Solution to Caron-Huot's problems

1. Consider $S(\omega) \simeq 1 + iC\omega^\lambda$ on the upper-half plane, thus writing ω as

$$\omega = |\omega|e^{i\theta}\,, \quad \theta \in (0,\pi)\,. \tag{0.1}$$

It is easy to find

$$|S(\omega)| \leq 1 \;\to\; \mathrm{Im}(C|\omega|^\lambda e^{i\lambda\theta}) \geq 0\,, \tag{0.2}$$

where we have ignored $|C\omega|^2$ term because it is comparably small. Expand the RHS, we find

$$2|\omega|\big(\mathrm{Re}\ C\ \sin(\lambda\theta) + \mathrm{Im}\ C\ \cos(\lambda\theta)\big) \geq 0\,. \tag{0.3}$$

For this inequality to have solution for C, we have to make sure that $\sin(\lambda\theta)$ doesn't change its sign. This requires

$$0 \leq |\lambda|\theta \leq \pi\,, \tag{0.4}$$

otherwise the contradiction will be created because sin flips sign crossing $|\lambda|\theta = \pi$ but cos doesn't flip sign. Thus we conclude

$$-1 \leq \lambda \leq 1\,. \tag{0.5}$$

When λ is very close to 1, for simplicity we can choose $\lambda = 1$, taking $\theta = \pi/2$ we can have

$$\mathrm{Re}\ C \geq 0\,. \tag{0.6}$$

However, λ is not necessarily to be one, any λ that $\lambda\theta = \pi/2$ can lead to positive constraint on Re C. Thus we have

$$\lambda = \frac{\pi}{2\theta} > \frac{1}{2}\,, \tag{0.7}$$

where in the second step we use $\theta < \pi$.

Now assume we take $\lambda = 1$, since $\cos\theta$ flips signs crossing $\pi/2$, to make the inequality valid for upper-half plane, we must set $\mathrm{Im}C = 0$.

2. Let's consider how signal is transformed

$$f_{\mathrm{out}}(t) = \int d\omega e^{-i\omega t} S(\omega) f_{\mathrm{in}}(\omega)\,. \tag{0.8}$$

Around $\omega \sim \omega_0$, we then have

$$f_{\mathrm{out}}(t) \simeq \int d\omega e^{-i\omega(t+\frac{a}{\Gamma^2})} e^{i\frac{\omega_0 a}{\Gamma^2} - \frac{a}{\Gamma}} f_{\mathrm{in}}(\omega) \simeq e^{i\frac{\omega_0 a}{\Gamma^2} - \frac{a}{\Gamma}} f_{\mathrm{in}}(t + \frac{a}{\Gamma^2})\,. \tag{0.9}$$

We thus see there is time advance. However, this does not violate causality because $e^{-a/\Gamma}$ highly suppresses the magnitude: the peak develops time advance, but it's allowed by causality because it's simply triggered by tail of input!

For numerical calculations, see Mathematica notebook.

3. Let's think more carefully about how to obtain k-subtracted dispersive sum rule.

$$\oint_\infty \frac{ds}{s(s(s+t))^{\frac{k}{2}}} M(s,t) = 0 \to$$
$$\mathrm{Res}_{s=0,s=-t} \frac{M_{\mathrm{low}}(s,t)}{s(s(s+t))^{\frac{k}{2}}} = \int_{M^2}^{\infty} ds \frac{\mathrm{Disc} M(s,t)}{s(s(s+t))^{\frac{k}{2}}} + \int_{-M^2-t}^{-\infty} ds \frac{\mathrm{Disc} M(s,t)}{s(s(s+t))^{\frac{k}{2}}} . \quad (0.10)$$

For the explicit evaluations, please check the Mathematica notebook. Note, for real scalar, we only have one spectral density, using crossing symmetry, the RHS boils down to

$$\int_{M^2}^{\infty} ds \frac{(2s+t)\mathrm{Disc} M(s,t)}{(s(s+t))^{\frac{k}{2}+1}} = 16 \sum_J (2J+1) \int_{M^2}^{\infty} \frac{dm^2}{m^2} \frac{2m^2+t}{(m^2(m^2+t))^{\frac{k+2}{2}}} \rho(m^2,J) P_J(x) , \quad (0.11)$$

where $x = 1 + 2t/m^2$ and $\rho \geq 0$. To have the first null constraint, we shall consider $k = 2$ and $k = 4$. This is equivalent to considering $k = 2$ for "full" subtracted sum rule, with crossing symmetry giving null constraint

$$\oint_\infty \frac{ds}{s-s'} \frac{M(s,t)}{s(s+t)} . \quad (0.12)$$

For convenience, we may consider this point of view for complex scalar. But it is easy to see equivalence.

For complex scalar, it is better to keep track of kinematic configuration. We have and denote

$$M_{\Phi\bar{\Phi}\bar{\Phi}\Phi} = f(s,u) , \quad M_{\Phi\Phi\bar{\Phi}\bar{\Phi}} = f(s,t) , \quad M_{\Phi\bar{\Phi}\Phi\bar{\Phi}} = f(u,t) . \quad (0.13)$$

because the first is $s-u$ symmetric. For twice-subtracted sum rules, we can define two different sum rules

$$\oint_\infty \frac{ds}{s-s'} \frac{f(s,u)}{s(s+t)} = 0 , \quad (0.14)$$
$$\oint_\infty \frac{ds}{s-s'} \frac{f(s,t)+f(u,t)}{s(s+t)} = 0 . \quad (0.15)$$

We can also define triple-subtracted sum rule

$$\oint_\infty \frac{ds}{s-s'} \frac{f(s,t)-f(u,t)}{(s+t)^3} . \quad (0.16)$$

For explicit evaluations, please refer the Mathematica notebook. In the end, we find at $1/m^6$, we have one null constraint, but at $1/m^8$, we have two null constraints.

If we allow single subtraction, we can define following sum rule

$$\oint_\infty \frac{ds}{s-s'} \frac{f(s,t)-f(u,t)}{s+t} \,. \tag{0.17}$$

We now have one null constraint at $1/m^4$, two null constraints at $1/m^6$, and three null constraints at $1/m^8$.

- 4, 5 see Mathematica noteebook.