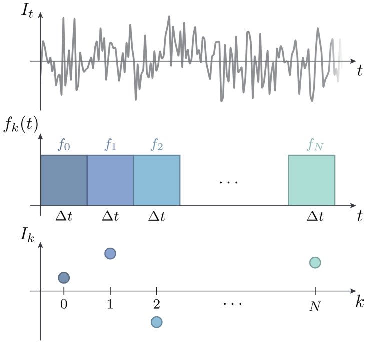
Digitized continuous quantum trajectories



continuous monitoring in the real world

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The team

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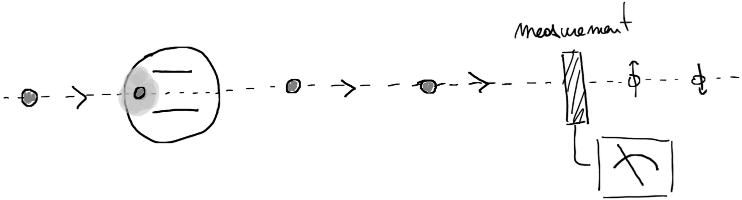
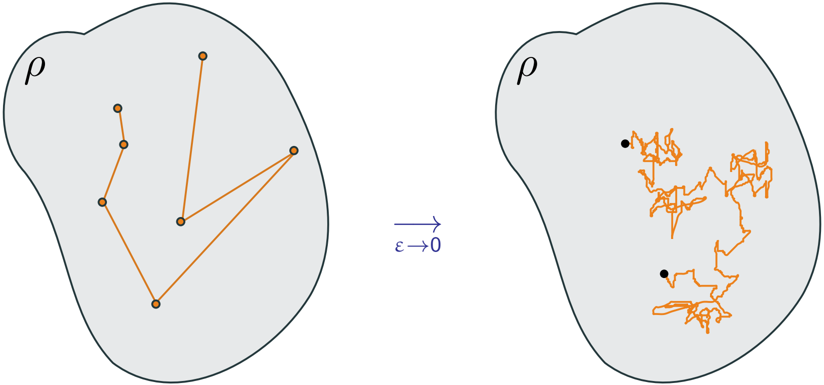
From discrete to continuous measurement

Continuous measurement – without Zeno effect



- time between ancillas $\Delta t \propto \varepsilon$
- interaction strength $\omega \propto \sqrt{\epsilon}$

From discrete to continuous measurement



Continuous measurement (diffusive / homodyne case)

For single measured operator L we have the SME:

$$d\rho_t = \underbrace{\mathcal{L}(\rho_t)\,dt}_{\texttt{Lindblad}} + \underbrace{\mathcal{M}(\rho_t, dY_t)}_{\texttt{Measurement}}$$

with the signal $I_t = \frac{dY_t}{dt}$

$$dY_t = \sqrt{\eta}\,\mathrm{tr}\left[(L + L^\dagger)\rho_t\right]dt + \underbrace{dW_t}_{\texttt{Wiener}}.$$

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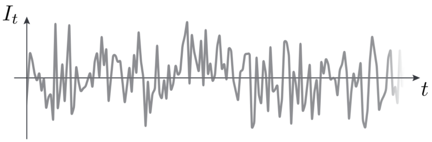
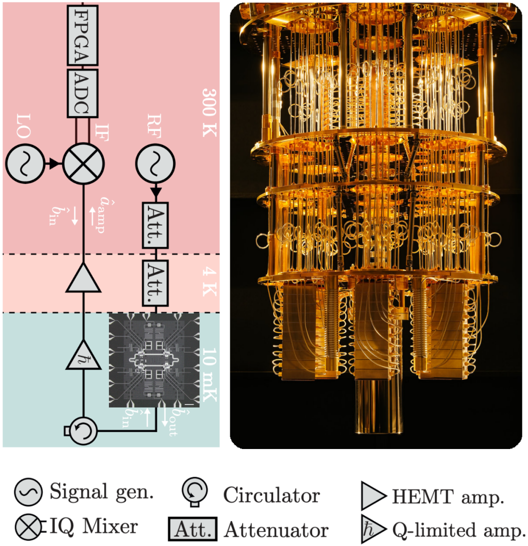
- today only homodyne – but applies to jump and heterodyne too

Continuous measurement signals are digitized



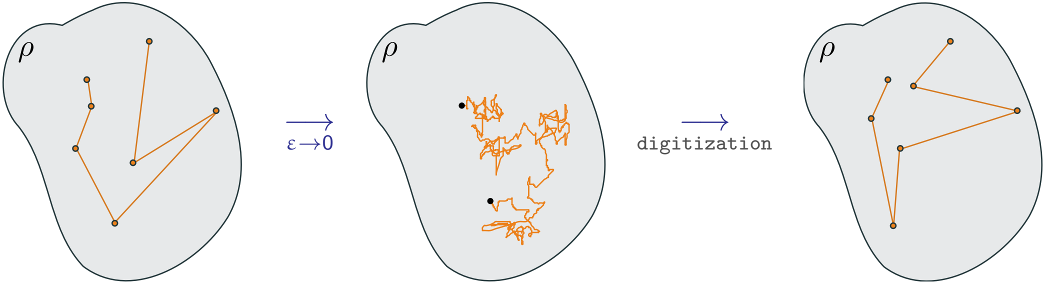
The *sharp* signal I_t is not empirically accessible:

- Formally it has the regularity of white noise (distribution)
- Practically, one can only store its average on time bins



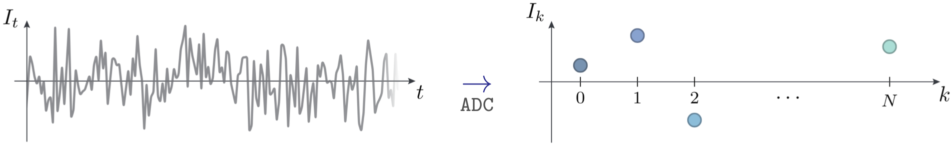
Blais et al. "Circuit quantum electrodynamics." Rev. Mod. Phys. (2021) – quantum computer image from IBM

From the discrete to the continuum and back



discrete time | continuous time | discretized continuous time

A problem in theory and practice



Currently: reconstruct with `Euler`, `Runge-Kutta`,or `Rouchon` scheme and

$$\frac{\mathrm{d}Y_t}{\mathrm{d}t} \simeq \frac{\Delta Y_t}{\Delta t} = \frac{I_k}{\Delta t} \quad \text{with} \quad I_k = \int_{(k-1)\Delta t}^{k\Delta t} \mathrm{d}Y_t$$

push ADC to the max so $\Delta t \ll$ relevant timescales

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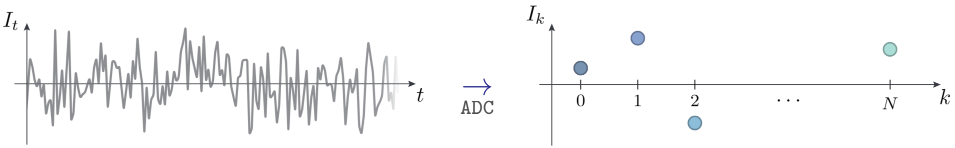

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push ADC to the max so $\Delta t \ll$ relevant timescales

Problems:

- As we probe faster timescales, Δt is not so small...
- Δt small $\implies$ lots of data: at 1GHz in `Float16`, 4GB/s per quadrature
- Theoretically, we would like to know the error!

2 Solutions

Tasks like parameter estimation and max-like tomography are sensitive to Δt

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Possible fix:

1. Rely on correlation functions as much as possible [Pierre Guilmin's talk]

$$\mathbb{E}\left[I_{k_1} I_{k_2} \cdots I_{k_n}\right]$$

since they are known exactly

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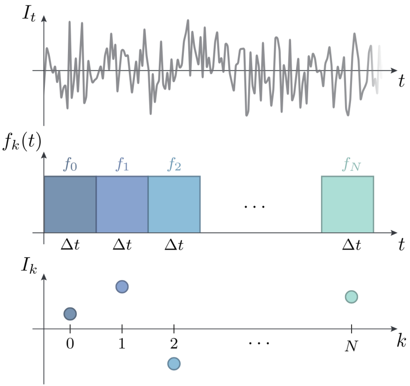
2. Construct the Kraus map $\Phi_{\Delta t}^{I_k}(\rho)$ for a finite time bin Δt

Setup:



All we know is the binned signal

$$I_k = \int_{(k-1)\Delta t}^{k\Delta t} \mathrm{d}Y_t$$

Some information is gone: we cannot know the true ρ_t beyond order Δt!

Definition

The "binned" conditional state $\bar{\rho}_k$

$$\bar{\rho}_k := \mathbb{E}\Big[\underbrace{\rho_{k\Delta t}}_{\texttt{true state}} \;\Big|\; \underbrace{I_k, I_{k-1}, \ldots, I_1}_{\texttt{digitized signal}}\Big]$$

- Clearly the best one can hope for $\rightarrow$ Bayesian optimum

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- In French we call it **robinet** (faucet) [binned ρ = rho binné = robinet]

Tool 1: Bayes' rule

$$
\begin{aligned}
\bar{\rho}_k &= \mathbb{E}\Big[\, \rho_{k\Delta t} \mid I_k, I_{k-1}, \ldots, I_1 \,\Big] && \texttt{Definition} \\
&= \mathbb{E}\Big[\, \rho_{k\Delta t} \mid I_k, \bar{\rho}_{k-1} \,\Big] && \texttt{Markov} \\
&= \frac{\mathbb{E}\left[\delta\left(I_k - \int_{k\Delta t}^{(k+1)\Delta t} \mathrm{d}Y_t\right) \rho_{k\Delta t} \mid \bar{\rho}_{k-1}\right]}{\mathbb{E}\left[\delta\left(I_k - \int_{(k-1)\Delta t}^{k\Delta t} \mathrm{d}Y_t\right) \mid \bar{\rho}_{k-1}\right]} && \texttt{Bayes}
\end{aligned}
$$

Tool 2: Fourier transform

Write the Dirac δ in Fourier:

$$\delta(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{d}p \, e^{ipx}$$

which implies

$$\delta\left(I_k - \int_{(k-1)\Delta t}^{k\Delta t} \mathrm{d}Y_t\right) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{d}p \, \exp\left[ip\left(I_k - \int_{(k-1)\Delta t}^{k\Delta t} \mathrm{d}Y_t\right)\right)$$

Tool 3: Tilted Lindbladian

Define the p-tilted ρ:

$$\rho_t^{(p)} = \mathbb{E}\left[\exp\left(-ip\int_0^t \mathrm{d}Y_t\right)\,\rho_t\right]$$

Using Itô's lemma, one can show it obeys the p-tilted Lindblad equation

$$\frac{\mathrm{d}\rho_t^{(p)}}{\mathrm{d}t} = \mathcal{L}\cdot\rho_t^{(p)} - ip\mathcal{C}_L\cdot\rho_t^{(p)} - \frac{p^2}{2}\rho_t^{(p)}$$

with $\mathcal{C}_L\cdot\rho = \sqrt{\eta}\left(L\rho + \rho L^\dagger\right)$

Final formula

Putting all together:

Exact Kraus map

$$\tilde{\rho}_k = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{d}p \;\; e^{ip\, I_k - \Delta t \frac{p^2}{2}} \;\; \mathcal{T}\mathrm{exp}\left(\int_{(k-1)\Delta t}^{k\Delta t} \mathcal{L}_t - i\, p\, \mathcal{C}_L \right) \cdot \bar{\rho}_{k-1}$$

$$\bar{\rho}_k = \frac{\tilde{\rho}_k}{\mathrm{tr}[\tilde{\rho}_k]}$$

- $\mathcal{T}\,\mathrm{exp}$ = time-ordered exponential = solution to linear ODE (regular exponential if $\mathcal{L}$ time-independent)

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- $\mathcal{T}\,\mathrm{exp}$ = time-ordered exponential = solution to linear ODE (regular exponential if $\mathcal{L}$ time-independent)
- Is it numerically tractable?

Large time bins with Gaussian quadratures

Numerical integration (Folklore)

$1d$ integrals of smooth functions **numerically exact** with ≈ 100 points

$$\int f(x)\mathrm{d}x \underset{\texttt{Float64}}{=} \sum_{j=1}^{\text{100-ish}} w_j\, f(x_j) \quad \texttt{[Gauss quadrature]}$$

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We include $\exp(-\Delta t \frac{p^2}{2})$ into the measure and use a Gauss-Hermite quadrature:

$$\tilde{\rho}_k = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{d}p \;\; e^{ip\, l_k - \Delta t \frac{p^2}{2}} \;\; \mathcal{T}\exp\left(\int_{(k-1)\Delta t}^{k\Delta t} \mathcal{L} - i\, p\; \mathcal{C}_L\right) \cdot \bar{\rho}_{k-1}$$

$$\underset{\texttt{Float64}}{=} \frac{1}{2\pi}\sum_j w_j\; e^{ip_j\, l_k} \;\underbrace{\mathcal{T}\exp\left(\int_{(k-1)\Delta t}^{k\Delta t} \mathcal{L} - i\, p_j\, \mathcal{C}_L\right) \cdot \bar{\rho}_{k-1}}_{\texttt{solution of linear ODE}}$$

$\Longrightarrow$ Solving (tilted) Lindblad equation on a time bin Δt for ≈ 100 different p_j

A first sanity check: signal probability for qubit

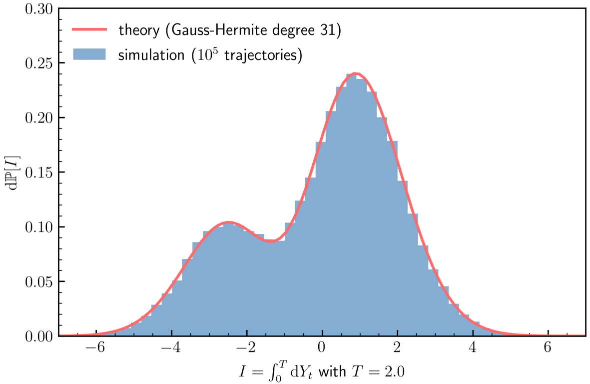
Empirical average with 10^6 very fine grained trajectories against analytical formula for 1 large time bin $\Delta t = T = 2$

$$H = \frac{1}{2}\sigma_x + \frac{1}{2}\sigma_y$$

$$L = \sigma_m$$

measure L with $\eta = 1$

$$\rho_0 = |e\rangle\langle e|$$



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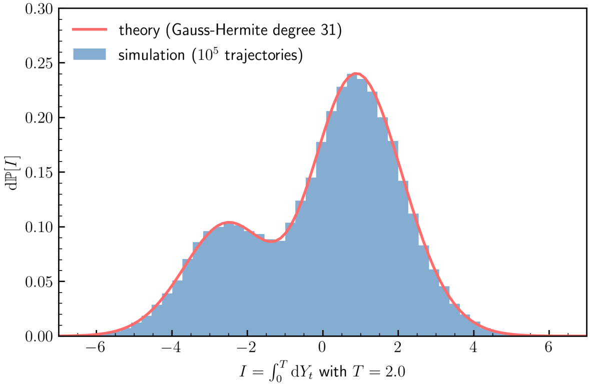

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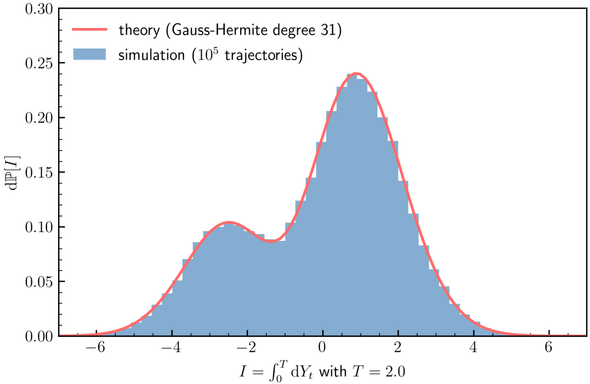


- with MC: `Qutip` 10min → `Dynamiqs` CPU 20s → `Dynamiqs` GPU 0.7s
- with Robinet and 32 quadrature points: 0.5ms

Deflating a coherent state with 2 photon loss

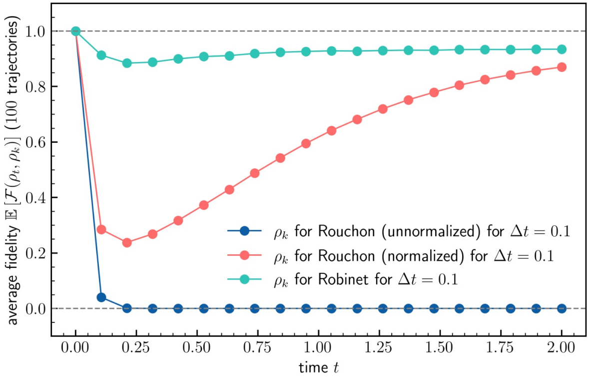
Simulate quantum trajectories with Robinet, Rouchon, Euler with coarse time bin $\Delta t = 0.1$ and compare fidelity with "true" state (simulated with $\delta t = 0.001$)

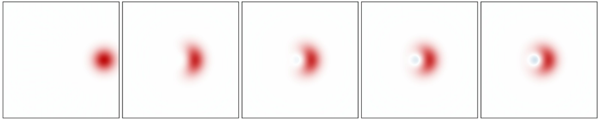
$H = 0$

$L = a^2$

measure L with $\eta = 0.5$

$\rho_0 = |\alpha\rangle\langle\alpha|$ with $\alpha = 3$





Deflating a coherent state with 2 photon loss

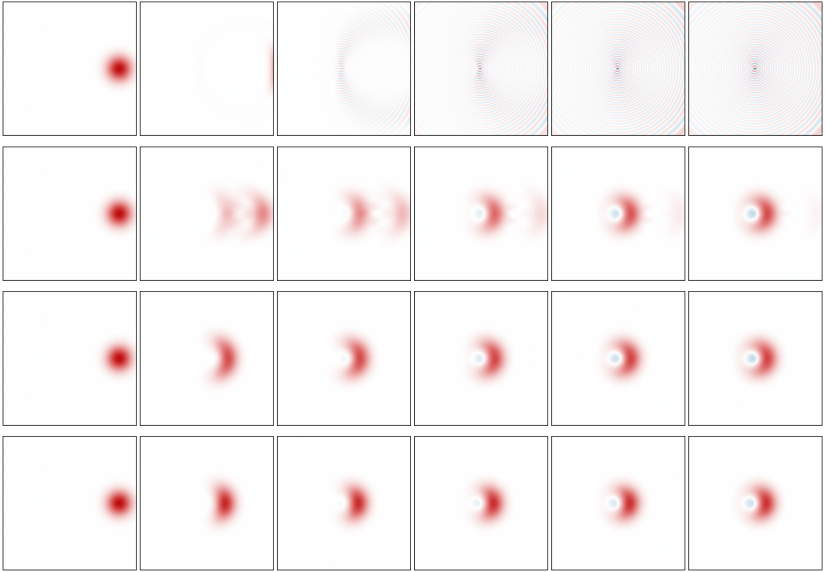
Averaged dynamics, qualitatively

Euler

Rouchon

Robinet

True



A perturbative expansion

What are the corrections to standard discretizations?

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If Δt is small:

- $I_k \sim \sqrt{\Delta t}$ $\to$ define $\mathcal{I}_k = I_k/\sqrt{\Delta t}$
- In the integral $p \sim 1/\sqrt{\Delta t}$ $\to$ define $u = \sqrt{\Delta t}\, p$

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Assuming time-independent $\mathcal{L}$:

$$\tilde{\rho}_k = \frac{1}{2\pi\sqrt{\Delta t}} \int_{\mathbb{R}} \mathrm{d}u \;\; e^{iu\,\mathcal{I}_k - \frac{u^2}{2}} \underbrace{\exp\left(\Delta t \mathcal{L} - i\sqrt{\Delta t}\, u\, \mathcal{C}_L\right)}_{\texttt{Taylorable}} \cdot \bar{\rho}_{k-1}$$

can be expanded in $(\sqrt{\Delta t})^n$

Perturbative expansion continued

$$
\begin{aligned}
\tilde{\rho}_k \simeq \int_{\mathbb{R}} \frac{\mathrm{d}u\, e^{iu\mathcal{J}_k - \frac{u^2}{2}}}{2\pi\sqrt{\Delta t}} \Bigg(1 &- \Delta t^{1/2}\ \left[i\,u\,\mathcal{C}_L \right] \\
&+ \Delta t \left[\mathcal{L} - \frac{u^2}{2}\mathcal{C}_L^2 \right] \\
&+ \Delta t^{3/2} \left[\frac{iu^3}{6}\mathcal{C}_L^3 - \frac{iu}{2}(\mathcal{C}_L\mathcal{L} + \mathcal{L}\mathcal{C}_L) \right] \\
&+ \Delta t^2 \left[\frac{u^4}{24}\mathcal{C}_L^4 + \frac{-u^2}{6}\left(\mathcal{C}_L^2\mathcal{L} + \mathcal{C}_L\mathcal{L}\mathcal{C}_L + \mathcal{L}\mathcal{C}_L^2\right) + \frac{1}{2}\mathcal{L}^2 \right] \Bigg) \bar{\rho}_{k-1}
\end{aligned}
$$

The integrals in u are Gaussian!

Perturbative expansion continued

Exact filter expanded to order $\sqrt{\Delta t}^4 = \Delta t^2$:

$$
\begin{aligned}
\tilde{\rho}_k \simeq \frac{e^{-\mathcal{J}_k^2/2}}{\sqrt{2\pi\Delta t}} \Bigg(& \mathbb{1} \\
& + \sqrt{\Delta t}^1 \; [\mathcal{J}_k \, \mathcal{C}_L] \\
& + \sqrt{\Delta t}^2 \left[\mathcal{L} - \frac{(1 - \mathcal{J}_k^2)}{2} \mathcal{C}_L^2 \right] \\
& + \sqrt{\Delta t}^3 \left[\frac{\mathcal{J}_k(\mathcal{J}_k^2 - 3)}{6} \mathcal{C}_L^3 + \frac{\mathcal{J}_k}{2} (\mathcal{C}_L \mathcal{L} + \mathcal{L} \mathcal{C}_L) \right] \\
& + \sqrt{\Delta t}^4 \left[\frac{(\mathcal{J}_k^4 - 6\mathcal{J}_k^2 + 3)}{24} \mathcal{C}_L^4 + \frac{(\mathcal{J}_k^2 - 1)}{6} \left(\mathcal{C}_L^2 \mathcal{L} + \mathcal{C}_L \mathcal{L} \mathcal{C}_L + \mathcal{L} \mathcal{C}_L^2 \right) + \frac{1}{2} \mathcal{L}^2 \right] \Bigg) \bar{\rho}_{k-1}
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Sampling trajectories

So far we assumed $I_1, \ldots, I_n$ experimentally given $\rightarrow$ reconstruction problem

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We can also sample:

$$\mathrm{d}\mathbb{P}\big[I_k \,|\, \bar{\rho}_{k-1}\big] = \mathrm{tr}\big[\tilde{\rho}_k\big]\mathrm{d}I_k$$

Using normalized $\mathcal{I}_k = I_k/\sqrt{\Delta t}$ this gives

$$\mathrm{d}\mathbb{P}[\mathcal{I}_k \,|\bar{\rho}_{k-1}] = \left[1 + A_1\mathcal{I}_k^1 + A_2\mathcal{I}_k^2 + A_3\mathcal{I}_k^3 + A_4\mathcal{I}_k^4 + \ldots\right] \frac{e^{-\mathcal{I}_k/2}}{\sqrt{2\pi}}\mathrm{d}\mathcal{I}_k$$

where A_ℓ are simple traces of operators applied to $\bar{\rho}_{k-1}$

$\rightarrow$ positive polynomial $\times$ Gaussian $=$ can be efficiently sampled

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$\to$ positive polynomial $\times$ Gaussian $=$ can be efficiently sampled

Question: Does the scheme fit the criteria of Wonglakhon, Wiseman, and Chantasri `2408.14105`?

Experimental tests of quantum trajectories

How to test quantum trajectory theory [Benjamin Huard's talk]:

- **ideally**: condition on full signal trajectory $\{I_k\}_{=1\cdots n}$, do tomography, compare [exponentially expensive]

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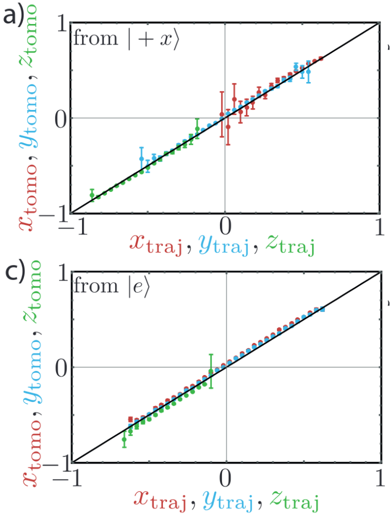
$\Longrightarrow$ powerful consistency check



Campagne-Ibarcq et al. PRX 2016

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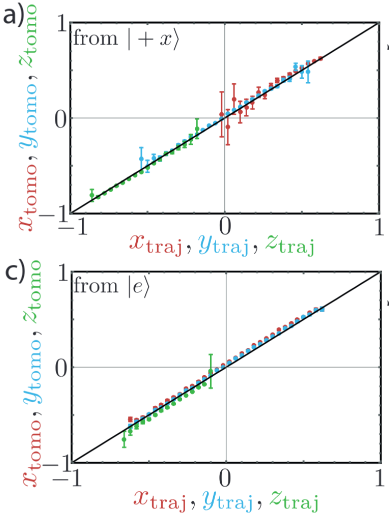

Campagne-Ibarcq et al. PRX 2016

Cannot in principle falsify poor use of signal (e.g. mix with noise, and use lower efficiency in SME)

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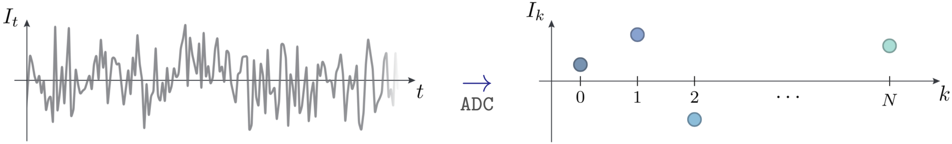
Any suboptimal use of the signal will give lower fidelity

Summary



1. Real signals are digitized / discretized / binned
2. The corresponding Kraus map expressible exactly $\rightarrow$ `Robinet`
3. It can be computed numerically exactly with Gaussian quadratures
4. Or expanded perturbatively, giving systematic corrections of order $\Delta t^{n/2}$ to standard schemes
5. It can be used to reconstruct state from data or direct simulation
6. It always gives physically sound states
7. It makes quantum trajectory theory testable in a slightly stronger sense