y(\rho) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\sigma}} e^{-\frac{(y-q)^2}{\sigma}} \mathrm{M}_q \rho \mathrm{M}_q^\dagger \, dq$$

Standard computations using

$$\mathrm{M}_q = \frac{1}{\pi^{1/4}} e^{-\frac{(q-\alpha \sin\theta)^2}{2}} |g\rangle\langle g| + \frac{1}{\pi^{1/4}} e^{-\frac{(q+\alpha \sin\theta)^2}{2}} |e\rangle\langle e|$$

show that

$$\mathcal{K}_y(\rho) = \frac{1}{\sqrt{\pi(1+\sigma)}} \Bigg(e^{-\frac{(y-\alpha \sin\theta)^2}{1+\sigma}} \langle g|\rho|g\rangle |g\rangle\langle g| + e^{-\frac{(y+\alpha \sin\theta)^2}{1+\sigma}} \langle e|\rho|e\rangle |e\rangle\langle e|$$
$$+ e^{-\frac{y^2}{1+\sigma} - (\alpha \sin\theta)^2} \left(\langle e|\rho|g\rangle |e\rangle\langle g| + \langle g|\rho|e\rangle |g\rangle\langle e| \right) \Bigg).$$

Density operator formulation (perfect detection)

$$\rho_{k+1} = \frac{M_{y_k}\rho_k M_{y_k}^\dagger}{\mathrm{Tr}\left(M_{y_k}\rho_k M_{y_k}^\dagger\right)} \quad \text{where } y_k \in [y, y+dy] \text{ with probability } \mathrm{Tr}\left(M_y\rho_k M_y^\dagger\right) dy$$

and measurement Kraus operators

$$M_y = \frac{1}{\pi^{1/4}} e^{-\frac{(y-\alpha\sin\theta)^2}{2}} |g\rangle\langle g| + \frac{1}{\pi^{1/4}} e^{-\frac{(y+\alpha\sin\theta)^2}{2}} |e\rangle\langle e|.$$

Since

$$\mathbb{E}\left(y_k \;\middle|\; \rho_k = \rho\right) \triangleq \overline{y} = -\alpha\sin\theta \;\; \mathrm{Tr}\left(\sigma_z\rho\right), \; \mathbb{E}\left(y_k^2 \;\middle|\; \rho_k = \rho\right) \triangleq \overline{y^2} = 1/2 + (\alpha\sin\theta)^2.$$

When $0 < \alpha\sin\theta = \epsilon \ll 1$, we have up-to third order terms versus ϵy,

$$\frac{M_y\rho M_y^\dagger}{\mathrm{Tr}\left(M_y\rho M_y^\dagger\right)} = \frac{(\cosh(\epsilon y) - \sinh(\epsilon y)\sigma_z)\rho(\cosh(\epsilon y) - \sinh(\epsilon y)\sigma_z)}{\cosh(2\epsilon y) - \sinh(2\epsilon y)\;\mathrm{Tr}\left(\sigma_z\rho\right)}$$

$$\approx \frac{\rho - \epsilon y(\sigma_z\rho + \rho\sigma_z) + (\epsilon y)^2(\rho + \sigma_z\rho\sigma_z)}{1 - 2\epsilon y\;\mathrm{Tr}\left(\sigma_z\rho\right) + 2(\epsilon y)^2}$$

$$\approx \rho + (\epsilon y)^2\left(\sigma_z\rho\sigma_z - \rho\right) + \left(\sigma_z\rho + \rho\sigma_z - 2\;\mathrm{Tr}\left(\sigma_z\rho\right)\rho\right)\left(-\epsilon y - 2(\epsilon y)^2\;\mathrm{Tr}\left(\sigma_z\rho\right)\right).$$

Replacing $\epsilon^2 y^2$ by its expectation value one gets, up to third order in ϵy and ϵ:

$$\frac{M_y \rho M_y^\dagger}{\text{Tr}\left(M_y \rho M_y^\dagger\right)} \approx \rho + \tfrac{\epsilon^2}{2}\Big(\sigma_z \rho \sigma_z - \rho\Big) + \Big(\sigma_z \rho + \rho \sigma_z - 2\ \text{Tr}\left(\sigma_z \rho\right) \rho\Big)\Big(-\epsilon y - \epsilon^2\ \text{Tr}\left(\sigma_z \rho\right)\Big).$$

Set $\epsilon^2 = 2dt$ and $\epsilon y = -2\ \text{Tr}\left(\sigma_z \rho\right) dt - dW$. Since by construction

$$\mathbb{E}\left(\epsilon y_k \;\middle|\; \rho_k = \rho\right) = -\epsilon^2\ \text{Tr}\left(\sigma_z \rho\right) \text{ and } \mathbb{E}\left((\epsilon y_k)^2 \;\middle|\; \rho_k = \rho\right) = \epsilon^2 + \epsilon^4$$

one has $\mathbb{E}\left(dW \;\middle|\; \rho\right) = 0$ and $\mathbb{E}\left(dW^2 \;\middle|\; \rho\right) = dt$ up to order 4 versus ϵ. Thus for dt very small, we recover the following diffusive SME[5]

$$\rho_{t+dt} = \rho_t + dt\Big(\sigma_z \rho_t \sigma_z - \rho\Big) + \Big(\sigma_z \rho_t + \rho_t \sigma_z - 2\ \text{Tr}\left(\sigma_z \rho_t\right) \rho\Big)\Big(dy_t - 2\ \text{Tr}\left(\sigma_z \rho_t\right) dt\Big)$$

with $dy_t = 2\ \text{Tr}\left(\sigma_z \rho_t\right) dt + dW_t$ replacing $-\epsilon y$ and $dy_t^2 = dW_t^2 = dt$ (Ito rules).

[5] Convergence in distribution when $dt \mapsto 0^+$: tightness property

$$\forall T > 0, \exists M > 0, \forall dt > 0, \forall k, k_1, k_2 \in \{0, \dots, [T/dt]\}, \mathbb{E}\left(\|\rho_{k_1} - \rho_k\|^2 \|\ \|\rho_{k_2} - \rho_k\|^2 \;\middle|\; \rho_0\right) \leq M(k_1 - k_2)\, dt,$$

and (Markov generator) convergence of $\frac{\mathbb{E}\left(f(\rho_{k+1} \;\middle|\; \rho_k = \rho\right) - f(\rho)}{dt}$ towards $\mathbb{E}\left(df_t \;\middle|\; \rho_t = \rho\right)/dt$ for any C^2 real function f.

With measurement errors parameterized by $\sigma > 0$, the partial Kraus map

$$\mathcal{K}_y(\rho) = \frac{1}{\sqrt{\pi(1+\sigma)}} \Big(e^{-\frac{(y-\epsilon)^2}{1+\sigma}} \langle g|\rho|g\rangle |g\rangle\langle g| + e^{-\frac{(y+\epsilon)^2}{1+\sigma}} \langle e|\rho|e\rangle |e\rangle\langle e| + e^{-\frac{y^2}{1+\sigma}-\epsilon^2}\big(\langle e|\rho|g\rangle |e\rangle\langle g| + \langle g|\rho|e\rangle |g\rangle\langle e|\big)\Big)$$

yields $\mathbb{E}\left(y_k \mid \rho_k\right) \triangleq \bar{y} = -\epsilon \operatorname{Tr}(\sigma_z \rho)$ and $\mathbb{E}\left(y_k^2 \mid \rho_k\right) \triangleq \overline{y^2} = (1+\sigma)/2 + \epsilon^2$.
Similar approximations with $\epsilon^2 = 2dt$ and dt very small, yield an SME with detection efficiency $\eta = \frac{1}{1+\sigma}$:

$$\rho_{t+dt} = \rho_t + dt\Big(\sigma_z\rho_t\sigma_z - \rho\Big) + \sqrt{\eta}\Big(\sigma_z\rho_t + \rho_t\sigma_z - 2\operatorname{Tr}(\sigma_z\rho_t)\,\rho\Big)dW_t$$

with $dy_t = \sqrt{\eta}\operatorname{Tr}(\sigma_z\rho_t + \rho_t\sigma_z)\,dt + dW_t \sim -\epsilon y/\sqrt{1+\sigma}$.
Convergence towards either $|g\rangle$ or $|e\rangle$ (QND measurement of the qubit) based on Lyapunov fonction $V(\rho) = \sqrt{1 - \operatorname{Tr}(\sigma_z\rho)^2}$ and Ito rules:

$$dV = -\frac{zdz}{\sqrt{1-z^2}} - \frac{dz^2}{2(1-z^2)^{3/2}} = -\frac{zdz}{\sqrt{1-z^2}} - 2\eta^2 Vdt$$

where $z = \operatorname{Tr}(\sigma_z\rho)$, $dz = 2\eta(1-z^2)dW$ and $dz^2 = 4\eta^2(1-z^2)^2dt$. Since $\mathbb{E}\left(dz \mid z\right) = 0$, $\bar{V}_t = \mathbb{E}\left(V(z_t) \mid z_0\right)$ solution of $\frac{d}{dt}\bar{V}_t = -2\eta^2\bar{V}_t$.

Outline

General form of diffusive SME with Ito formulation:

$$d\rho_t = \left(-i[\mathsf{H}, \rho_t] + \sum_\nu \mathsf{L}_\nu \rho_t \mathsf{L}_\nu^\dagger - \frac{1}{2}(\mathsf{L}_\nu^\dagger \mathsf{L}_\nu \rho_t + \rho_t \mathsf{L}_\nu^\dagger \mathsf{L}_\nu) \right) dt$$

$$+ \sum_\nu \sqrt{\eta_\nu} \left(\mathsf{L}_\nu \rho_t + \rho_t \mathsf{L}_\nu^\dagger - \operatorname{Tr}\left((\mathsf{L}_\nu + \mathsf{L}_\nu^\dagger)\rho_t \right) \rho_t \right) dW_{\nu,t},$$

$$dy_{\nu,t} = \sqrt{\eta_\nu} \operatorname{Tr}\left(\mathsf{L}_\nu \rho_t + \rho_t \mathsf{L}_\nu^\dagger \right) dt + dW_{\nu,t}$$

with efficiencies $\eta_\nu \in [0,1]$ and $dW_{\nu,t}$ being independent Wiener processes. Equivalent formulation with Ito rules:

$$\rho_{t+dt} = \frac{\mathsf{M}_{dy_t} \rho_t \mathsf{M}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu) \mathsf{L}_\nu \rho_t \mathsf{L}_\nu^\dagger dt}{\operatorname{Tr}\left(\mathsf{M}_{dy_t} \rho_t \mathsf{M}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu) \mathsf{L}_\nu \rho_t \mathsf{L}_\nu^\dagger dt \right)}$$

with $\mathsf{M}_{dy_t} = \mathsf{I} + \left(-i\mathsf{H} - \frac{1}{2}\sum_\nu \mathsf{L}_\nu^\dagger \mathsf{L}_\nu \right) dt + \sum_\nu \sqrt{\eta_\nu} dy_{\nu,t} \mathsf{L}_\nu$. Moreover $dy_{\nu,t} = s_{\nu,t}\sqrt{dt}$ follows the following probability density knowing ρ_t:

$$\mathbb{P}\Big((s_{\nu,t} \in [s_\nu, s_\nu + ds_\nu])_\nu \mid \rho_t \Big) = \operatorname{Tr}\left(\mathsf{M}_{s\sqrt{dt}}\, \rho_t \mathsf{M}_{s\sqrt{dt}}^\dagger + \sum_\nu (1-\eta_\nu) \mathsf{L}_\nu \rho_t \mathsf{L}_\nu^\dagger dt \right) \prod_\nu \frac{e^{-\frac{s_\nu^2}{2}} ds_\nu}{\sqrt{2\pi}}.$$

[6] A. Barchielli and M. Gregoratti. *Quantum Trajectories and Measurements in Continuous Time: the Diffusive Case*. Springer Verlag, 2009.

Linearity/positivity/trace preserving numerical integration scheme for

$$d\rho_t = \left(-i[\mathrm{H},\rho_t] + \sum_\nu \mathrm{L}_\nu \rho_t \mathrm{L}_\nu^\dagger - \frac{1}{2}(\mathrm{L}_\nu^\dagger \mathrm{L}_\nu \rho_t + \rho_t \mathrm{L}_\nu^\dagger \mathrm{L}_\nu)\right) dt$$
$$+ \sum_\nu \sqrt{\eta_\nu}\left(\mathrm{L}_\nu\rho_t + \rho_t \mathrm{L}_\nu^\dagger - \mathrm{Tr}\left((\mathrm{L}_\nu + \mathrm{L}_\nu^\dagger)\rho_t\right)\rho_t\right) dW_{\nu,t},$$

$$dy_{\nu,t} = \sqrt{\eta_\nu}\ \mathrm{Tr}\left(\mathrm{L}_\nu\rho_t + \rho_t\mathrm{L}_\nu^\dagger\right) dt + dW_{\nu,t}$$

With $\mathrm{M}_0 = \mathrm{I} + \left(-i\mathrm{H} - \frac{1}{2}\sum_\nu \mathrm{L}_\nu^\dagger \mathrm{L}_\nu\right)dt$, $\quad \mathrm{S} = \mathrm{M}_0^\dagger \mathrm{M}_0 + \left(\sum_\nu \mathrm{L}_\nu^\dagger \mathrm{L}_\nu\right) dt$ set

$$\widetilde{\mathrm{M}}_0 = \mathrm{M}_0 \mathrm{S}^{-1/2}, \quad \widetilde{\mathrm{L}}_\nu = \mathrm{L}_\nu \mathrm{S}^{-1/2}.$$

Sampling of $dy_{\nu,t} = s_{\nu,t}\sqrt{dt}$ according to the following probability law:

$$\mathbb{P}\left(\left(s_{\nu,t} \in [s_\nu, s_\nu + ds_\nu]\right)_\nu \mid \rho_t\right) = \mathrm{Tr}\left(\widetilde{\mathrm{M}}_{s\sqrt{dt}}\rho_t\widetilde{\mathrm{M}}_{s\sqrt{dt}}^\dagger + \sum_\nu (1-\eta_\nu)\widetilde{\mathrm{L}}_\nu\rho_t\widetilde{\mathrm{L}}_\nu^\dagger dt\right) \prod_\nu \frac{e^{-\frac{s_\nu^2}{2}} ds_\nu}{\sqrt{2\pi}}.$$

where $\widetilde{\mathrm{M}}_{dy_t} = \widetilde{\mathrm{M}}_0 + \sum_\nu \sqrt{\eta_\nu} dy_{\nu,t}\widetilde{\mathrm{L}}_\nu$. Exact Kraus-map formulation:

$$\rho_{t+dt} = \frac{\widetilde{\mathrm{M}}_{dy_t}\rho_t\widetilde{\mathrm{M}}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu)\widetilde{\mathrm{L}}_\nu\rho_t\widetilde{\mathrm{L}}_\nu^\dagger dt}{\mathrm{Tr}\left(\widetilde{\mathrm{M}}_{dy_t}\rho_t\widetilde{\mathrm{M}}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu)\widetilde{\mathrm{L}}_\nu\rho_t\widetilde{\mathrm{L}}_\nu^\dagger dt\right)}.$$

[7] A. Jordan, A. Chantasri, PR, and B.Huard. Anatomy of fluorescence: quantum trajectory statistics from continuously measuring spontaneous emission. *Quantum Studies: Mathematics and Foundations*, 3(3):237–263, 2016.

Outline

MINES PARIS | PSL

Probe photon is in the vacuum state $|0\rangle$. Composite qubit/photon wave function $|\Psi\rangle$ before D:

$$\Bigg(|g\rangle\langle g| \cos(\theta\sqrt{\mathrm{n}}) + |e\rangle\langle e| \cos(\theta\sqrt{\mathrm{n}+\mathrm{I}}) + |g\rangle\langle e| \frac{\sin(\theta\sqrt{\mathrm{n}})}{\sqrt{\mathrm{n}}}\mathrm{a}^\dagger - |e\rangle\langle g| \mathrm{a}\frac{\sin(\theta\sqrt{\mathrm{n}})}{\sqrt{\mathrm{n}}} \Bigg) |\psi\rangle|0\rangle$$

$$= \big(\langle g|\psi\rangle\,|g\rangle + \cos\theta\,\langle e|\psi\rangle\,|e\rangle \big)|0\rangle + \sin\theta\,\langle e|\psi\rangle\,|g\rangle|1\rangle.$$

With measurement observable $\mathrm{n} = \sum_{n\geq 0} n|n\rangle\langle n|$, outcome $y \in \{0,1\}$ reads (density operator formulation)

$$\rho_{k+1} = \begin{cases} \frac{\mathrm{M}_0\rho_k\mathrm{M}_0^\dagger}{\mathrm{Tr}\left(\mathrm{M}_0\rho_k\mathrm{M}_0^\dagger\right)} & \text{if } y_k = 0 \text{ with probability } \mathrm{Tr}\left(\mathrm{M}_0\rho_k\mathrm{M}_0^\dagger\right); \\ \frac{\mathrm{M}_1\rho_k\mathrm{M}_1^\dagger}{\mathrm{Tr}\left(\mathrm{M}_1\rho_k\mathrm{M}_1^\dagger\right)} & \text{if } y_k = 1 \text{ with probability } \mathrm{Tr}\left(\mathrm{M}_1\rho_k\mathrm{M}_1^\dagger\right); \end{cases}$$

measurement Kraus operators $\mathrm{M}_0 = |g\rangle\langle g| + \cos\theta|e\rangle\langle e|$ and $\mathrm{M}_1 = \sin\theta|g\rangle\langle e|$. Almost convergence analysis when $\cos^2(\theta) < 1$ towards $|g\rangle$ via the Lyapunov function (super martingale)

$$V(\rho) = \mathrm{Tr}\left(|e\rangle\langle e|\rho\right) \text{ since } \mathbb{E}\left(V(\rho_{k+1}) \,\middle|\, \rho_k \right) = \cos^2\theta\ V(\rho_k).$$

Outline

Since $\mathrm{Tr}\left(\mathrm{M}_0 \rho \mathrm{M}_0^\dagger\right) = 1 - \sin^2\theta \,\mathrm{Tr}\left(\sigma_- \rho \sigma_+\right)$ and $\mathrm{Tr}\left(\mathrm{M}_1 \rho \mathrm{M}_1^\dagger\right) = \sin^2\theta \,\mathrm{Tr}\left(\sigma_- \rho \sigma_+\right)$, one gets with $\sin^2\theta = dt$ and $y \sim dN$, an SME driven by Poisson process $dN_t \in \{0, 1\}$ of expectation value $\mathrm{Tr}\left(\sigma_- \rho_t \sigma_+\right) dt$ knowing ρ_t:

$$d\rho_t = \left(\sigma_- \rho_t \sigma_+ - \tfrac{1}{2}(\sigma_+\sigma_- \rho_t + \rho_t \sigma_+ \sigma_-)\right) dt \\ + \left(\frac{\sigma_- \rho_t \sigma_+}{\mathrm{Tr}\left(\sigma_- \rho_t \sigma_+\right)} - \rho_t\right)\left(dN_t - \left(\mathrm{Tr}\left(\sigma_- \rho_t \sigma_+\right)\right) dt\right).$$

At each time-step, one has the following choice:

- with probabilty $1 - \mathrm{Tr}\left(\sigma_- \rho_t \sigma_+\right) dt$, $dN_t = N_{t+dt} - N_t = 0$ and

$$\rho_{t+dt} = \frac{\mathrm{M}_0 \rho_t \mathrm{M}_0^\dagger}{\mathrm{Tr}\left(\mathrm{M}_0 \rho_t \mathrm{M}_0^\dagger\right)}$$

  with $\mathrm{M}_0 = \mathrm{I} - \frac{dt}{2}\sigma_+\sigma_-$.

- with probability $\mathrm{Tr}\left(\sigma_- \rho_t \sigma_+\right) dt$, $dN_t = N_{t+dt} - N_t = 1$ and

$$\rho_{t+dt} = \frac{\mathrm{M}_1 \rho_t \mathrm{M}_1^\dagger}{\mathrm{Tr}\left(\mathrm{M}_1 \rho_t \mathrm{M}_1^\dagger\right)}$$

  with $\mathrm{M}_1 = \sqrt{dt}\, \sigma_-$.

With left stochastic matrix $\begin{pmatrix} 1-\bar\theta dt & 1-\bar\eta \\ \bar\theta dt & \bar\eta \end{pmatrix}$ including dark counts of rate $\bar\theta \geq 0$ and detection efficiency $\bar\eta \in [0,1]$:

- $dN_t = N_{t+dt} - N_t = 0$ and

$$\rho_{t+dt} = \frac{(1-\bar\theta dt)\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + (1-\bar\eta)\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger}{\mathrm{Tr}\left((1-\bar\theta dt)\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + (1-\bar\eta)\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger\right)}$$
$$= \frac{\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + (1-\bar\eta)\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger}{\mathrm{Tr}\left(\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + (1-\bar\eta)\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger\right)} + O(dt^2).$$

  with probability

$$1-\left(\bar\theta+\bar\eta\ \mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right)\right)dt = \mathrm{Tr}\left((1-\bar\theta dt)\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + (1-\bar\eta)\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger\right) + O(dt^2)$$

  and where $\mathrm{M}_0 = \mathrm{I} - \frac{dt}{2}\sigma_+\sigma_-$ and $\mathrm{M}_1 = \sqrt{dt}\ \sigma_-$.
- $dN_t = N_{t+dt} - N_t = 1$ and

$$\rho_{t+dt} = \frac{\bar\theta\, dt\, \mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + \bar\eta\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger}{\mathrm{Tr}\left(\bar\theta\, dt\, \mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + \bar\eta\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger\right)} = \frac{\bar\theta\rho_t + \bar\eta\sigma_-\rho_t\sigma_+}{\bar\theta + \bar\eta\ \mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right)} + O(dt)$$

  with probability

$$\left(\bar\theta + \bar\eta\ \mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right)\right)dt = \mathrm{Tr}\left(\bar\theta\, dt\, \mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + \bar\eta\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger\right) + O(dt^2)$$

Towards jump SME (3)

Jump SME with dark count rate $\bar{\theta}$ and detection efficiency $\bar{\eta}$

$$d\rho_t = \left(\sigma_- \rho_t \sigma_+ - \tfrac{1}{2}(\sigma_+ \sigma_- \rho_t + \rho_t \sigma_+ \sigma_-)\right)\, dt \\ + \left(\frac{\bar{\theta}\rho_t + \bar{\eta}\sigma_-\rho_t\sigma_+}{\mathrm{Tr}\left(\bar{\theta}\rho_t + \bar{\eta}\sigma_-\rho_t\sigma_+\right)} - \rho_t \right) \left(dN_t - \left(\bar{\theta} + \bar{\eta}\ \mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right)\right) dt \right).$$

corresponds to the following choices

- $dN_t = N_{t+dt} - N_t = 0$

$$\rho_{t+dt} = \frac{\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + (1-\bar{\eta})\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger}{\mathrm{Tr}\left(\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + (1-\bar{\eta})\mathrm{M}_1\rho_t\mathrm{M}_1^\dagger\right)}$$

  with probability $1 - \left(\bar{\theta} + \bar{\eta}\ \mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right)\right) dt$,

- $dN_t = N_{t+dt} - N_t = 1$ and

$$\rho_{t+dt} = \frac{\bar{\theta}\rho_t + \bar{\eta}\sigma_-\rho_t\sigma_+}{\bar{\theta} + \bar{\eta}\ \mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right)}$$

  with probability $\left(\bar{\theta} + \bar{\eta}\ \mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right)\right) dt$,

where $\mathrm{M}_0 = \mathrm{I} - \frac{dt}{2}(\sigma_+\sigma_- + \mathrm{I})$ and $\mathrm{M}_1 = \sqrt{dt}\ \sigma_-$.

Jump SME in continuous-time[8] (1)

General structure of a Jump SME in continuous time with counting process N_t with increment expectation value knowing ρ_t given by $\langle dN_t \rangle = \left(\bar{\theta} + \bar{\eta}\ \mathrm{Tr}\left(V\rho_t V^\dagger\right)\right) dt$, with $\bar{\theta} \geq 0$ (dark count rate) and $\bar{\eta} \in [0,1]$ (detection efficiency):

$$d\rho_t = \left(-i[\mathrm{H},\rho_t] + \mathrm{V}\rho_t\mathrm{V}^\dagger - \tfrac{1}{2}(\mathrm{V}^\dagger\mathrm{V}\rho_t + \rho_t\mathrm{V}^\dagger\mathrm{V})\right) dt$$
$$+ \left(\frac{\bar{\theta}\rho_t + \bar{\eta}\mathrm{V}\rho_t\mathrm{V}^\dagger}{\bar{\theta} + \bar{\eta}\ \mathrm{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)} - \rho_t\right)\left(dN_t - \left(\bar{\theta} + \bar{\eta}\ \mathrm{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)\right) dt\right).$$

Here H and V are operators on an underlying Hilbert space $\mathcal{H}$, H being Hermitian. At each time-step between t and $t+dt$, one has the following recipe

- $dN_t = 0$ with probability $1 - \left(\bar{\theta} + \bar{\eta}\ \mathrm{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)\right) dt$

$$\rho_{t+dt} = \frac{\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + (1-\bar{\eta})\mathrm{V}\rho_t\mathrm{V}^\dagger dt}{\mathrm{Tr}\left(\mathrm{M}_0\rho_t\mathrm{M}_0^\dagger + (1-\bar{\eta})\mathrm{V}\rho_t\mathrm{V}^\dagger dt\right)}$$

  where $\mathrm{M}_0 = I - \left(iH + \frac{1}{2}\mathrm{V}^\dagger\mathrm{V}\right) dt$.
- $dN_t = 1$ with probability $\left(\bar{\theta} + \bar{\eta}\ \mathrm{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)\right) dt$,

$$\rho_{t+dt} = \frac{\bar{\theta}\rho_t + \bar{\eta}\mathrm{V}\rho_t\mathrm{V}^\dagger}{\bar{\theta} + \bar{\eta}\ \mathrm{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)}.$$

[8] J. Dalibard, Y. Castin, and K. Mølmer. Wave-function approach to dissipative processes in quantum optics. *Phys. Rev. Lett.*, 68(5):580–583, 1992.

$$d\rho_t = \left(-i[\mathrm{H},\rho_t] + \mathrm{V}\rho_t\mathrm{V}^\dagger - \tfrac{1}{2}(\mathrm{V}^\dagger\mathrm{V}\rho_t + \rho_t\mathrm{V}^\dagger\mathrm{V})\right)\,dt$$
$$+ \left(\frac{\bar\theta\rho_t + \bar\eta\mathrm{V}\rho_t\mathrm{V}^\dagger}{\bar\theta + \bar\eta\,\mathrm{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)} - \rho_t\right)\left(dN_t - \left(\bar\theta + \bar\eta\,\mathrm{Tr}\left(\mathrm{V}\rho_t\mathrm{V}^\dagger\right)\right)dt\right).$$

Take a discretization step $dt > 0$ and set $\mathrm{M}_0 = I - \left(iH + \frac{1}{2}\mathrm{V}^\dagger\mathrm{V}\right)dt$, $\widetilde{\mathrm{M}}_0 = \mathrm{M}_0\mathrm{S}^{-1/2}$ and $\widetilde{\mathrm{V}} = \mathrm{V}\mathrm{S}^{-1/2}$ with $\mathrm{S} = \mathrm{M}_0^\dagger\mathrm{M}_0 + \mathrm{V}^\dagger\mathrm{V}dt$. Use the following numerical CP scheme:

- $dN_t = 0$ with probability $\mathrm{Tr}\left(e^{-\bar\theta dt}\widetilde{\mathrm{M}}_0\rho_t\widetilde{\mathrm{M}}_0^\dagger + (1-\bar\eta)dt\widetilde{\mathrm{V}}\rho_t\widetilde{\mathrm{V}}^\dagger\right)$

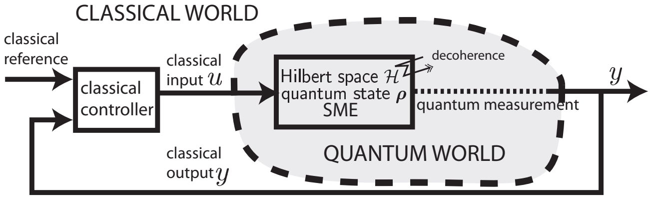
$$\rho_{t+dt} = \frac{e^{-\bar\theta dt}\widetilde{\mathrm{M}}_0\rho_t\widetilde{\mathrm{M}}_0^\dagger + (1-\bar\eta)dt\widetilde{\mathrm{V}}\rho_t\widetilde{\mathrm{V}}^\dagger}{\mathrm{Tr}\left(e^{-\bar\theta dt}\widetilde{\mathrm{M}}_0\rho_t\widetilde{\mathrm{M}}_0^\dagger + (1-\bar\eta)dt\widetilde{\mathrm{V}}\rho_t\widetilde{\mathrm{V}}^\dagger\right)}.$$

- $dN_t = 1$ with probability $\mathrm{Tr}\left(\left(1-e^{-\bar\theta dt}\right)\widetilde{\mathrm{M}}_0\rho_t\widetilde{\mathrm{M}}_0^\dagger + \bar\eta dt\widetilde{\mathrm{V}}\rho_t\widetilde{\mathrm{V}}^\dagger\right)$

$$\rho_{t+dt} = \frac{\left(1-e^{-\bar\theta dt}\right)\widetilde{\mathrm{M}}_0\rho_t\widetilde{\mathrm{M}}_0^\dagger + \bar\eta dt\widetilde{\mathrm{V}}\rho_t\widetilde{\mathrm{V}}^\dagger}{\mathrm{Tr}\left(\left(1-e^{-\bar\theta dt}\right)\widetilde{\mathrm{M}}_0\rho_t\widetilde{\mathrm{M}}_0^\dagger + \bar\eta dt\widetilde{\mathrm{V}}\rho_t\widetilde{\mathrm{V}}^\dagger\right)}.$$

Probabilities are preserved exactly: for any ρ_t, $\bar\theta \geq 0$, $\bar\eta \in [0,1]$

$$\mathrm{Tr}\left(e^{-\bar\theta dt}\widetilde{\mathrm{M}}_0\rho_t\widetilde{\mathrm{M}}_0^\dagger + (1-\bar\eta)dt\widetilde{\mathrm{V}}\rho_t\widetilde{\mathrm{V}}^\dagger\right) + \mathrm{Tr}\left(\left(1-e^{-\bar\theta dt}\right)\widetilde{\mathrm{M}}_0\rho_t\widetilde{\mathrm{M}}_0^\dagger + \bar\eta dt\widetilde{\mathrm{V}}\rho_t\widetilde{\mathrm{V}}^\dagger\right) \equiv 1$$



- P-controller (Markovian feedback[9]) for $u_t \; dt = k \; dy_t$, the ensemble average closed-loop dynamics of ρ remains governed by a linear Lindblad master equation.
- PID controller: no Lindblad master equation in closed-loop for dynamics output feedback
- Nonlinear hidden-state stochastic systems: Lyapunov state-feedback[10]; many open issues on convergence rates, delays, robustness, ...
- **Short sampling times limit feedback complexity**

[9] H. Wiseman, G. Milburn (2009). Quantum Measurement and Control. Cambridge University Press.

[10] See e.g.: C. Ahn et. al (2002): Continuous quantum error correction via quantum feedback control. Phys. Rev. A 65;
M. Mirrahimi, R. Handel (2007): Stabilizing feedback controls for quantum systems. SIAM Journal on Control and Optimization, 46(2), 445-467;
W. Liang, Weichao, N. Amini and P. Mason (2019): On Exponential Stabilization of N-Level Quantum Angular Momentum Systems. SIAM Journal on Control and Optimization 57(6):3939-3960.

Four modeling features[11]:

1. Schrödinger equations defining unitary transformations.
2. **Randomness**, irreversibility and dissipation induced by the measurement of observables with degenerate spectra.
3. Entanglement and tensor product for composite systems.
4. Classical probability (Bayesian inference) to include classical noises, measurement errors and uncertainties.

$\Rrightarrow$ **Hidden-state controlled Markov system**

Control input u, state ρ (density op.), measured output y:

$$\rho_{t+1} = \frac{\mathcal{K}_{u_t,y_t}(\rho_t)}{\mathrm{Tr}\big(\mathcal{K}_{u_t,y_t}(\rho_t)\big)}, \text{ with proba. } \mathbb{P}\Big(y_t \;/\rho_t, u_t\Big) = \mathrm{Tr}\left(\mathcal{K}_{u_t,y_t}(\rho_t)\right)$$

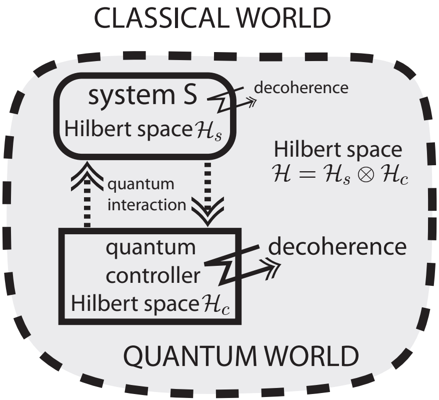
where $\mathcal{K}_{u,y}(\rho) = \sum_{\mu=1}^{m} \eta_{y,\mu} M_{u,\mu}\rho M_{u,\mu}^\dagger$ with **left stochastic matrix** $(\eta_{y,\mu})$ and Kraus operators $M_{u,\mu}$ satisfying $\sum_\mu M_{u,\mu}^\dagger M_{u,\mu} = I$.

Kraus map $\mathcal{K}_u$ (ensemble average, quantum channel)

$$\mathbb{E}\left(\rho_{t+1}|\rho_t \;\Big|\; =\right) \mathcal{K}_u(\rho_t) = \sum_y \mathcal{K}_{u,y}(\rho_t) = \sum_\mu M_{u,\mu}\rho_t M_{u,\mu}^\dagger.$$

[11] See, e.g., books: E.B Davies in 1976; S. Haroche with J.M. Raimond in 2006; C. Gardiner with P. Zoller in 2014/2015.

Quantum analogue of Watt speed governor: a **dissipative** mechanical system controls another mechanical system [12]



Optical pumping (Kastler 1950), coherent population trapping (Arimondo 1996)

Dissipation engineering, autonomous feedback: (Zoller, Cirac, Wolf, Verstraete, Devoret, Schoelkopf, Siddiqi, Martinis, Mølmer, Raimond, Brune,. . . , Lloyd, Viola, Ticozzi, Leghtas, Mirrahimi, Sarlette, PR, . . .)

(S,L,H) theory and linear quantum systems: quantum feedback networks based on stochastic Schrödinger equation, Heisenberg picture (Gardiner, Yurke, Mabuchi, Genoni, Serafini, Milburn, Wiseman, Doherty, . . . , Gough, James, Petersen, Nurdin, Yamamoto, Zhang, Dong, . . .)

Stability analysis: Kraus maps and Lindblad propagators are always contractions (non commutative diffusion and consensus).

[12] J.C. Maxwell (1868): On governors. Proc. of the Royal Society, No.100.

The closed-loop Lindblad master equation on $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_c$:

$$\frac{d}{dt}\rho = -i\Big[\mathsf{H}_s \otimes \mathsf{I}_c + \mathsf{I}_s \otimes \mathsf{H}_c + \mathsf{H}_{sc} \;,\; \rho\Big] + \sum_{\nu} \mathbb{D}_{\mathsf{L}_{s,\nu}\otimes \mathsf{I}_c}(\rho) + \sum_{\nu'} \mathbb{D}_{\mathsf{I}_s \otimes \mathsf{L}_{c,\nu'}}(\rho)$$

with $\mathbb{D}_{\mathsf{L}}(\rho) = \mathsf{L}\rho \mathsf{L}^\dagger - \frac{1}{2}\left(\mathsf{L}^\dagger \mathsf{L}\rho + \rho \mathsf{L}^\dagger \mathsf{L}\right)$ and operators made of tensor products.

• Typical goal in autonomous quantum error correction. Consider a convex subset $\overline{\mathcal{D}}_s$ of steady-states for the decoherence-free ideal system S: each density operator $\overline{\rho}_s$ on $\mathcal{H}_s$ belonging to $\overline{\mathcal{D}}_s$ satisfies $i[\mathsf{H}_s, \overline{\rho}_s] = 0$.

• Designing a **realistic** quantum controller C $(\mathsf{H}_c, \mathsf{L}_{c,\nu'})$ and coupling Hamiltonian H_{sc} stabilizing $\overline{\mathcal{D}}_s$ is non trivial. **Realistic** means in particular relying on physical time-scales and constraints:

- Fastest time-scales attached to H_s and H_c (Bohr frequencies) and averaging approximations: $\|\mathsf{H}_s\|, \|\mathsf{H}_c\| \gg \|\mathsf{H}_{sc}\|$,
- High-quality oscillations: $\|\mathsf{H}_s\| \gg \|\mathsf{L}_{s,\nu}^\dagger \mathsf{L}_{s,\nu}\|$ and $\|\mathsf{H}_c\| \gg \|\mathsf{L}_{c,\nu'}^\dagger \mathsf{L}_{c,\nu'}\|$.
- Decoherence rates of S much slower than those of C: $\|\mathsf{L}_{s,\nu}^\dagger \mathsf{L}_{s,\nu}\| \ll \|\mathsf{L}_{c,\nu'}^\dagger \mathsf{L}_{c,\nu'}\|$: model reduction by quasi-static approximations (adiabatic elimination, singular/regular perturbations).

$$\rho_{k+1} = \mathsf{U}_{u_k} \frac{\mathcal{K}_{y_k}(\rho_k)}{\mathrm{Tr}\left(\mathcal{K}_{y_k}(\rho_k)\right)} \mathsf{U}_{u_k}^\dagger, \quad \text{with prob.} \mathbb{P}_{y_k}(\rho_k) = \mathrm{Tr}\left(\mathcal{K}_{y_k}(\rho_k)\right)$$

where u_k and y_k are input/output at step k.
With the static output feedback $u_k = f(y_k)$ the closed-loop dynamics read

$$\rho_{k+1} = \mathsf{U}_{f(y_k)} \frac{\mathcal{K}_{y_k}(\rho_k)}{\mathrm{Tr}\left(\mathcal{K}_{y_k}(\rho_k)\right)} \mathsf{U}_{f(y_k)}^\dagger, \quad \text{with prob. } \mathbb{P}_{y_k}(\rho_k) = \mathrm{Tr}\left(\mathcal{K}_{y_k}(\rho_k)\right).$$

The closed-loop ensemble average reads

$$\rho_{k+1} = \overline{\mathcal{K}}(\rho_k) = \sum_y \mathsf{U}_{f(y)} \mathcal{K}_y(\rho_k) \mathsf{U}_{f(y)}^\dagger, \quad \rho_0 = \rho_0$$

Closed-loop Kraus map $\overline{\mathcal{K}}$ differs in general from $\mathcal{K} = \sum_y \mathcal{K}_y$.

Controlled qubit with diffusive fluorescence measurement:

$$\rho_{t+dt} = e^{-iu_t dt\ \sigma_y}\Big(\rho_t + \kappa\big(\sigma_-\rho_t\sigma_+ - \tfrac{1}{2}\sigma_+\sigma_-\rho_t - \tfrac{1}{2}\rho_t\sigma_+\sigma_-\big)dt \ldots$$

$$+ \sqrt{\eta\kappa}\big(\sigma_-\rho_t + \rho_t\sigma_+ - \operatorname{Tr}(\sigma_x\rho_t)\,\rho_t\big)dW_t\Big)e^{+iudt\ \sigma_y}$$

$$dy_t = \sqrt{\eta\kappa}\operatorname{Tr}(\sigma_x\rho_t)\,dt + dW_t$$

Open-loop ensemble-average with $u = 0$ converge to $|g\rangle\langle g|$

$$\frac{d}{dt}\rho = \kappa\big(\sigma_-\rho_t\sigma_+ - \tfrac{1}{2}\sigma_+\sigma_-\rho_t - \tfrac{1}{2}\rho_t\sigma_+\sigma_-\big)$$

and also the stochastic dynamics.
Closed-loop Markovian feedback with $u_t dt = g dy_t$ requires to use the Ito correction in $e^{\pm igdy_t\ \sigma_y}$:

$$e^{\pm igdy_t\ \sigma_y} = 1 + \big(\pm ig\sqrt{\eta\kappa}\operatorname{Tr}(\sigma_x\rho_t) - \tfrac{g^2}{2}\big)dt \pm igdW_t\sigma_y.$$

[13] H.M. Wiseman, G.J. Milburn: Quantum Measurement and Control. Cambridge University Press (2009)

This yields to the following closed-loop SME

$$d\rho_t = \rho_{t+dt} - \rho_t = \left(\sum_{\nu=1}^{2} \mathsf{L}_\nu \rho_t \mathsf{L}_\nu^\dagger - \tfrac{1}{2} \mathsf{L}_\nu^\dagger \mathsf{L}_\nu \rho_t - \tfrac{1}{2} \rho_t \mathsf{L}_\nu^\dagger \mathsf{L}_\nu \right) dt \ldots$$
$$+ \sqrt{\eta} \left((\mathsf{L}_1 \rho_t + \rho_t \mathsf{L}_1^\dagger - \operatorname{Tr}\left(\mathsf{L}_1 \rho_t + \rho_t \mathsf{L}_1^\dagger \right) \rho_t) \right) dW_t \ldots$$
$$+ \sqrt{1-\eta} \left((\mathsf{L}_2 \rho_t + \rho_t \mathsf{L}_2^\dagger - \operatorname{Tr}\left(\mathsf{L}_2 \rho_t + \rho_t \mathsf{L}_2^\dagger \right) \rho_t) \right) dW_t$$

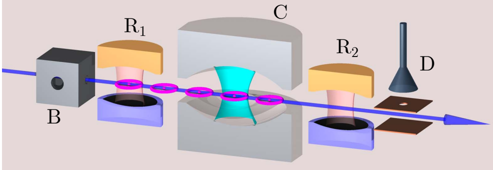
with $\mathsf{L}_1 = \sqrt{\kappa}\sigma_- - ig\sqrt{\eta}\sigma_y$ and $\mathsf{L}_2 = -ig\sqrt{1-\eta}\sigma_y$.

When $\eta = 1$ and $g = -\sqrt{\kappa}$, one has $\mathsf{L}_1 = \sqrt{\kappa}\sigma_+$, $\mathsf{L}_2 = 0$ and

$$d\rho_t = \kappa \left(\sigma_+ \rho_t \sigma_- - \tfrac{1}{2} \sigma_- \sigma_+ \rho_t - \tfrac{1}{2} \rho_t \sigma_- \sigma_+ \right) dt$$
$$+ \sqrt{\kappa} \left((\sigma_+ \rho_t + \rho_t \sigma_- - \operatorname{Tr}\left(\sigma_x \rho_t \right) \rho_t) \right) dW_t.$$

Thus the closed-loop system converges towards the excited state $|e\rangle$.
Multiple-input multiple-output (MIMO) experiment in [14]

[14] P.Campagne-Ibarcq, ..., B. Huard:Using Spontaneous Emission of a Qubit as a Resource for Feedback Control. PRL 2016.



Take two scalar control inputs (u, v) with $M_{g,(u,v)} = \cos(u + vN)$ and $M_{e,(u,v)} = \sin(u + vN)$ in

$$\rho_{k+1} = \frac{M_{y_k,(u_k,v_k)} \rho_k M_{y_k,(u_k,v_k)}}{\mathrm{Tr}\left(M_{y_k,(u_k,v_k)} \rho_k M_{y_k,(u_k,v_k)}\right)}$$

where $y_k = y \in \{g, e\}$ with probability $\mathrm{Tr}\left(M_{y,(u_k,v_k)} \rho_k M_{y,(u_k,v_k)}\right)$.
Assume support of ρ_0 in $\mathrm{span}\{|0\rangle, |1\rangle, \ldots, |2^m - 1\rangle\}$ for some integer $m > 0$. Then the following closed-loop dynamics

$$\rho_{k+1} = \frac{M_{y_k,(u_k,v_k)} \rho_k M^\dagger_{y_k,(u_k,v_k)}}{\mathrm{Tr}\left(M_{y_k,(u_k,v_k)} \rho_k M^\dagger_{y_k,(u_k,v_k)}\right)}$$

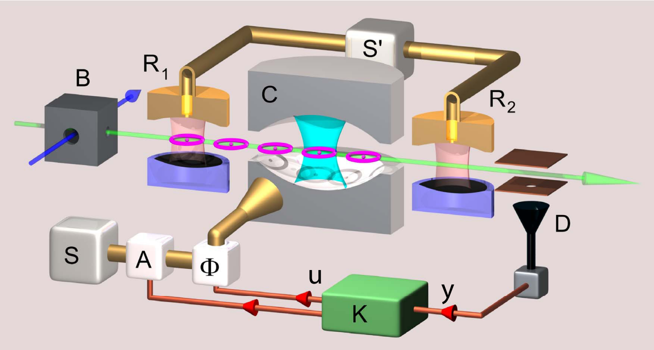
where (u_k, v_k) depends on $(y_{k-1}, \ldots, y_0)$ as follows ($f(g) = 0$ and $f(e) = 1$)

$$u_k = -\frac{\pi}{2^{k+1}} \left(\sum_{\ell=0}^{k-1} f(y_\ell) 2^\ell\right), \quad v_k = \frac{\pi}{2^{k+1}}$$

converges in m step towards the Fock state $n = \sum_{\ell=0}^{m-1} f(y_\ell) 2^\ell$.

[15] Haroche/Raimond/Brune: Measuring photon numbers in a cavity by atomic interferometry: optimizing the convergence procedure. J. Phys. II France , 2(4):659–670 (1992).)

The photon box of the Laboratoire Kastler-Brossel (LKB):
group of S.Haroche, J.M.Raimond and M. Brune. [16]



Stabilization of a quantum state with exactly $n = 0, 1, 2, 3, \ldots$ photon(s).

Experiment: C. Sayrin et. al., Nature 477, 73-77, September 2011.

Theory: I. Dotsenko et al., Physical Review A, 80: 013805-013813, 2009.
R. Somaraju et al., Rev. Math. Phys., 25, 1350001, 2013.
H. Amini et. al., Automatica, 49 (9): 2683-2692, 2013.

[16] Courtesy of Igor Dotsenko. **Sampling period** $80\ \mu s$.

Input u: classical amplitude of a coherent micro-wave pulse.
State ρ: the density operator of the photon(s) trapped in the cavity.
Output y: quantum projective measurement of the probe atom.
The ideal model reads

$$\rho_{k+1} = \begin{cases} \frac{D_{u_k} M_g \rho_k M_g^\dagger D_{u_k}^\dagger}{\mathrm{Tr}\left(M_g \rho_k M_g^\dagger\right)} & y_k = g \text{ with probability } \mathbb{P}_{g,k} = \mathrm{Tr}\left(M_g \rho_k M_g^\dagger\right) \\ \frac{D_{u_k} M_e \rho_k M_e^\dagger D_{u_k}^\dagger}{\mathrm{Tr}\left(M_e \rho_k M_e^\dagger\right)} & y_k = e \text{ with probability } \mathbb{P}_{e,k} = \mathrm{Tr}\left(M_e \rho_k M_e^\dagger\right) \end{cases}$$

- Displacement unitary operator ($u \in \mathbb{R}$): $D_u = e^{u a^\dagger - u a}$ with $a = \text{upper diag}(\sqrt{1}, \sqrt{2}, \ldots)$ the photon annihilation operator.
- Measurement Kraus operators in the linear dispersive case $M_g = \cos\left(\frac{\phi_0 N + \phi_R}{2}\right)$ and $M_e = \sin\left(\frac{\phi_0 N + \phi_R}{2}\right)$: $M_g^\dagger M_g + M_e^\dagger M_e = I$ with $N = a^\dagger a = \text{diag}(0, 1, 2, \ldots)$ the photon number operator.

With a sampling time of 80 μs, the controller is classical

- Goal: stabilization of the steady-state $|\bar{n}\rangle\langle\bar{n}|$ (controller set-point).
- At each time step k:
  1. read y_k the measurement outcome for probe atom k.
  2. update the quantum state estimation $\rho_{\mathrm{k}-1}$ to ρ_{k} from y_k
  3. compute u_k as a function of ρ_{k} (state feedback).
  4. apply the micro-wave pulse of amplitude u_k.

Observer/controller exploiting the **quantum separation principle**[17]:

1. real-time state estimation based on asymptotic observer: here quantum filtering techniques;
2. state feedback stabilization towards a stationary regime: here control Lyapunov techniques constructed with open-loop martingales $\mathrm{Tr}\left(g(\mathrm{N})\rho\right)$ and inversion of a Laplacian matrix.

[17] L. Bouten and R. van Handel: On the separation principle of quantum control. In *Quantum Stochastics and Information: Statistics, Filtering and Control*, V. P Belavkin and M. I. Guta (Eds.) World Scientific, 2008.

Experimental closed-loop data

C. Sayrin et. al., Nature 477, 73-77, Sept. 2011.

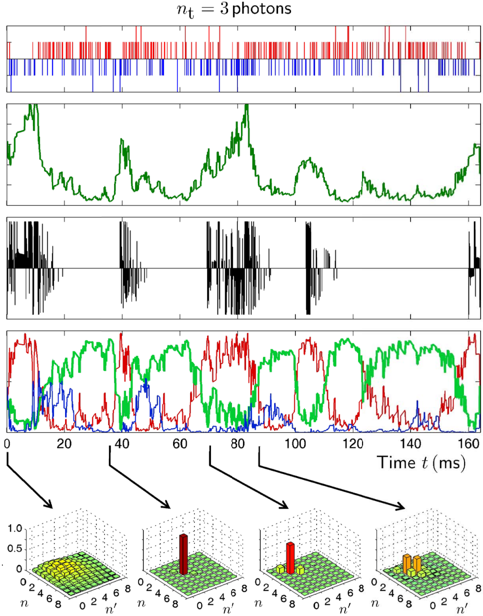
Decoherence due to finite photon life time around 70 ms)

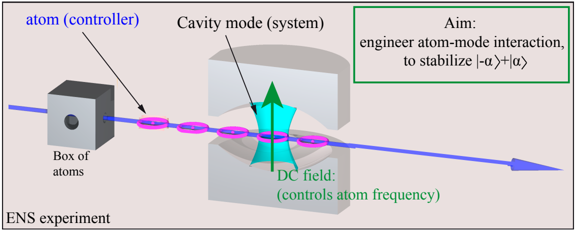
Detection efficiency 40%
Detection error rate 10%
Delay 4 sampling periods

The quantum filter takes into account cavity decoherence, measure imperfections and delays (Bayesian inference).

Truncation to 9 photons

Stabilization around 3-photon state





Jaynes-Cumming Hamiltionian

$$H(t)/\hbar = \omega_c a^\dagger a \otimes I_M + \omega_q(t) I_S \otimes \sigma_z/2 + i\Omega(t)\left(a^\dagger \otimes \sigma_- - a \otimes \sigma_+\right)/2$$

with the open-loop control $t \mapsto \omega_q(t)$ combining dispersive $\omega_q \neq \omega_c$ and resonant $\omega_q = \omega_c$ interactions.

Key issues: **convergence** of $\rho_{k+1} = \mathcal{K}(\rho_k) = M_g \rho_k M_g^\dagger + M_e \rho_k M_e^\dagger$

[18] A. Sarlette et al: Stabilization of Nonclassical States of the Radiation Field in a Cavity by Reservoir Engineering. Physical Review Letters, Volume 107, Issue 1, 2011.

Iterations $\rho_{k+1} = \mathrm{K}(\rho_k) = \mathrm{M}_g \rho_k \mathrm{M}_g^\dagger + \mathrm{M}_e \rho_k \mathrm{M}_e^\dagger$ in the Kerr frame $\rho = e^{-i h^{\mathrm{Kerr}}} \rho^{\mathrm{Kerr}} e^{i h^{\mathrm{Kerr}}}$ yields

$$\rho_{k+1}^{\mathrm{Kerr}} = \mathrm{K}^{\mathrm{Kerr}}(\rho_k^{\mathrm{Kerr}}) = \mathrm{M}_g^{\mathrm{Kerr}} \rho_k^{\mathrm{Kerr}} (\mathrm{M}_g^{\mathrm{Kerr}})^\dagger + \mathrm{M}_e^{\mathrm{Kerr}} \rho_k^{\mathrm{Kerr}} (\mathrm{M}_e^{\mathrm{Kerr}})^\dagger.$$

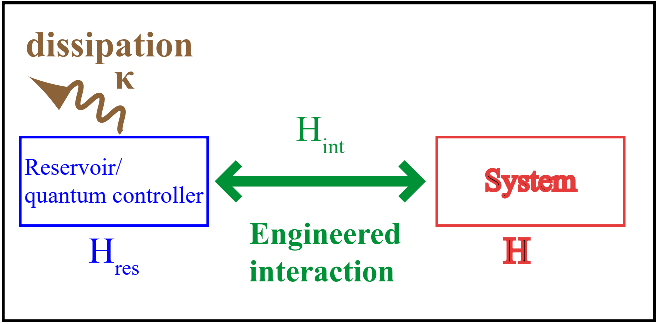
with $\mathrm{M}_g^{\mathrm{Kerr}} = \cos(\frac{u}{2}) \cos(\theta_{\mathrm{N}}/2) + \sin(\frac{u}{2}) \frac{\sin(\theta_{\mathbf{N}}/2)}{\sqrt{\mathrm{N}}} \mathrm{a}^\dagger$ and $\mathrm{M}_e^{\mathrm{Kerr}} = \sin(\frac{u}{2}) \cos(\theta_{\mathrm{N}+1}/2) - \cos(\frac{u}{2}) \mathrm{a} \frac{\sin(\theta_{\mathbf{N}}/2)}{\sqrt{\mathrm{N}}}$.

Assume $|u| \leq \pi/2$, $\theta_0 = 0$, $\theta_n \in]0, \pi[$ for $n > 0$ and $\lim_{n \mapsto +\infty} \theta_n = \pi/2$, then (Zaki Leghtas, PhD thesis (2012))

- exists a unique common eigen-state $|\psi^{\mathrm{Kerr}}\rangle$ of $\mathrm{M}_g^{\mathrm{Kerr}}$ and $\mathrm{M}_e^{\mathrm{Kerr}}$: $\rho_\infty^{\mathrm{Kerr}} = |\psi^{\mathrm{Kerr}}\rangle\langle\psi^{\mathrm{Kerr}}|$ fixed point of $\mathrm{K}^{\mathrm{Kerr}}$.
- if, moreover $n \mapsto \theta_n$ is increasing, $\lim_{k \mapsto +\infty} \rho_k^{\mathrm{Kerr}} = \rho_\infty^{\mathrm{Kerr}}$.

For well chosen experimental parameters, $\rho_\infty^{\mathrm{Kerr}} \approx |\alpha_\infty\rangle\langle\alpha_\infty|$ and $h^{\mathrm{Kerr}} \approx \pi \mathrm{N}^2/2$. Since $e^{-i\frac{\pi}{2}\mathrm{N}^2} |\alpha_\infty\rangle = \frac{e^{-i\pi/4}}{\sqrt{2}} (|\alpha_\infty\rangle + i|\text{-}\alpha_\infty\rangle)$:

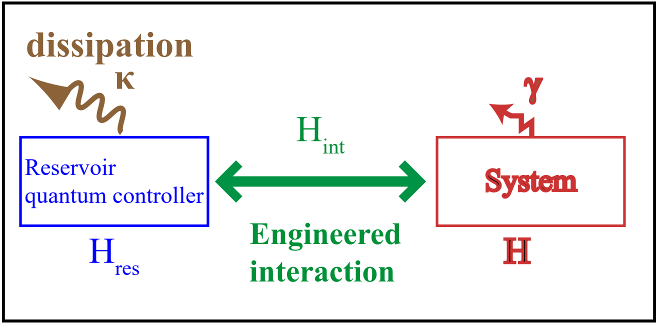
$$\lim_{k \mapsto +\infty} \rho_k = \tfrac{1}{2} \Big(|\alpha_\infty\rangle + i|\text{-}\alpha_\infty\rangle \Big) \Big(\langle\alpha_\infty| + i\langle\text{-}\alpha_\infty| \Big)$$
$$\neq \tfrac{1}{2} |\alpha_\infty\rangle\langle\alpha_\infty| + \tfrac{1}{2} |\text{-}\alpha_\infty\rangle\langle\text{-}\alpha_\infty|.$$



$$\mathbf{H} = \mathbf{H}_{res} + \mathbf{H}_{int} + \mathbf{H}$$

If $\rho \underset{t\to\infty}{\to} \rho_{res} \otimes |\bar{\psi}\rangle\langle\bar{\psi}|$ exponentially with rate $1/\tau > 0$ then ......

[19] See, e.g., the lectures of H. Mabuchi delivered at the "Ecole de physique des Houches", July 2011.



$$\mathbf{H} = \mathbf{H}_{res} + \mathbf{H}_{int} + \mathbf{H}$$

$$\ldots\ldots \quad \rho \underset{t\to\infty}{\rightarrow} \rho_{res} \otimes |\bar{\psi}\rangle\langle\bar{\psi}| + \overline{\delta\rho}, \text{ if } \tau\gamma \ll 1 \text{ then } |\overline{\delta\rho}| \ll 1$$

Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) master equation:

$$\frac{d}{dt}\rho = -i[H_0 + uH_1, \rho] + \sum_\nu \left(L_\nu \rho L_\nu^\dagger - \frac{1}{2}(L_\nu^\dagger L_\nu \rho + \rho L_\nu^\dagger L_\nu) \right)$$

- Preservation of trace, hermiticity and positivity: ρ lies in the set of Hermitian and trace-class operators that are non-negative and of trace one.
- Invariance under unitary transformations.
  A time-varying change of frame $\rho \mapsto U_t^\dagger \rho U_t$ with U_t unitary.
  The new density operator obeys to a similar master equation where $H_0 + uH_1 \mapsto U_t^\dagger (H_0 + uH_0) U_t + iU_t^\dagger \left(\frac{d}{dt} U_t\right)$ and $L_\nu \mapsto U_t^\dagger L_\nu U_t$.
- "L^1-contraction" properties. Such master equations generate contraction semi-groups for many distances (nuclear distance[20], Hilbert metric on the cone of non negative operators[21]).
- If the Hermitian operator A satisfies the operator inequality

$$i[H_0 + uH_1, A] + \sum_\nu \left(L_\nu^\dagger A L_\nu - \frac{1}{2}(L_\nu^\dagger L_\nu A + A L_\nu^\dagger L_\nu) \right) \leq 0$$

then $V(\rho) = \operatorname{Tr}(A\rho)$ is a Lyapunov function when $A \geq 0$.

[20] D.Petz (1996). Monotone metrics on matrix spaces. Linear Algebra and its Applications

[21] R. Sepulchre, A. Sarlette, PR (2010). Consensus in non-commutative spaces. IEEE-CDC.

- Quantum error corrrection requires redundancy.
- Bosonic code: instead of encoding a logical qubit in N physical qubits living in $\mathbb{C}^{2^N}$, encode a logical qubit in an harmonic oscillator living in Fock space $\text{span}\{|0\rangle, |1\rangle, \ldots, |n\rangle, \ldots\} \sim L^2(\mathbb{R}, \mathbb{C})$ of infinite dimension.
- Cat-qubit [22]: $|\psi_L\rangle \in \text{span}\{|\alpha\rangle, |\text{-}\alpha\rangle\}$ where $|\alpha\rangle$ is the coherent state of real amplitude α: $\mathrm{a}|\alpha\rangle = \alpha|\alpha\rangle$ with $\mathrm{a} = (\mathrm{q} + i\mathrm{p})/\sqrt{2}$ and $[\mathrm{q}, \mathrm{p}] = i$:

$$|\psi\rangle \sim \psi(q) \in L^2(\mathbb{R}, \mathbb{C}),\ \mathrm{q}|\psi\rangle \sim q\psi(q),\ \mathrm{p}|\psi\rangle \sim -i\frac{d\psi}{dq}(q),\ |\alpha\rangle \sim \frac{\exp\left(-\frac{(q-\alpha\sqrt{2})^2}{2}\right)}{\sqrt{2\pi}}.$$

- Stabilisation of cat-qubit via a single Lindblad dissipator $\mathrm{L} = \mathrm{a}^2 - \alpha^2$. For any initial density operator $\rho(0)$, the solution $\rho(t)$ of

$$\frac{d}{dt}\rho = \mathrm{L}\rho\mathrm{L}^\dagger - \frac{1}{2}(\mathrm{L}^\dagger\mathrm{L}\rho + \rho\mathrm{L}^\dagger\mathrm{L})$$

converges exponentially towards a steady-state density operator since

$$\frac{d}{dt}\,\mathrm{Tr}\left(\mathrm{L}^\dagger\mathrm{L}\rho\right) \leq -2\,\mathrm{Tr}\left(\mathrm{L}^\dagger\mathrm{L}\rho\right), \quad \ker(\mathrm{L}) = \text{span}\{|\alpha\rangle, |\text{-}\alpha\rangle\}.$$

Any density operator with support in $\text{span}\{|\alpha\rangle, |\text{-}\alpha\rangle\}$ is a steady-state.

[22] M. Mirrahimi, Z. Leghtas, . . . , M. Devoret: Dynamically protected cat-qubits: a new paradigm for universal quantum computation. 2014, New Journal of Physics.

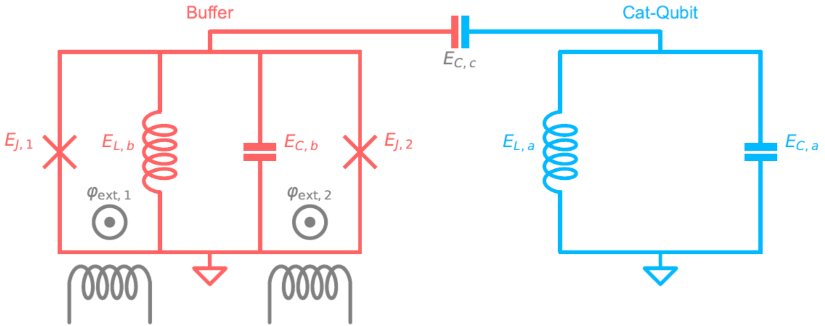


Figure S3. Equivalent circuit diagram. The cat-qubit (blue), a linear resonator, is capacitively coupled to the buffer (red). One recovers the circuit of Fig. 2 by replacing the buffer inductance with a 5-junction array and by setting $\varphi_\Sigma = (\varphi_{\mathrm{ext},1} + \varphi_{\mathrm{ext},2})/2$ and $\varphi_\Delta = (\varphi_{\mathrm{ext},1} - \varphi_{\mathrm{ext},2})/2$. Not shown here: the buffer is capacitively coupled to a transmission line, the cat-qubit resonator is coupled to a transmon qubit

[23] R. Lescanne, M. Villiers, Th. Peronnin, . . . , M. Mirrahimi and Z. Leghtas: Exponential suppression of bit-flips in a qubit encoded in an oscillator. 2020, Nature Physics

Oscillator a with **quantum controller based on a damped oscillator** b:

$$\frac{d}{dt}\rho = g_2\Big[(\mathrm{a}^2-\alpha^2)\mathrm{b}^\dagger - ((\mathrm{a}^\dagger)^2-\alpha^2)\mathrm{b}\ ,\ \rho\Big] + \kappa_b\Big(\mathrm{b}\rho\mathrm{b}^\dagger - (\mathrm{b}^\dagger\mathrm{b}\rho + \rho\mathrm{b}^\dagger\mathrm{b})/2\Big)$$

with $\alpha\in\mathbb{R}$ such that $\alpha^2 = u/g_2$, the drive amplitude $u\in\mathbb{R}$ applied to mode b and $1/\kappa_b > 0$ the life-time of photon in mode b.
Any density operators $\bar\rho = \bar\rho_a\otimes|0\rangle\langle 0|_b$ **is a steady-state as soon as the support of $\bar\rho_a$ belongs to the two dimensional vector space spanned by the quasi-classical wave functions $|\alpha\rangle$ and $|\text{-}\alpha\rangle$** (range$(\bar\rho_a)\subset$ span$\{|\alpha\rangle, |\text{-}\alpha\rangle\}$)

Usually $\kappa_b \gg |g_2|$, mode b relaxes rapidly to vaccuum $|0\rangle\langle 0|_b$, can be eliminated adiabatically (**singular perturbations**, second order corrections) to provides the slow evolution of mode a

$$\frac{d}{dt}\rho_a = \tfrac{4|g_2|^2}{\kappa_b}\Big(\mathrm{L}\rho\mathrm{L}^\dagger - \frac{1}{2}(\mathrm{L}^\dagger\mathrm{L}\rho + \rho\mathrm{L}^\dagger\mathrm{L})\Big) \text{ with } \mathrm{L} = \mathrm{a}^2-\alpha^2.$$

Convergence via the exponential Lyapunov function $V(\rho) = \operatorname{Tr}\left(\mathrm{L}^\dagger\mathrm{L}\rho\right)$ [24]

[24] For a mathematical proof of convergence analysis in an adapted Banach space, see :R. Azouit, A. Sarlette, PR: Well-posedness and convergence of the Lindblad master equation for a quantum harmonic oscillator with multi-photon drive and damping. 2016, ESAIM: COCV.

Since $\langle\alpha|\text{-}\alpha\rangle = e^{-2\alpha^2} \approx 0$:

$$|0_L\rangle \approx |\alpha\rangle,\ |1_L\rangle \approx |\text{-}\alpha\rangle,\ |+_L\rangle \propto \tfrac{|\alpha\rangle+|\text{-}\alpha\rangle}{\sqrt{2}},\ |-_L\rangle \propto \tfrac{|\alpha\rangle-|\text{-}\alpha\rangle}{\sqrt{2}}.$$

Photon loss as dominant error channel (dissipator a with $0 < \kappa_1 \ll 1$):

$$\frac{d}{dt}\rho_a = \mathcal{D}_{a^2-\alpha^2}(\rho) + \kappa_1\mathcal{D}_a(\rho)$$

with $\mathcal{D}_L(\rho) = L\rho L^\dagger - \frac{1}{2}(L^\dagger L\rho + \rho L^\dagger L)$.

- if $\rho(0) = |0_L\rangle\langle 0_L|$ or $|1_L\rangle\langle 1_L|$, $\rho(t)$ converges to a statistical mixture of quasi-classical states close to $\frac{1}{2}|\alpha\rangle\langle\alpha| + \frac{1}{2}|\text{-}\alpha\rangle\langle\text{-}\alpha|$ in a time

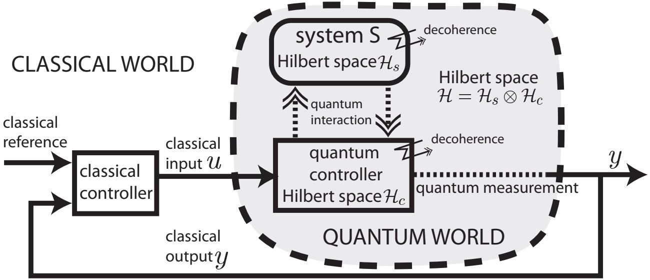
  $$T_{bit-flip} \sim \frac{e^{2\alpha^2}}{\kappa_1}$$

  since $a|0_L\rangle \approx \alpha|0_L\rangle$ and $a|1_L\rangle \approx -\alpha|1_L\rangle$.
- if $\rho(0) = |+_L\rangle\langle +_L|$ or $|-_L\rangle\langle -_L|$, $\rho(t)$ converges also to the same statistical mixture in a time

  $$T_{phase-flip} \sim \frac{1}{\kappa_1\alpha^2}$$

  since $a|+_L\rangle = \alpha|-L\rangle$ and $a|-_L\rangle = \alpha|+L\rangle$.

Take α large to ignore bit-flip and to correct only the phase-flip with $1D$ repetition code: important overhead reduction investigated by the startup **Alice&Bob** and also by **AWS**.



To protect quantum information stored in system S:

- fast stabilization and protection mainly achieved by **quantum controllers** (autonomous feedback stabilizing decoherence-free sub-spaces);
- slow decoherence and perturbations, parameter estimation mainly tackled by **classical controllers and estimation algorithms** (measurement-based feedback and estimation "finishing the job")

Need of adapted mathematical and numerical methods for high-precision dynamical modeling and control based on (stochastic) master equations.

Quantic research group ENS/Inria/Mines/CNRS, June 2023