Gravitational Wave Tails from Soft Theorem

Ashoke Sen

International Center for Theoretical Sciences, Bengaluru, India

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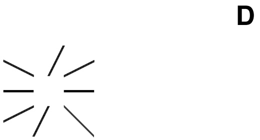
Consider a violent explosion in space



A stationary system breaks apart into fragments.

This process emits gravitational wave.

Detector D placed far away detects this wave as ripples in space-time.

Examples: Supernova

A more general situation: collision

A set of objects come together, interact, and fly apart, possibly exchanging mass, energy and momentum during this process.

This will also produce gravitational wave.

Example: Bullet cluster

A supercluster of galaxies passing through another supercluster of galaxies, each weighing about 10^{14} times the mass of the sun, at a relative speed of about 1% of the speed of light.

In general, computing gravitational waves produced during such processes is complicated.

1. When the objects are close, they may undergo complicated, non-gravitational interactions, as in the case of explosion of supernova.

2. Gravity is described by non-linear partial differential equations

– even if the interactions were purely gravitational, e.g. in the case of black hole merger, the analysis is complicated.

Surprisingly, some results in quantum theory of gravity, known as soft graviton theorem, can be used to get analytical results on some aspects of classical gravitational wave.

This will be the focus of this talk.

Collaborators

Alok Laddha
Biswajit Sahoo
Arnab Priya Saha
Debanjan Karan
Babli Khatun

Introduction

Gravitational radiation

In Einstein's general theory of relativity, a gravitational field is a symmetric two index field $h_{\mu\nu}$ that measures distortion of space-time

– sourced by mass / energy

Our convention:

$$h_{\mu\nu} \equiv (g_{\mu\nu} - \eta_{\mu\nu})/2, \qquad 0 \leq \mu, \nu \leq 3$$

Gravitational waves involve time and space-varying $h_{\mu\nu}$

Key distinguishing feature of gravitational radiation:

At a distance R from the source, the fields fall off as 1/R and not as $1/R^2$ or faster.

h$_{\mu\nu}$ can be measured by a gravitational wave detector like LIGO

Goal of a theorist: For a given process, compute h$_{\mu\nu}$ at a far away detector.

Conventions

c will denote the speed of light.

G will denote Newton's gravitational constant

Space-time coordinates: x^μ for $0 \le \mu \le 3$

$x^0 = c\,t$,   t is the time coordinate,

$\vec{x} = (x^1, x^2, x^3)$ are space coordinates

4-momentum p^μ: p^0 is (energy / c), (p^1, p^2, p^3) are momenta

$\eta_{\mu\nu}$ and $\eta^{\mu\nu}$: 4×4 diagonal matrix diag (-1, 1, 1, 1)

$$a.b \equiv \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} a^\mu b^\nu = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$$

$$p^2 \equiv p.p = -(p^0)^2 + \vec{p}^2 = -E^2/c^2 + \vec{p}^2 = -m^2 c^2$$

Summation convention: An index repeated twice is summed over from 0 to 3, e.g.

$$\eta_{\mu\nu}\mathbf{p}^{\mu} = \sum_{\mu=0}^{3} \eta_{\mu\nu}\mathbf{p}^{\mu}$$

Indices are raised and lowered by η

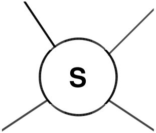
$$\mathbf{b}_{\mu} \equiv \eta_{\mu\rho}\mathbf{b}^{\rho}, \qquad \mathbf{a}^{\mu} \equiv \eta^{\mu\rho}\mathbf{a}_{\rho}$$

$$\Rightarrow \quad \mathbf{b}_0 = -\mathbf{b}^0, \quad \mathbf{b}_1 = \mathbf{b}^1, \quad \mathbf{b}_2 = \mathbf{b}^2, \quad \mathbf{b}_3 = \mathbf{b}^3$$

Summary of results

Consider a scattering in space

A set of objects of four momenta $p'_1, \cdots p'_m$ come together, interact, and disperse as a set of objects with four momenta $p_1, \cdots p_n$.



We shall choose the origin of space-time to be inside the region S where the scattering event takes place

The peak of the $h_{\mu\nu}$ signal reaches a detector D placed at a far way point $\vec{x}$ at some time t_0:

$$t_0 = R/c + \text{correction}, \quad R \equiv |\vec{x}|$$

The correction is due to the gravitational drag on the gravitational wave.

Define retarded time:

$$\mathbf{u} \equiv \mathbf{t} - \mathbf{t_0}$$

Our focus will be on the late and early time tail of the wave – the value of $h_{\mu\nu}$ at D at large positive u and large negative u.

Define $e_{\mu\nu}$ via:

$$e_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\,\eta^{\rho\sigma}\,h_{\rho\sigma} \quad \Leftrightarrow \quad h_{\mu\nu} = e_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\,\eta^{\rho\sigma}\,e_{\rho\sigma}$$

Terms proportional to R^{-1} have the form,

$$e_{\mu\nu} = \frac{1}{u}\,C_{\mu\nu} + G_{\mu\nu}u^{-2}\ln|u| + \mathcal{O}(u^{-2}), \quad \text{for large negative u}$$

$$e_{\mu\nu} = A_{\mu\nu} + \frac{1}{u}\,B_{\mu\nu} + F_{\mu\nu}u^{-2}\ln u + \mathcal{O}(u^{-2}), \quad \text{for large positive u}$$

$A_{\mu\nu}, B_{\mu\nu}, C_{\mu\nu}, F_{\mu\nu}, G_{\mu\nu}$ are given solely by the 4-momenta of the ingoing and outgoing objects without requiring any knowledge of the details of the scattering process.

$$A^{\mu\nu} = \frac{2\,G}{R\,c^3}\left[-\sum_{a=1}^{n} p_a^{\mu}\,p_a^{\nu}\,\frac{1}{n.p_a} + \sum_{a=1}^{m} p_a'^{\mu}\,p_a'^{\nu}\,\frac{1}{n.p_a'}\right], \quad R \equiv |\vec{x}|, \quad n \equiv (1, \vec{x}/R)$$

$$B^{\mu\nu} = -\frac{4\,G^2}{R\,c^7}\left[\sum_{a=1}^{n}\sum_{\substack{b=1\\ b\neq a}}^{n} \frac{p_a.p_b}{\{(p_a.p_b)^2 - m_a^2 m_b^2 c^4\}^{3/2}}\left\{\frac{3}{2}m_a^2 m_b^2 c^4 - (p_a.p_b)^2\right\}\right.$$

$$\times\,\frac{p_a^{\mu}}{n.p_a}\,(n.p_b\,p_a^{\nu} - n.p_a\,p_b^{\nu})$$

$$\left. -\left\{\sum_{b=1}^{n} p_b.n \sum_{a=1}^{n}\frac{1}{p_a.n}\,p_a^{\mu}p_a^{\nu} - \sum_{b=1}^{m} p_b'.n\sum_{a=1}^{m}\frac{1}{p_a'.n}\,p_a'^{\mu}p_a'^{\nu}\right\}\right]$$

$$C^{\mu\nu} = \frac{4\,G^2}{R\,c^7}\left[\sum_{a=1}^{m}\sum_{\substack{b=1\\ b\neq a}}^{m}\frac{p_a'.p_b'}{\{(p_a'.p_b')^2 - m_a'^2 m_b'^2 c^4\}^{3/2}}\left\{\frac{3}{2}m_a'^2 m_b'^2 c^4 - (p_a'.p_b')^2\right\}\right.$$

$$\left.\times\,\frac{p_a'^{\mu}}{n.p_a'}\,(n.p_b'\,p_a'^{\nu} - n.p_a'\,p_b'^{\nu})\right].$$

$$F^{\mu\nu} = 2\frac{G^3}{R\,c^{11}}\Bigg[4\Bigg\{\sum_{b=1}^{n} p_b.n \sum_{d=1}^{n} p_d.n \sum_{a=1}^{n} \frac{p_a^\mu p_a^\nu}{p_a.n} - \sum_{b=1}^{m} p'_b.n \sum_{d=1}^{n} p'_d.n \sum_{a=1}^{m} \frac{p_a'^\mu p_a'^\nu}{p'_a.n}\Bigg\}$$

$$+4\sum_{d=1}^{n} p_d.n \sum_{a=1}^{n} \sum_{\substack{b=1\\ b\neq a}}^{n} \frac{1}{p_a.n} \frac{p_a.p_b}{\{(p_a.p_b)^2 - m_a^2 m_b^2 c^4\}^{3/2}} \{2(p_a.p_b)^2 - 3m_a^2 m_b^2 c^4\}\{n.p_b\, p_a^\mu\, p_a^\nu - n.p_a\, p_a^\mu\, p_b^\nu\}$$

$$+2\sum_{d=1}^{n} p'_d.n \sum_{a=1}^{m} \sum_{\substack{b=1\\ b\neq a}}^{m} \frac{1}{p'_a.n} \frac{p'_a.p'_b}{\{(p'_a.p'_b)^2 - m_a'^2 m_b'^2 c^4\}^{3/2}} \{2(p'_a.p'_b)^2 - 3m_a'^2 m_b'^2 c^4\}\{n.p'_b\, p_a'^\mu\, p_a'^\nu - n.p'_a\, p_a'^\mu\, p_b'^\nu\}$$

$$+\sum_{a=1}^{n} \sum_{\substack{b=1\\ b\neq a}}^{n} \sum_{\substack{d=1\\ c\neq a}}^{n} \frac{1}{p_a.n} \frac{p_a.p_b}{\{(p_a.p_b)^2 - m_a^2 m_b^2 c^4\}^{3/2}} \{2(p_a.p_b)^2 - 3m_a^2 m_b^2 c^4\} \frac{p_a.p_d}{\{(p_a.p_d)^2 - m_a^2 m_d^2 c^4\}^{3/2}}$$

$$\{2(p_a.p_d)^2 - 3m_a^2 m_d^2 c^4\}\{n.p_b\, p_a^\mu - n.p_a\, p_b^\mu\}\,\{n.p_d\, p_a^\nu - n.p_a\, p_d^\nu\}\Bigg],$$

$$G^{\mu\nu} = -2\frac{G^3}{R\,c^{11}}\Bigg[2\sum_{d=1}^{m} p'_d.n \sum_{a=1}^{m} \sum_{\substack{b=1\\ b\neq a}}^{m} \frac{1}{p'_a.n} \frac{p'_a.p'_b}{\{(p'_a.p'_b)^2 - m_a'^2 m_b'^2 c^4\}^{3/2}} \{2(p'_a.p'_b)^2 - 3m_a'^2 m_b'^2 c^4\}$$

$$\{n.p'_b\, p_a'^\mu\, p_a'^\nu - n.p'_a\, p_a'^\mu\, p_b'^\nu\}$$

$$-\sum_{a=1}^{m} \sum_{\substack{b=1\\ b\neq a}}^{m} \sum_{\substack{d=1\\ c\neq a}}^{m} \frac{1}{p'_a.n} \frac{p'_a.p'_b}{\{(p'_a.p'_b)^2 - m_a'^2 m_b'^2 c^4\}^{3/2}} \{2(p'_a.p'_b)^2 - 3m_a'^2 m_b'^2 c^4\} \frac{p'_a.p'_d}{\{(p'_a.p'_d)^2 - m_a'^2 m_d'^2 c^4\}^{3/2}}$$

$$\{2(p'_a.p'_d)^2 - 3m_a'^2 m_d'^2 c^4\}\{n.p'_b\, p_a'^\mu - n.p'_a\, p_b'^\mu\}\,\{n.p'_d\, p_a'^\nu - n.p'_a\, p_d'^\nu\}\Bigg].$$

$$\mathbf{e}_{\mu\nu} = \mathbf{A}_{\mu\nu} + \frac{1}{\mathbf{u}}\,\mathbf{B}_{\mu\nu} + \mathbf{F}_{\mu\nu}\mathbf{u}^{-2}\ln\mathbf{u} + \mathcal{O}(\mathbf{u}^{-2}), \quad \textbf{for large positive u}$$

$$\mathbf{e}_{\mu\nu} = \frac{1}{\mathbf{u}}\,\mathbf{C}_{\mu\nu} + \mathbf{G}_{\mu\nu}\mathbf{u}^{-2}\ln|\mathbf{u}| + \mathcal{O}(\mathbf{u}^{-2}), \quad \textbf{for large negative u}$$

$\mathbf{A}_{\mu\nu}$: memory term

– a permanent change in the state of the detector after the passage of gravitational wave

Zeldovich, Polnarev; Braginsky, Grishchuk; Braginsky, Thorne; Strominger; $\cdots$

$\mathbf{B}_{\mu\nu}, \mathbf{C}_{\mu\nu}, \mathbf{F}_{\mu\nu}, \mathbf{G}_{\mu\nu}$: tail terms Laddha, A.S.; Sahoo, A.S.; Saha, Sahoo, A.S.; Sahoo

1. $\mathbf{A}_{\mu\nu}, \mathbf{B}_{\mu\nu}, \mathbf{C}_{\mu\nu}, \mathbf{F}_{\mu\nu}, \mathbf{G}_{\mu\nu}$ can be expressed in terms of the momenta of incoming and outgoing objects without knowing what forces operated and how the objects moved during the scattering

– consequence of 'soft graviton theorem'

In contrast, if we were to compute $\mathbf{h}_{\mu\nu}$ at finite time, it will depend on the details of the scattering process and will involve very complicated calculations.

2. In the expressions for A$^{\mu\nu}$, B$^{\mu\nu}$ and F$^{\mu\nu}$, the sum over final state particles a,b includes integration over outgoing flux of gravitational radiation, regarded as a flux of massless particles.

For A$^{\mu\nu}$ this gives the 'non-linear memory' term

Christodoulou; Thorne; Blanche, Damour; Bieri, Garfinkle; $\cdots$

Due to some miraculous cancellation, in B$^{\mu\nu}$ and F$^{\mu\nu}$ the contribution from massless final states can be expressed in terms of massive state momenta.

In $B_{\mu\nu}$, drop massless particles / radiation contribution in the sum over final states, and add

$$-\frac{4\,G^2}{R\,c^7}\left[P_F^\mu P_F^\nu - P_I^\mu P_I^\nu\right]$$

P_I: total incoming momentum

P_F: total outgoing momentum carried by massive particles

In $F_{\mu\nu}$, drop massless particles / radiation contribution in the sum over final states, and add

$$-\frac{8\,G^3}{R\,c^{11}}\left[n.P_F\,P_F^\mu\,P_F^\nu - n.P_I\,P_I^\mu P_I^\nu\right]$$

Note: These are not new formulæ but follow from manipulating the results shown earlier

3. If the incoming or outgoing objects carry charge then there are additional terms in the formula (known)

4. Explosion can be regarded as a special case of scattering when the initial state has just one object.

In this case $C_{\mu\nu}$ and $G_{\mu\nu}$ vanish and $e_{\mu\nu}$ takes the form:

$$e_{\mu\nu} = A_{\mu\nu} + \frac{1}{u} B_{\mu\nu} + F_{\mu\nu}\, u^{-2} \ln u + \mathcal{O}(u^{-2}), \quad \text{for large positive u}$$

$$e_{\mu\nu} = 0, \quad \text{for large negative u}$$

5. The results are statements in classical theory of gravity.

However they are easier to understand as classical limits of some results in quantum theory of gravity

– known as quantum soft graviton theorem.

In fact the tail terms were first predicted from quantum soft graviton theorem and then a fully classical derivation was found.

From now onwards, we set c=1

Outline of the derivation from soft graviton theorem

In quantum theory of fields the scattering amplitude A gives the probability amplitude for transition of one state to another.

$|A|^2$: transition probability from one state to another.

Suppose we know the scattering amplitude A for transition from some incoming state P to some outgoing state Q.

Both P and Q contain sets of particles, in general different

Soft graviton theorem gives the transition amplitude for

$$P \Rightarrow Q + \text{some low energy gravitons}$$

in terms of the original transition amplitude A. Weinberg; $\cdots$

Gravitons are massless particles representing quanta of gravitational wave

– just like photons are quanta of electromagnetic wave.

Classical limit:

1. Take the states P and Q to consist of very massive objects.

2. Compute the transition probability for

P $\Rightarrow$ Q $+$ M low energy gravitons, each of energy $\hbar\omega$ and momentum $\hbar\omega\hat{\mathbf{n}}$

– takes the form

$$\mathrm{R\,S^M/M!} \longrightarrow \text{bose statistics}$$

• R depends on the details of the scattering process

• S is the known 'soft factor' that depends only on initial and final momenta due to soft graviton theorem.

3. This has a sharp maximum at M=S for large S

$\rightarrow$ S is the expected 'classical number' of low energy gravitons.

$\Rightarrow$ energy carried by these gravitons is S $\hbar\,\omega$

4. Classical flux of energy carried by gravitational wave of frequency ω is

$$\mathbf{S}\hbar\omega$$

The energy flux can be translated into the time Fourier transform of the gravitational wave-form $h_{\mu\nu}$, expressed as function of ω.

Inverse Fourier transform gives the late and early time profile, leading to the quoted results

Some shortcomings:

1.The usual soft factor S is infrared divergent

– we need to regulate the infrared divergence by ω^{-1}

Laddha, A.S.; Sahoo, A.S.

2. Since this analysis proceeds via computation of the energy flux in different frequencies, it is insensitive to the frequency dependent 'phase' of the wave-form

– needs to be determined by separate computation

Direct classical analysis is more complicated, but does not have these shortcomings

Direct classical derivation

Goal: Find $h_{\mu\nu}$, or equivalently $e_{\mu\nu}$, generated during the collision process.

In de Donder gauge, Einstein's equation determining $e_{\mu\nu}$ can be written as:

$$\Box\, e^{\mu\nu}(x) = -8\,\pi\, G\, T^{\mu\nu}(x)$$

$T^{\mu\nu}$ on the right hand side has explicit dependence on $e^{\rho\sigma}$, and also the trajectories of the various objects involved in the scattering

The trajectories of the objects in turn depend on $e^{\rho\sigma}$

– set of complicated non-linear partial differential equations

– may not even be known if the interactions are unknown

Define

$$\tilde{e}^{\mu\nu}(\omega, \vec{x}) = e^{-i\omega|\vec{x}|} \int dt\, e^{i\omega t}\, e^{\mu\nu}(t, \vec{x})$$

$e^{\mu\nu} \to$ constant for large u corresponds to $\tilde{e}^{\mu\nu} \sim \omega^{-1}$ for small ω

$e^{\mu\nu} \sim 1/u$ for large u corresponds to $\tilde{e}^{\mu\nu} \sim \ln\omega$ for small ω

$e^{\mu\nu} \sim \ln u/u^2$ corresponds to $\tilde{e}^{\mu\nu} \sim \omega(\ln\omega)^2$

These are non-analytic at $\omega = 0$

$\Box\, e^{\mu\nu} = -8\,\pi\, G\, T^{\mu\nu}$ can be formally 'solved' as:

$$e^{\mu\nu}(x) = -8\,\pi\, G \int d^4y\, G_r(x, y)\, T^{\mu\nu}(y)$$

$G_r(x, y)$: retarded Green's function in flat space-time

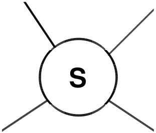
Using the explicit expression for G_r, one finds that for large $|\vec{x}|$

$$\tilde{e}^{\mu\nu}(\omega, \vec{x}) = \frac{2G}{|\vec{x}|} \int d^4y\, e^{-ik.y}\, T^{\mu\nu}(y), \qquad k = \omega(1, \vec{x}/|\vec{x}|)$$

$$\tilde{e}^{\mu\nu}(\omega, \vec{x}) = \frac{2G}{|\vec{x}|} \int d^4y\, e^{-ik.y}\, T^{\mu\nu}(y), \qquad k = \omega(1, \vec{x}/|\vec{x}|)$$

We divide the integration region over y into two parts:

1. Scattering region: A region S of large size around y=0.

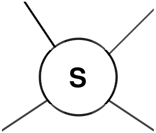


2. Asymptotic region: Complement of S

Non-analytic terms in $\tilde{e}^{\mu\nu}$ cannot come from the integral of a finite quantity over a finite region

$\Rightarrow$ the behaviour of $e^{\mu\nu}$ for large $|u|$ is insensitive to the contribution from the finite region S in $\int d^4y$.

$$\tilde{e}^{\mu\nu}(\omega, \vec{x}) = \frac{2G}{|\vec{x}|} \int d^4y \, e^{-ik.y} \, T^{\mu\nu}(y), \qquad k = \omega(1, \vec{x}/|\vec{x}|)$$



One can restrict the y integral in the region outside S.

In this region, only long range forces are important.

Furthermore, $e^{\mu\nu}$ is small and as a result $T^{\mu\nu}$ simplifies.

The equations can be solved iteratively and give the result quoted earlier.

Saha, Sahoo, A.S.; Sahoo

$$T^{X\mu\nu}(x) \equiv \sum_{a=1}^{n} m_a \int_0^{\infty} d\tau\, \delta^{(4)}(x - X_a(\tau))\, \frac{dX_a^{\mu}}{d\tau}\, \frac{dX_a^{\nu}}{d\tau} \ \rightarrow\ \text{outgoing}$$

$$+ \sum_{a=1}^{m} m'_a \int_{-\infty}^{0} d\tau\, \delta^{(4)}(x - X'_a(\tau))\, \frac{dX_a'^{\mu}}{d\tau}\, \frac{dX_a'^{\nu}}{d\tau} \ \leftarrow\ \text{incoming}\,,$$

$$T^{\mu\nu}(x) = T^{X\mu\nu}(x) + T^{h\mu\nu}(x),$$

$$\Box\, e^{\mu\nu} = -8\,\pi\, G\, T^{\mu\nu}\,,$$

$$\frac{d^2X_a^{\mu}}{d\tau^2} = -\Gamma^{\mu}_{\nu\rho}(X(\tau))\, \frac{dX_a^{\nu}}{d\tau}\, \frac{dX_a^{\rho}}{d\tau}, \qquad \frac{d^2X_a'^{\mu}}{d\tau^2} = -\Gamma^{\mu}_{\nu\rho}(X'(\tau))\, \frac{dX_a'^{\nu}}{d\tau}\, \frac{dX_a'^{\rho}}{d\tau}\,,$$

Boundary conditions:

$$\lim_{\tau\rightarrow\infty} \frac{dX_a^{\mu}}{d\tau} = v_a^{\mu} = \frac{1}{m_a}\, p_a^{\mu},$$

$$\lim_{\tau\rightarrow-\infty} \frac{dX_a'^{\mu}}{d\tau} = v_a'^{\mu} = \frac{1}{m'_a}\, p_a'^{\mu}\,.$$

Spin dependence

Our results are independent of the spins of the incoming and outgoing objects

Soft theorem $\Rightarrow$ spin dependent terms arise at order u^{-2}

Unfortunately the existence of u^{-1} term makes the order u^{-2} terms ambiguous

However for scattering via purely gravitational interaction, the ambiguity appears at order G^3 while the spin dependent terms arise at order G^2

$\Rightarrow$ at order G^2 the spin dependent order u^{-2} terms can be determined unambiguously.

Ghosh, Sahoo

Comments:

The classical analysis is straightforward and does not involve the complications of infrared divergence and phase ambiguity that the quantum soft theorems suffer from.

However, it is the quantum theory that teaches us the right question for which we have simple answer

In the usual classical set up, we specify the initial configuration and then evolve it to find the final state and the radiation emitted during the process

– a very complicated problem with no universal answer

It is the quantum theory that teaches us to specify both the initial and final state of matter and then ask for radiation emitted during the process

Possible generalizations

There is reason to believe that the coefficients of $u^{-n-1}(\ln|u|)^n$ terms may also have universal properties, i.e. they depend only on the momenta of the particles

Sahoo; Alessio, Di Vecchia, Heissenberg

Can we compute them?

Soft theorem is not useful any more for $n > 1$ but direct classical analysis may be possible

Debanjan Karan, Babli Khatun, Biswajit Sahoo, A.S., in progress

Ultimate goal: Use these results in reverse

By measuring the gravitational wave-form in a detector, we can measure the coefficient of $u^{-n-1}(\ln u)^n$ terms.

Can we use this data to reconstruct the initial and final states of the event that led to the production of the gravitational wave?

Few details of the ‘derivation’ using soft graviton theorem

What is soft graviton theorem?

Take a general coordinate invariant quantum theory of gravity coupled to matter fields

Consider an S-matrix element involving

– arbitrary number N of external particles of finite momentum $p_1, \cdots p_N$

– M external gravitons carrying small momentum $k_1, \cdots k_M$.

Soft graviton theorem: Expansion of this amplitude in power series in $k_1, \cdots k_M$ in terms of the amplitude without the soft gravitons

– can be proved in general with some assumptions to be stated below

A.S.; Laddha, A.S.; Chakrabarti, Kashyap, Sahoo, A.S., Verma

Assumptions

1. The scattering is described by a general coordinate invariant one particle irreducible (1PI) effective action

– tree amplitudes computed from this give the full quantum results

2. The interaction terms do not contribute powers of soft momentum in the denominator

– breaks down in D=4

We'll first review the results in D $>$ 4

Result:

Let Γ be the scattering amplitude of any set of finite energy (hard) particles.

Scattering amplitude of the same set of states with M additional soft gravitons of polarization $\{\varepsilon_r\}$ and momentum $\{k_r\}$ ($1 \leq r \leq M$) takes the form

$$S(\{\varepsilon_r\}, \{k_r\})\,\Gamma$$

up to subleading order in expansion in powers of soft momentum.

$S(\{\varepsilon_r\}, \{k_r\})$: Known, universal operator, involving derivatives with respect to momenta of hard particles and matrices acting on the polarization of the hard particles.

We have not explicitly displayed the dependence of $S(\{\varepsilon_r\}, \{k_r\})$ on the momenta and angular momenta of hard particles.

Classical limit

We take the limit in which the energy of each hard particle becomes large (compared to M_{pl})

– represented by wave-packets with sharply peaked distribution of position, momentum, spin etc.

In this limit the soft factor $S(\{\varepsilon_r\}, \{k_r\})$ becomes a multiplicative factor $\prod_r S(\varepsilon_r, k_r)$ instead of a differential operator

The probability of producing M soft gravitons with given quantum number (ε, k) is $\propto |S(\varepsilon, k)|^{2M}/M!\times$ phase space factors

– sharply peaked around $M = |S(\varepsilon, k)|^2\times$ phase space factor $\equiv M_{cl}(k)$

$\Rightarrow$ flux of classical soft radiation

$\Rightarrow$ classical gravitational wave-form $\propto S(\varepsilon, k)$ (up to a phase).

Result

$$(\mathbf{h}^{\mu\nu}(\vec{\mathbf{x}},\omega))^{\mathbf{TT}} = -\frac{1}{2\omega^2}\left(\frac{\omega}{2\pi i R}\right)^{(D-2)/2} \sum_{a} \eta_a \, (p_a \cdot n)^{-1} \left[p_a^{\mu} p_a^{\nu} - i\,\omega\, n_{\rho}\, J_a^{\rho(\nu} p_a^{\mu)}\right]^{TT}$$

up to terms higher order in ω

TT: transverse traceless part

$\eta_a = 1$ for outgoing and -1 for incoming particles.

$n = \left(1, \vec{x}/|\vec{x}|\right), \quad R = |\vec{x}|, \quad k = \omega\,\left(1, \vec{x}/|\vec{x}|\right)$

If in the far past / future the object has trajectory

$$x_a^{\mu} = c_a^{\mu} + m_a^{-1}\, p_a^{\mu} \tau_a$$

then

$$J_a^{\mu\nu} = (x_a^{\mu} p_a^{\nu} - x_a^{\nu} p_a^{\mu}) + \text{spin} = (c_a^{\mu} p_a^{\nu} - c_a^{\nu} p_a^{\mu}) + \text{spin}$$

D=4

The S-matrix suffers from IR divergence, making the soft factor ill-defined.

Bern, Davies, Nohle

However we can still try to use the 'soft formula' for the classical wave-form.

Naive guess: Classical wave-form is still given by the same formulæ:

$$(\mathbf{h}^{\mu\nu}(\vec{\mathbf{x}},\omega))^{\mathrm{TT}} = -\frac{1}{2\omega^2}\left(\frac{\omega}{2\pi i R}\right)\sum_a \eta_a\,(\mathbf{p}_a\cdot\mathbf{n})^{-1}\left[\mathbf{p}_a^{\mu}\mathbf{p}_a^{\nu} - i\,\omega\,\mathbf{n}_\rho\,\mathbf{J}_a^{\rho(\nu}\mathbf{p}_a^{\mu)}\right]^{\mathrm{TT}}$$

Problem: Due to long range force on the initial / final trajectories due to other particles, the trajectory of the a-th particle takes the form:

$$x_a^\mu = c_a^\mu + m_a^{-1} p_a^\mu \tau_a + b_a^\mu \ln|\tau_a|$$

for some computable constants b_a^μ.

$$J_a^{\mu\nu} = (x_a^\mu p_a^\nu - x_a^\nu p_a^\mu) = (c_a^\mu p_a^\nu - c_a^\nu p_a^\mu) + (b_a^\mu p_a^\nu - b_a^\nu p_a^\mu)\ln|\tau_a|$$

Due to the $\ln|\tau_a|$ term, the soft factors do not have well defined $|\tau_a| \to \infty$ limit

Guess: Large τ_a divergence is cut-off at $\tau_a = \omega^{-1}$

$$J_a^{\mu\nu} = (x_a^\mu p_a^\nu - x_a^\nu p_a^\mu) = (c_a^\mu p_a^\nu - c_a^\nu p_a^\mu) + (b_a^\mu p_a^\nu - b_a^\nu p_a^\mu)\ln|\omega^{-1}|$$

In any given scattering process, the b_a^μ's can be computed by knowing the long range force between the objects.

With the $\ln|\tau| \Rightarrow \ln\omega^{-1}$ rule, the low frequency component of the gravitational wave-form is given by the TT component of:

$$\frac{2\,G}{i\,R}\sum_a \eta_a \frac{p_a^\mu p_a^\nu}{p_a.n}\left\{-\frac{1}{\omega}+2\,i\,G\,\ln(\omega^{-1}R^{-1})\sum_{b,\eta_b=-1} n.p_b\right\}$$

$$+2\,\frac{G^2}{R}\ln\omega^{-1}\sum_a \sum_{\substack{b\neq i\\ \eta_a\eta_b=1}} \frac{n_\rho p_a^{(\nu}}{p_a.n}\,(p_a^{\mu)}p_b^\rho - p_b^{\mu)}p_a^\rho)$$

$$\times\frac{p_b.p_a}{\{(p_b.p_a)^2-m_a^2m_b^2\}^{3/2}}\;\{2(p_b.p_a)^2-3m_a^2m_b^2\}+\text{finite}\,.$$

η_a: +1 if a is incoming, -1 if a is outgoing.

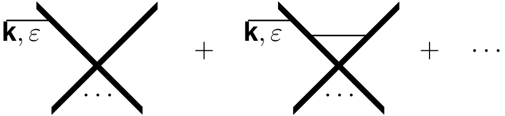
$n=(1,\hat{n}),\quad \hat{n}=\vec{x}/|\vec{x}|$

Sahoo, A.S.

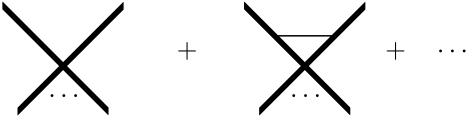
$2\,i\,G\,\ln(\omega^{-1}R^{-1})\sum_{b,\eta_b=-1}k.p_b$ factor is a pure phase and absent in the original soft factor, but present in the amplitude calculation described below.

With this new insight we can now go back to the S-matrix and see if we can reproduce the $\ln\omega$ terms in the soft factor.

We consider all diagrams up to one loop



and try to express this as a multiplicative factor $\varepsilon^{\mu\nu}\mathbf{S}_{\mu\nu}(\mathbf{k})$ times



Thin lines are gravitons and thick lines are scalars with an n-point contact interaction.

After suitable normalization, $\mathbf{S}_{\mu\nu}(\mathbf{k})$ should give the graviton profile in frequency space.

Results:

One loop correction to the soft factor indeed reproduces the logarithmic corrections obtained by $\ln \tau \to \ln \omega^{-1}$ rules in the soft factor.

Added bonus: This computation can distinguish ln($\omega - i\epsilon$) from ln($\omega + i\epsilon$)

ln $(\omega \pm i\epsilon)$ as $\omega \to 0$ translates to

$$\begin{cases} \mp 1/u \quad \text{as} \quad u \to \pm\infty \\ 0 \quad \text{as} \quad u \to \mp\infty \end{cases}$$

leading to the results mentioned at the beginning.

The 'phase' comes from loop momentum $<< \omega$ while the other terms come from loop momenta $>> \omega$.

Details of Classical Analysis

$$\mathbf{T}^{\mathbf{X}\mu\nu}(\mathbf{x}) \equiv \sum_{a=1}^{n} m_a \int_0^{\infty} d\tau\, \delta^{(4)}(\mathbf{x} - \mathbf{X}_a(\tau))\, \frac{d X_a^{\mu}}{d\tau}\, \frac{d X_a^{\nu}}{d\tau} \ \rightarrow\ \textbf{outgoing}$$

$$+ \sum_{a=1}^{m} m'_a \int_{-\infty}^{0} d\tau'\, \delta^{(4)}(\mathbf{x} - \mathbf{X}'_a(\tau'))\, \frac{d X_a'^{\mu}}{d\tau'}\, \frac{d X_a'^{\nu}}{d\tau'} \ \leftarrow\ \textbf{incoming}\,,$$

$$\mathbf{T}^{\mu\nu}(\mathbf{x}) = \mathbf{T}^{\mathbf{X}\mu\nu}(\mathbf{x}) + \mathbf{T}^{\mathbf{h}\mu\nu}(\mathbf{x}),$$

$$\Box\, e^{\mu\nu} = -8\,\pi\, G\, T^{\mu\nu}\,,$$

$$\frac{d^2 X_a^{\mu}}{d\tau^2} = -\Gamma^{\mu}_{\nu\rho}(\mathbf{X}(\tau))\, \frac{d X_a^{\nu}}{d\tau}\, \frac{d X_a^{\rho}}{d\tau}, \qquad \frac{d^2 X_a'^{\mu}}{d\tau^2} = -\Gamma^{\mu}_{\nu\rho}(\mathbf{X}'(\tau))\, \frac{d X_a'^{\nu}}{d\tau}\, \frac{d X_a'^{\rho}}{d\tau}\,,$$

Boundary conditions:

$$\lim_{\tau\to\infty} \frac{d X_a^{\mu}}{d\tau} = v_a^{\mu} = \frac{1}{m_a}\, p_a^{\mu}\,,$$

$$\lim_{\tau\to-\infty} \frac{d X_a'^{\mu}}{d\tau} = v_a'^{\mu} = \frac{1}{m'_a}\, p_a'^{\mu}\,.$$

We shall use two kinds of Fourier transforms:

Time Fourier transform:

$$\widetilde{\mathbf{F}}(\omega, \mathbf{x}) = \mathbf{e}^{-\mathbf{i}\omega|\vec{\mathbf{x}}|} \int \mathbf{dt}\, \mathbf{e}^{\mathbf{i}\omega\mathbf{t}} \mathbf{F}(\mathbf{t}, \vec{\mathbf{x}})\mathbf{dt}$$

Full Fourier transform:

$$\widehat{\mathbf{F}}(\mathbf{k}) = \int \mathbf{d}^4\mathbf{x}\, \mathbf{e}^{-\mathbf{ik.x}}\, \mathbf{F}(\mathbf{x})$$

$$\tilde{e}^{\mu\nu}(\omega,\vec{x}) = e^{-i\omega|\vec{x}|} \int dt\, e^{i\omega t}\, e^{\mu\nu}(t,\vec{x})$$

$$e^{\mu\nu}(x) = -8\,\pi\, G \int d^4y\, G_r(x,y)\, T^{\mu\nu}(y)$$

$$\widetilde{e}^{\mu\nu}(\omega,x) = -8\,\pi\, G \int d^4y\, T^{\mu\nu}(y) \int dt\, e^{i\omega t - i\omega|\vec{x}|}$$

$$\times \int \frac{d^4\ell}{(2\pi)^4}\, e^{i\omega(y^0-t)+i\vec{\ell}.(\vec{x}-\vec{y})}\, \frac{1}{(\omega+i\epsilon)^2-\vec{\ell}^{\,2}}$$

$$= \quad -8\,\pi\, G \int d^4y\, T^{\mu\nu}(y) \int \frac{d^3\ell}{(2\pi)^3}\, e^{i\omega y^0+i\vec{\ell}.(\vec{x}-\vec{y})}\, \frac{1}{(\omega+i\epsilon)^2-\vec{\ell}^{\,2}}\,.$$

For large $|\vec{x}|$ one can do the $\ell_{||}$ integral using Cauchy's theorem and $\ell_\perp$ integral using saddle point approximation and get:

$$\tilde{e}^{\mu\nu}(\omega,\vec{x}) = \frac{2G}{|\vec{x}|}\, \widehat{T}^{\mu\nu}(k) \quad \text{for large } |\vec{x}|, \qquad k = \omega(1, \vec{x}/|\vec{x}|)$$

$$\widehat{T}^{\mu\nu}(k) \equiv \int d^4y\, e^{-ik.y}\, T^{\mu\nu}(y)$$

Goal: Compute $\widehat{T}^{\mu\nu}(k)$

Strategy: Write out all the equations in momentum space and solve them iteratively.

At the leading order $\widehat{T}^{\mu\nu}(k) = \widehat{T}^{X\mu\nu}(k)$

$$T^{X\mu\nu}(x) \equiv \sum_{a=1}^{n} m_a \int_0^\infty d\tau\, \delta^{(4)}(x - X_a(\tau)) \frac{dX_a^\mu}{d\tau} \frac{dX_a^\nu}{d\tau} + \sum_{a=1}^{m} m'_a \int_{-\infty}^0 d\tau\, \delta^{(4)}(x - X'_a(\tau)) \frac{dX_a'^\mu}{d\tau} \frac{dX_a'^\nu}{d\tau},$$

We shall treat the incoming particles as outgoing particles with momentum $-p'_a$ and proper time $-\tau$

$$T^{X\mu\nu}(x) \equiv \sum_{a=1}^{m+n} m_a \int_0^\infty d\sigma\, \delta^{(4)}(x - X_a(\sigma)) \frac{dX_a^\mu}{d\sigma} \frac{dX_a^\nu}{d\sigma}$$

$$\sigma = \begin{cases} \tau \text{ for outgoing} \\ -\tau \text{ for incoming} \end{cases}$$

Leading order result

$$X_a^\mu = \frac{p_a^\mu}{m_a}\sigma + r_a$$

$$\widehat{T}^{\mu\nu}(k) = \widehat{T}^{X\mu\nu}(k)$$

$$\begin{aligned}
\widehat{T}^{\mu\nu}(k) &= \int d^4x\, e^{-ik.x} \sum_{a=1}^{m+n} m_a \int_0^\infty d\sigma\, \delta^{(4)}(x - X_a(\sigma)) \frac{dX_a^\mu}{d\sigma} \frac{dX_a^\nu}{d\sigma} \\
&= \sum_{a=1}^{m+n} m_a \int_0^\infty d\sigma\, e^{-ik.X_a(\sigma)} \frac{dX_a^\mu}{d\sigma} \frac{dX_a^\nu}{d\sigma} \\
&= \sum_{a=1}^{m+n} m_a^{-1} \int_0^\infty d\sigma\, e^{-ik.(m_a^{-1}p_a\,\sigma + r_a)}\, p_a^\mu p_a^\nu \\
&= \sum_{a=1}^{m+n} \frac{1}{i(k.p_a - i\epsilon)} e^{-ik.r_a}\, p_a^\mu p_a^\nu \\
&\simeq \sum_{a=1}^{m+n} p_a^\mu\, p_a^\nu \frac{1}{i(k.p_a - i\epsilon)} \qquad \text{for small } k
\end{aligned}$$

$$\widehat{T}^{\mu\nu}(k) \simeq \sum_{a=1}^{m+n} p_a^\mu \, p_a^\nu \, \frac{1}{i(k.p_a - i\epsilon)} \quad \text{for small } k, \quad k = \omega(1, \hat{n})$$

$$k.p_a - i\epsilon = -\omega \, p_a^0 - i\epsilon + \omega \, \hat{n}.\vec{p}_a$$

$$\tilde{e}^{\mu\nu}(\omega, \vec{x}) = \frac{2G}{|\vec{x}|} \widehat{T}^{\mu\nu}(k) = \frac{2G}{|\vec{x}|} \sum_{a=1}^{m+n} p_a^\mu \, p_a^\nu \, \frac{1}{i(-\omega \, p_a^0 - i\epsilon + \omega \, \hat{n}.\vec{p}_a)}$$

$$e^{\mu\nu}(t, \vec{x}) = e^{i\omega R} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, e^{-i\omega t} \, \tilde{e}^{\mu\nu}(\omega, \vec{x}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, e^{-i\omega u} \, \tilde{e}^{\mu\nu}(\omega, \vec{x})$$

Note: For outgoing particles $p_a^0 > 0$ and the pole is in the lower half ωplane

For incoming particles $p_a^0 = -p_a'^0 < 0$ and the pole is in the upper half ω plane

For $u > 0$ we can close the contour in the lower half ω plane, picking contribution from the outgoing particles

For $u < 0$ we can close the contour in the upper half ω plane, picking contribution from the incoming particles

$$\tilde{e}^{\mu\nu}(\omega,\vec{x}) = \frac{2G}{|\vec{x}|}\,\widehat{T}^{\mu\nu}(k) = \frac{2G}{|\vec{x}|}\sum_{a=1}^{m+n} p_a^\mu\, p_a^\nu\, \frac{1}{i(-\omega\, p_a^0 - i\epsilon + \omega\,\hat{n}.\vec{p}_a)}$$

$$e^{\mu\nu}(t,\vec{x}) = e^{i\omega R}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,e^{-i\omega t}\,\tilde{e}^{\mu\nu}(\omega,\vec{x}) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,e^{-i\omega u}\,\tilde{e}^{\mu\nu}(\omega,\vec{x})$$

Result after evaluating the ω integral by Cauchy's theorem:

$$\begin{aligned} e^{\mu\nu}(t,\vec{x}) &= -\frac{2\,G}{R}\sum_{a=1}^{n} p_a^\mu\, p_a^\nu\,\frac{1}{n.p_a} \quad \text{for large positive u} \\ &= -\frac{2\,G}{R}\sum_{a=1}^{m} p_a'^\mu\, p_a'^\nu\,\frac{1}{n.p_a'} \quad \text{for large negative u} \end{aligned}$$

It is conventional to make $e^{\mu\nu}$ vanish for large negative u by adding a constant to it by coordinate transformation

$$\begin{aligned} e^{\mu\nu}(t,\vec{x}) &= -\frac{2\,G}{R}\left[\sum_{a=1}^{n} p_a^\mu\, p_a^\nu\,\frac{1}{n.p_a} - \sum_{a=1}^{m} p_a'^\mu\, p_a'^\nu\,\frac{1}{n.p_a'}\right] \quad \text{for } u\to\infty \\ &= 0 \quad \text{for } u\to-\infty \end{aligned}$$

$\rightarrow$ memory effect

At the end of the first order analysis, we have, without taking large $|\vec{x}|$ limit,

$$\widehat{e}_{\mu\nu}(k) = -8\,\pi\,G\,G_r(k)\,\widehat{T}_{\mu\nu}(k)$$

$$= -8\,\pi\,G \sum_{a=1}^{m+n} p_{a\mu}\,p_{a\nu}\,e^{-ik.r_a}\,G_r(k)\,\frac{1}{i(k.p_a - i\epsilon)},$$

$$G_r(k) \equiv \frac{1}{(k^0 + i\epsilon)^2 - \vec{k}^2}\,.$$

$$e_{\mu\nu}(x) = -8\,\pi\,G \sum_b \int \frac{d^4\ell}{(2\pi)^4}\,e^{i\ell.x} G_r(\ell)\,p_{b\mu}\,p_{b\nu}\,\frac{1}{i(\ell.p_b - i\epsilon)},$$

$$h_{\mu\nu}(x) = e_{\mu\nu}(x) - \frac{1}{2}\,\eta_{\mu\nu}\,\eta^{\rho\sigma} e_{\rho\sigma}(x)\,,$$

Use this to compute $\Gamma^{\mu}_{\nu\rho}(x)$, correction to the trajectories and correction to the gravitational contribution $T^{h\mu\nu}$ to $T^{\mu\nu}$

$$\frac{d^2X^{\mu}_a}{d\tau^2} = -\Gamma^{\mu}_{\nu\rho}(X(\tau))\,\frac{dX^{\nu}_a}{d\tau}\,\frac{dX^{\rho}_a}{d\tau}$$

$$\delta X^{\mu}_a(\sigma) = \int_0^{\sigma} d\sigma' \int_{\sigma'}^{\infty} d\sigma''\,\Gamma^{\mu}_{\nu\rho}(v_a\,\sigma'' + r_a)\,p^{\nu}_a\,p^{\rho}_a/m^2_a$$

$$\delta X_a^\mu(\sigma) = \int_0^\sigma d\sigma' \int_{\sigma'}^\infty d\sigma'' \, \Gamma^\mu_{\nu\rho}(v_a \, \sigma'' + r_a) \, p_a^\nu \, p_a^\rho / m_a^2$$

$$e_{\mu\nu}(x) = -8\,\pi\,G \sum_b \int \frac{d^4\ell}{(2\pi)^4} \, e^{i\ell.x} G_r(\ell) \, p_{b\mu} \, p_{b\nu} \, \frac{1}{i(\ell.p_b - i\epsilon)},$$

$$h_{\mu\nu}(x) = e_{\mu\nu}(x) - \frac{1}{2} \, \eta_{\mu\nu} \, \eta^{\rho\sigma} e_{\rho\sigma}(x) \, ,$$

Substitute into the expression for

$$\widehat{T}^{\mu\nu}(k) = \sum_{a=1}^{m+n} m_a \int_0^\infty d\sigma \, e^{-ik.X(\sigma)} \, \frac{dX_a^\mu}{d\sigma} \, \frac{dX_a^\nu}{d\sigma}$$

$$\widehat{\delta T}^{\mu\nu}(k) = 2 \sum_{a=1}^{m+n} m_a \int_0^\infty d\sigma \, e^{-ik.X(\sigma)} \, \frac{dX_a^\mu}{d\sigma} \, \frac{d\delta X_a^\nu}{d\sigma}$$

Once we express Γ as a momentum space integral, the integration over σ'', σ' and σ can be performed with ease, leaving us with a momentum space integral.

$$
\begin{aligned}
\delta\widehat{T}^{X\mu\nu}(k) \quad = \quad & -8\,\pi\,G\,\sum_{a=1}^{m+n}\sum_{b\neq a}\int\frac{d^4\ell}{(2\pi)^4}\,\frac{1}{\ell.p_b - i\epsilon}\,G_r(\ell)\,e^{-ik.r_a - i\ell.(r_b - r_a)} \\
& \Bigg[\Big(2\,p_a.p_b\,k.p_b\,p_a.\ell - k.\ell\,(p_a.p_b)^2 - p_b^2\,p_a.k\,p_a.\ell + \frac{1}{2}k.\ell\,p_a^2\,p_b^2\Big)\,p_a^\nu\,p_a^\mu \\
& \qquad \times\frac{1}{\ell.p_a + i\epsilon}\,\frac{1}{k.p_a - i\epsilon}\,\frac{1}{(\ell-k).p_a + i\epsilon} \\
& -\Big\{2\,p_a.p_b\,\ell.p_a\,\left(p_a^\nu p_b^\mu + p_a^\mu p_b^\nu\right) - (p_a.p_b)^2\left(\ell^\mu p_a^\nu + \ell^\nu p_a^\mu\right) \\
& -2\,p_b^2\,\ell.p_a\,p_a^\mu\,p_a^\nu + \frac{1}{2}\,p_a^2\,p_b^2\,\left(\ell^\mu p_a^\nu + \ell^\nu p_a^\mu\right)\Big\}\frac{1}{\ell.p_a + i\epsilon}\,\frac{1}{(\ell-k).p_a + i\epsilon}\Bigg].
\end{aligned}
$$

$$
G_r(k) \equiv \frac{1}{(k^0 + i\epsilon)^2 - \vec{k}^2}
$$

Goal: Evaluate this in the small k limit

Result:

$$\delta\widehat{T}^{X\mu\nu}(k) = 2\,G \sum_{a=1}^{m+n} \sum_{\substack{b\neq a \\ \eta_a\eta_b=1}} \frac{\ln\{L(\omega+i\epsilon\eta_a)\}}{\{(p_a.p_b)^2 - p_a^2 p_b^2\}^{3/2}}$$

$$\left[\frac{k.p_b}{k.p_a}\, p_a^\mu p_a^\nu\, p_a.p_b \left\{\frac{3}{2} p_a^2 p_b^2 - (p_a.p_b)^2\right\} + \frac{1}{2} p_a^\mu p_a^\nu\, p_a^2\, (p_b^2)^2\right.$$

$$\left. -\{p_a^\mu p_b^\nu + p_a^\nu p_b^\mu\}\, p_a.p_b \left\{\frac{3}{2} p_a^2 p_b^2 - (p_a.p_b)^2\right\}\right].$$

$L \sim |r_a - r_b|$ **acts as UV cut-off (size of the scattering region)**

$\eta_b = 1$ **for b outgoing,** -1 **for b incoming**

Contribution from the gravitational energy momentum tensor:

$$\begin{aligned}
8\pi\, G\, T^{h}_{\mu\nu} \quad = \quad & -2\Big[\frac{1}{2}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} + h^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta} - h^{\alpha\beta}\partial_\nu\partial_\beta h_{\alpha\mu} \\
& - h^{\alpha\beta}\partial_\mu\partial_\beta h_{\alpha\nu} + h^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} + \partial^\beta h^\alpha_\nu \partial_\beta h_{\alpha\mu} - \partial^\beta h^\alpha_\nu \partial_\alpha h_{\beta\mu}\Big] \\
& + h_{\mu\nu}\partial_\rho\partial^\rho h - 2h_{\mu\rho}\partial^\sigma\partial_\sigma h_\nu^{\ \rho} - 2h_{\nu\rho}\partial^\sigma\partial_\sigma h_\mu^{\ \rho} \\
& + \eta_{\mu\nu}\Big[\frac{3}{2}\partial^\rho h_{\alpha\beta}\partial_\rho h^{\alpha\beta} + 2h^{\alpha\beta}\partial^\rho\partial_\rho h_{\alpha\beta} - \partial^\beta h^{\alpha\rho}\partial_\alpha h_{\beta\rho}\Big] \\
& + h\Big[\partial^\rho\partial_\rho h_{\mu\nu} - \frac{1}{2}\partial^\rho\partial_\rho h\, \eta_{\mu\nu}\Big] + O(h^3)\,,
\end{aligned}$$

From the expression for $\widehat{h}_{\mu\nu}$ we can find the expression for $\widehat{T}^{h}_{\mu\nu}$.

$$\widehat{\mathbf{T}}^{h\mu\nu}(\mathbf{k}) = -8\,\pi\,\mathbf{G}\sum_{\mathbf{a},\mathbf{b}} \mathbf{e}^{-i\mathbf{k}.\mathbf{r_a}} \int \frac{\mathbf{d}^4\ell}{(2\pi)^4}\,\mathbf{e}^{i\ell.(\mathbf{r_a}-\mathbf{r_b})}\,\mathbf{G_r}(\mathbf{k}-\ell)\mathbf{G_r}(\ell)$$

$$\frac{1}{\mathbf{p_b}.\ell - i\epsilon}\;\frac{1}{\mathbf{p_a}.(\mathbf{k}-\ell) - i\epsilon}$$

$$\times\left\{\mathbf{p}_{\mathbf{b}\alpha}\mathbf{p}_{\mathbf{b}\beta} - \frac{1}{2}\mathbf{p}_\mathbf{b}^2\eta_{\alpha\beta}\right\}\mathcal{F}^{\mu\nu,\alpha\beta,\rho\sigma}(\mathbf{k},\ell)\left\{\mathbf{p}_{\mathbf{a}\rho}\mathbf{p}_{\mathbf{a}\sigma} - \frac{1}{2}\mathbf{p}_\mathbf{a}^2\eta_{\rho\sigma}\right\}$$

$$\begin{aligned}
&\mathcal{F}^{\mu\nu,\alpha\beta,\rho\sigma}(\mathbf{k},\ell)\\
= \;& 2\Big[\frac{1}{2}\ell^\mu(\mathbf{k}-\ell)^\nu\eta^{\rho\alpha}\eta^{\sigma\beta} + (\mathbf{k}-\ell)^\mu(\mathbf{k}-\ell)^\nu\eta^{\rho\alpha}\eta^{\sigma\beta}\\
&-(\mathbf{k}-\ell)^\nu(\mathbf{k}-\ell)^\beta\eta^{\rho\alpha}\eta^{\sigma\mu} - (\mathbf{k}-\ell)^\mu(\mathbf{k}-\ell)^\beta\eta^{\rho\alpha}\eta^{\sigma\nu}\\
&+(\mathbf{k}-\ell)^\alpha(\mathbf{k}-\ell)^\beta\eta^{\rho\mu}\eta^{\sigma\nu} + (\mathbf{k}-\ell).\ell\,\eta^{\beta\nu}\eta^{\alpha\rho}\eta^{\sigma\mu} - \ell^\rho(\mathbf{k}-\ell)^\alpha\eta^{\beta\nu}\eta^{\sigma\mu}\\
&-\frac{1}{2}(\mathbf{k}-\ell)^2\eta^{\alpha\mu}\eta^{\beta\nu}\eta^{\rho\sigma} + \eta^{\alpha\mu}\eta^{\beta\rho}\eta^{\nu\sigma}(\mathbf{k}-\ell)^2 + \eta^{\alpha\nu}\eta^{\beta\rho}\eta^{\mu\sigma}(\mathbf{k}-\ell)^2\Big]\\
&-\eta^{\mu\nu}\Big[\frac{3}{2}(\mathbf{k}-\ell).\ell\,\eta^{\rho\alpha}\eta^{\sigma\beta} + 2(\mathbf{k}-\ell)^2\eta^{\rho\alpha}\eta^{\sigma\beta} - \ell^\sigma(\mathbf{k}-\ell)^\alpha\eta^{\rho\beta}\Big]\\
&-\eta^{\alpha\beta}(\mathbf{k}-\ell)^2\eta^{\rho\mu}\eta^{\sigma\nu} + \frac{1}{2}\eta^{\alpha\beta}(\mathbf{k}-\ell)^2\eta^{\rho\sigma}\eta^{\mu\nu}\,.
\end{aligned}$$

Goal: Evaluate $\widehat{\mathbf{T}}^{h\mu\nu}(\mathbf{k})$ for small k

Result:

$$
\begin{aligned}
\widehat{T}^{h\mu\nu}(k) &= 2\,G\,\ln\{(\omega+i\epsilon)R\}\sum_{a=1}^{m+n}\sum_{b=1}^{n}\frac{1}{p_a.k-i\epsilon}\,\frac{1}{p_b.k-i\epsilon} \\
&\left\{p_a.p_b\,k.p_a\,k.p_b\,\eta^{\mu\nu}-\frac{1}{2}\,p_b^2\,(k.p_a)^2\,\eta^{\mu\nu}-(k.p_b)^2\,p_a^\mu\,p_a^\nu\right\} \\
&+G\sum_{a=1}^{m+n}\ln\{L(\omega+i\epsilon\eta_a)\}\sum_{\substack{b=1\\ b\neq a,\,\eta_a\eta_b=1}}^{m+n}\frac{1}{\{(p_a.p_b)^2-p_a^2p_b^2\}^{3/2}} \\
&\left[-p_b^\mu p_b^\nu\,(p_a^2)^2\,(p_b^2)+\{p_a^\mu p_b^\nu+p_a^\nu p_b^\mu\}\,p_a.p_b\,\left\{\frac{3}{2}p_a^2p_b^2-(p_a.p_b)^2\right\}\right].
\end{aligned}
$$

$$e^{\mu\nu}(t,\vec{x}) = \frac{2G}{|\vec{x}|}\, e^{i\omega R} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\, [\widehat{T}^{X\mu\nu}(k) + \widehat{T}^{h\mu\nu}(k)]$$

$$k = \omega(1,\hat{n}) \qquad \hat{n} = \vec{x}/|\vec{x}|$$

This gives the 1/u terms in $e_{\mu\nu}$ using

$$\int \frac{d\omega}{2\pi}\, e^{-i\omega u} \ln(\omega + i\epsilon) \simeq \begin{cases} -\frac{1}{u} & \text{for } u \to \infty \\ 0 & \text{for } u \to -\infty \end{cases} .$$

$$\int \frac{d\omega}{2\pi}\, e^{-i\omega u} \ln(\omega - i\epsilon) \simeq \begin{cases} 0 & \text{for } u \to \infty \\ \frac{1}{u} & \text{for } u \to -\infty \end{cases} .$$