1. Plane curves

We are interested in polynomials in two variables x, y with real or complex coefficients. Let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively.

Some examples of polynomials of the kind we will study are:

$$x^2 + y^2 - 1, xy^{10} - \pi x^4 y^4 + 100x^{12} + \sqrt{2}y^{71}.$$

For non-negative integers i, j and a complex number c, a term of the form $cx^i y^j$ is called a *monomial*. The *degree* of such a monomial is defined to be $i + j$. A polynomial is thus simply a sum of monomials. The degree of a polynomial is the degree of the largest degree monomial which appears in it. The degree of the two polynomials above are 2 and 71, respectively.

Exercise 1.1. Give a few examples of polynomials and note their degrees.

We are interested in *zeros* or *roots* of polynomials. Let $f(x, y)$ be a polynomial. An element of (a, b) of $\mathbb{C}^2$ is called a *zero* of $f(x, y)$ if $f(a, b) = 0$.

For example, $(0, 1)$ is a zero of the polynomial $x^2 - 1 = 0$ as well as of $xy^2 + y - 1$. On the other hand, $(0, 1)$ is not a zero of $x^2y^3 + 1$.

Definition 1.2. The set of zeros of a polynomial is called a *plane curve*.

Note that plane curves are subsets of $\mathbb{C}^2$. If a plane curve C is the zero set of a polynomial f, we also say that C *is defined by* f. If C is defined by f, we also write $C = V(f)$.

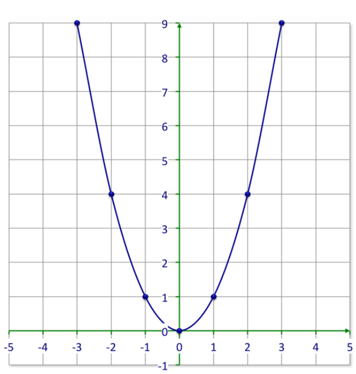
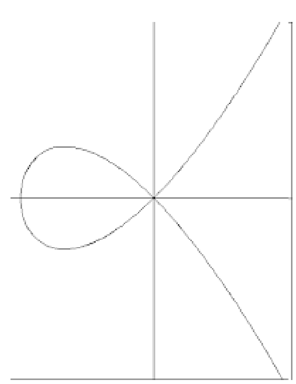
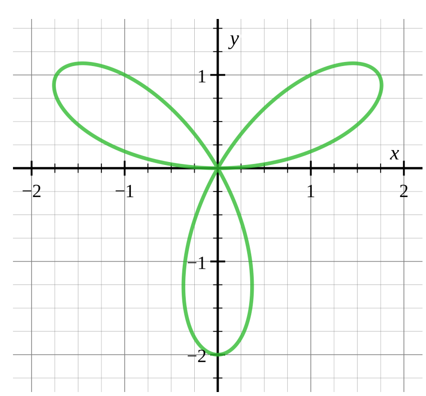
We have special names for plane curves of small degrees.

Lines: A line is the zero set of a polynomial of degree 1.
Conics: Zeros of a polynomial of degree 2 form a *conic*.
Cubics: *Cubics* are zeroes of degree 3 polynomials.
Curves of degree d: A *curve of degree* d is the zero set of a polynomial of degree d.

Exercise 1.3. Draw the curves in $\mathbb{R}^2$ defined by the following polynomials.

(1) $2x + 6y - 5$.
(2) $x^2 + y^2$.
(3) $x^2 + y^2 - 1$.
(4) $x^2 - y^2$.
(5) $x^2 + y^2 + 1$.
(6) $x^2 - y$.
(7) $\frac{x^2}{4} + \frac{y^2}{9} - 1$.
(8) $y^2 - x^3$.
(9) $y^2 - x^2(x + 1)$.

Note that $(0, 0)$ is contained in each of the following three curves:

Exercise 1.4. Can you identify the differences in how $(0,0)$ sits inside these curves?

Definition 1.5. Let (a, b) is a point in $\mathbb{C}^2$. Let C be a plane curve defined by a polynomial $f(x, y)$. Suppose that m is the largest non-negative integer such that all the partial derivatives of f of order up to m vanish at (a, b). The *multiplicity* of C at (a, b) is defined to be $m + 1$.

The multiplicity of a curve C a point p is denoted by $\mathrm{mult}_p C$.

Example 1.6. The multiplicities of the curves defined by $x^2 + y^2 - 1, x^2 + y^2 + 2x, y^2 - x^3, y^9 + x^{20}$ at $(0,0)$ are $0, 1, 2$ and 9, respectively.

Exercise 1.7. Find the multiplicity of the following curves at the indicated points.

(1) $x + y$ at $(3, 3)$.
(2) $x + y$ at $(10, -10)$.
(3) x^{23} at $(0, 0)$.
(4) x^9y^2 at $(0, 0)$.
(5) x^9y^2 at $(1, 0)$.
(6) x^9y^2 at $(0, 1)$.
(7) $x^2y^{17} + xy^2 - 5x^{10}$ at $(0, 0)$.

Exercise 1.8. Let C be a plane curve of degree d. Let p be a point on C and let m be the multiplicity of C at p. Show that $1 \leqslant m \leqslant d$. Give examples to show that both the extreme values can be attained.

Now we will study how to *measure* the set of polynomials of a given degree d. Note that for any $d \geqslant 0$, the set of polynomials of degree d is infinite. However, we can describe these sets with finitely many parameters.

Let $d = 0$. Note that a polynomial of degree 0 is simply an element a of $\mathbb{C}$. So every such polynomial is a complex multiple of 1. So we can say that 1 is enough to describe all the polynomials of degree 0. Since only one monomial (namely, 1) is needed to describe them, we say that the *dimension* of the set of polynomials of degree 0 is 1.

Let $d = 1$. An arbitrary polynomial of degree 1 is of the form $a + bx + cy$ where a, b, c are complex numbers. So we say that $1, x, y$ describe the set of all the polynomials of degree 1. The dimension of the set of polynomials of degree 1 is 3.

Similarly, the monomials $1, x, y, x^2, y^2, xy$ can describe any degree 2 polynomial and the dimension of the set of polynomials of degree 2 is 6.

Exercise 1.9. For any non-negative integer d, show that set of all the curves of degree d has dimension $\frac{(d+2)(d+1)}{2}$ and list the monomials which can express any polynomial of degree d.

Now we want to study the following question.

Exercise 1.10 (Main Exercise)**.** Fix r points $p_1, \ldots, p_r$ in $\mathbb{C}^2$. Let $d, m \geqslant 1$ be integers. Is there a curve of degree d which has multiplicity at least m at p_i for each $i = 1, \ldots, r$?

In order to study this, let us look at some specific cases of the above question.

Exercise 1.11. Given integers $d, m \geqslant 1$, is there a curve of degree d which has multiplicity at least m at $(0,0)$?

Next, generalise to any one point p in $\mathbb{C}^2$:

Exercise 1.12. Fix a point p in $\mathbb{C}^2$. Let $d, m \geqslant 1$ be integers. Is there a curve of degree d which has multiplicity at least m at p? Can you find some conditions on d and m so that the answer is YES?

Exercise 1.13. Suppose $r = 2$. That is, we are given two points p_1, p_2 in $\mathbb{C}^2$. Is there a curve of degree d which has multiplicity at least m at both p_1 and p_2, in the following cases?

$d = 1, m = 1$; $d = 1, m = 2$; $d = 2, m = 2$; $d = 3, m = 2$.

Now the same question, in general, for $r = 2$:

Exercise 1.14. Fix 2 points p_1, p_2 in $\mathbb{C}^2$. Let $d, m \geqslant 1$ be integers. Under what conditions on d and m, is there a curve of degree d which has multiplicity at least m at p_i for each $i = 1, 2$?

Exercise 1.15. Is there a conic through any given five points of $\mathbb{C}^2$? What about through any given six points?

The answer to Main Exercise 1.10 is given by the following.

Exercise 1.16. Let $p_1, \dots, p_r \in \mathbb{C}^2$ be distinct points. Let $d > 0, m_1, \dots, m_r \geqslant 0$ be integers.

If $\frac{(d+2)(d+1)}{2} - \sum_{i=1}^{r} \frac{(m_i+1)m_i}{2} > 0$, show that there exists a degree d curve passing through p_i with multiplicity at least m_i for $i = 1, 2, \dots, r$.

Using Exercise 1.16, show the following.

Exercise 1.17.

(1) There is a conic through any given 5 points in $\mathbb{C}^2$.
(2) There is a cubic through any given 9 points in $\mathbb{C}^2$.
(3) Let $p_1, p_2, \dots, p_7$ be distinct points in $\mathbb{C}^2$. There exists a cubic passing through p_1 with multiplicity 2 and passing through other points $p_2, \dots, p_7$.

1.1. **Infimum and supremum.** Let S be a non-empty set of real numbers. We say that S is *bounded above* if there exists an integer N such that $s \leqslant N$ for every $s \in S$. Similarly, S is *bounded below* if there exists an integer M such that $s \geqslant M$ for every $s \in S$.

Suppose that S is bounded below. Then a real number x is called a *lower bound* of S if $s \geqslant x$ for all $s \in S$. The *greatest lower bound* or *infimum* of S is a real number x satisfying the following conditions:

- x is a lower bound of S, and
- if y is a lower bound of S then $x \geqslant y$.

If S is bounded above, the *least upper bound* or *supremum* of S is defined similarly.

It is a fact that every bounded above set of real numbers has a supremum and every bounded below set of real numbers has an infimum. The infimum and supremum of S are denoted by $\inf S$ and $\sup S$, respectively.

Exercise 1.18. Let S be a bounded below set. If x, y are both infimums of S, show that $x = y$. Similarly, the supremum of a bounded above set is unique.

Exercise 1.19. Find the infimum and supremum of the following sets, when applicable.

(1) $S_1 = \{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 1\}$.
(2) $S_2 = \{x \in \mathbb{R} \mid 0 < x < 1\}$.
(3) $S_3 = \{x \in \mathbb{Q} \mid 0 < x < 1\}$.
(4) $S_4 = \{\frac{a}{b} \in \mathbb{Q} \mid a, b$ are positive integers such that $a + b \leqslant 4\}$.
(5) $S_5 = \{x \in \mathbb{Q} \mid x^2 \leqslant 2\}$.

Note that the infimum or the supremum of a set S need not belong to S.

1.2. **An invariant.**

Definition 1.20. Let $p_1, \dots, p_r$ in $\mathbb{C}^2$ be distinct points. Then we define the following number:

$$A(p_1, p_2, \dots, p_r) = \inf \frac{d}{m},$$

where the infimum is taken over all ratios d/m such that there is a degree d curve passing through each p_i with multiplicity at least m.

Exercise 1.21.

(1) Show that $A(p) = 1$ for any point p in $\mathbb{C}^2$.
(2) Show that $A(p_1, p_2) = 1$ for any distinct points p_1, p_2 in $\mathbb{C}^2$.
(3) Let $p_1 = (1, 0), p_2 = (2, 0), p_3 = (3, 0)$ and $q = (1, 1)$. Show that $A(p_1, p_2, p_3) = 1$ and $A(p_1, p_2, q) \leqslant \frac{3}{2}$.

We now give some upper bounds for $A(p_1, \dots, p_r)$.

Exercise 1.22. Let r be a positive integer. Show that $A(p_1, \dots, p_r) \leqslant \sqrt{r}$, for all points $p_1, \dots, p_r \in \mathbb{C}^2$. Use Exercise 1.16.

Hints for Exercise 1.22. By Exercise 1.16, there exists a curve of degree d passing through r points with multiplicity at least m if

$$\frac{(d+2)(d+1)}{2} - r\frac{(m+1)m}{2} > 0.$$

This is equivalent to $(d+2)(d+1) - r(m+1)m > 0$. Now write $t = d/m$, or $d = tm$. Then the above condition is equivalent to

$$(t^2 - r)m^2 + (3t - r)m + 2 > 0. \tag{1.1}$$

For which values of t is this inequality valid?

Suppose that $t^2 > r$. For m *large enough*, show that (1.1) is valid by following the argument below:

Consider the graph of the **quadratic** function in the variable m on the left hand side of (1.1). It goes to infinity as m goes to infinity, provided the leading constant is positive.

We say that the term $(t^2 - r)m^2$ *dominates* left hand side of (1.1).

So for any $d/m = t > \sqrt{r}$, there exists a curve of degree d passing through $p_1, p_2, \dots, p_r$ with multiplicity at least m. Thus $A(p_1, \dots, p_r) \leqslant \sqrt{r}$.

We are now able to give the following definition.

Definition 1.23. Let r be a positive integer. Then we define

$$A_r = \sup A(p_1, \dots, p_r),$$

where the supremum is taken over all sets of distinct points $p_1, \dots, p_r$ in $\mathbb{C}^2$.

2. Bézout's theorem

Let C, D be plane curves defined by polynomials f and g, respectively. We are interested in the intersection $C \cap D$, i.e., the set of common points of C and D. So $C \cap D$ consists of points p in $\mathbb{C}^2$ such that $f(p) = g(p) = 0$. In general, $C \cap D$ can be finite or infinite.

Definition 2.1. We say that C and D have *proper* intersection if their intersection is a finite set. If the intersection is not proper, we say C and D have *improper* intersection.

Exercise 2.2. In each of the following cases determine if the intersection of the two curves is proper or improper.

(1) $C = V(x)$ and $D = V(y)$.
(2) $C = V(xy)$ and $D = V(y)$.
(3) $C = V(y^2 - x^3)$ and $D = V(x)$.
(4) $C = V(y^2 - x^3)$ and $D = V(y^2 - x^3 - x^2)$.
(5) $C = V(y^2 - x(x-2)(x+1))$ and $D = V(y^2 + x^2 - 2x)$.
(6) $C = V(x^2 + y^2 + x^3 + y^3)$ and $D = V(x^3 + y^3 - 2xy)$.
(7) $C = V(y^5 - x(y^2 - x)^2)$ and $D = V(y^4 + y^3 - x^2)$.
(8) $C = V((x^2 + y^2)^2 + 3x^2y - y^3)$ and $D = V((x^2 + y^2)^3 - 4x^2y^2)$.

The following is a well-known classical result about plane curves.

Theorem 2.3 (Bézout's theorem [1])**.** Let C, D be plane curves which intersect properly. Then

$$(\deg C)(\deg D) \geqslant \sum_{p \in C \cap D} (\mathrm{mult}_p C)(\mathrm{mult}_p D).$$

[1]The statement given here suffices for our purposes, but it is significantly weaker than the classical Bézout's theorem. The full and much stronger version requires the notion of *intersection multiplicity* of two plane curves.

Exercise 2.4. In each of the problems in Exercise 2.2 for which the intersection is proper, verify that Theorem 2.3 is true.

We now use Bézout's theorem to compute the value of A_r for small r.

Exercise 2.5. Show the following.
$A_1 = A_2 = 1$;
$A_3 = 3/2$;
$A_4 = A_5 = 2$;
$A_6 = 12/5$;
$A_7 = 21/8$;
$A_8 = 48/17$; and
$A_9 = 3$.

2.1. **Nagata Conjecture.** The following is a famous *open problem*:

Conjecture 2.6 (Nagata Conjecture, 1958)**.** $A_r = \sqrt{r}$ for any integer $r \geqslant 9$.

Exercise 2.7. It is easy to see that if the points are *special*, then the equality in Conjecture 2.6 does not hold. For example, for any collinear points $p_1, \dots, p_r$, calculate $A(p_1, \dots, p_r)$.

Remark 2.8. Nagata proved, in 1958, that the above conjecture is true when $r = s^2$ for an integer $s \geqslant 3$. The conjecture is open in all other cases, starting from $r = 10$.