# Quantum unitary evolution interspersed with repeated non-unitary interactions at random times


**Debraj Das**

ICTP - International Centre for Theoretical Physics, IT


Quantum Trajectories, ICTS, IN
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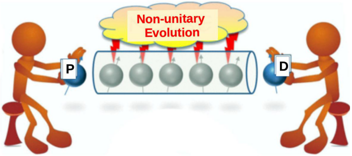
Collaborators: **S. Gupta** (TIFR, IN) and **S. Dattagupta** (SNU, IN)

# Introduction

- Isolated quantum systems: Unitary dynamics ⇒ coherent evolution, essential for preserving information
- Perfect isolation never possible;
  isolation only for finite time intervals until external effects kick in
- External effects due to interactions with external environment or measuring apparatus ⇒ Non-unitary evolution ⇒ Decoherence




**P: Preparation, D: Detection**

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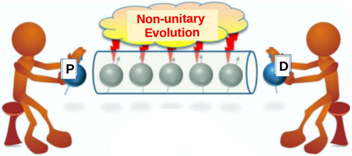


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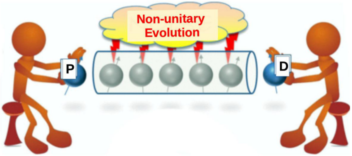


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- **Study Unitary Evolution interspersed with Non-unitary Interactions at Random Times (classical randomness)**

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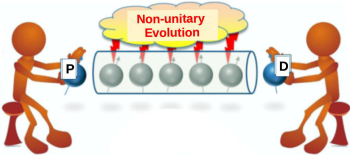
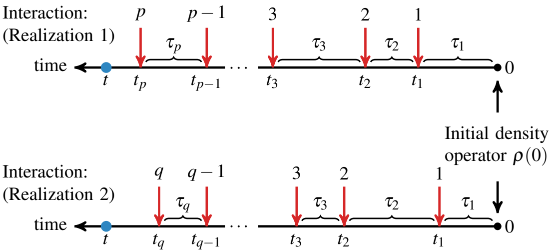



**P: Preparation, D: Detection**

- Typically interactions kick in at random times
- **Study Unitary Evolution interspersed with Non-unitary Interactions at Random Times (classical randomness)**
- **Main Question:**
  **Interplay of Classical and Quantum Randomness ⇒ Consequences ??**

# Evolution scheme

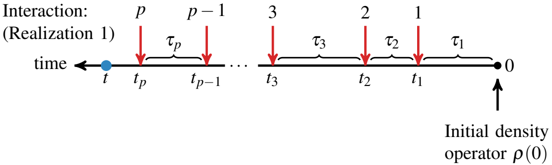
- Generic Hamiltonian $H$
- Interaction operator: $T$




- Time gaps $\tau_p$: independently sampled from a distribution $p(\tau)$
- $t$: observation time
- **Interplay of Classical and Quantum randomness !**

# Averaged density operator

- Consider a typical realization, say realization 1:




- $\rho^{(1)}(t) = \mathrm{e}^{-\mathrm{i}\mathcal{L}(t-t_p)}\,T\mathrm{e}^{-\mathrm{i}\mathcal{L}(t_p-t_{p-1})}\,T\ldots\mathrm{e}^{-\mathrm{i}\mathcal{L}(t_2-t_1)}\,T\mathrm{e}^{-\mathrm{i}\mathcal{L}t_1}\rho(0)$
- $\mathcal{L}\rightarrow$ unitary evolution:
  $\rho(t'>t'') = \mathrm{e}^{-\mathrm{i}\mathcal{L}(t'-t'')}\rho(t'') = \mathrm{e}^{-\mathrm{i}H(t'-t'')}\rho(t'')\mathrm{e}^{\mathrm{i}H(t'-t'')}$
- Density operator averaged over realizations of random times of interactions:
  $\overline{\rho}(t) = \sum_{p=0}^{\infty}\int_0^t \mathrm{d}t_p\int_0^{t_p}\mathrm{d}t_{p-1}\ldots\int_0^{t_3}\mathrm{d}t_2\int_0^{t_2}\mathrm{d}t_1$

$$F(t-t_p)\mathrm{e}^{-\mathrm{i}\mathcal{L}(t-t_p)}\,T\,p(t_p-t_{p-1})\mathrm{e}^{-\mathrm{i}\mathcal{L}(t_p-t_{p-1})}\,T..p(t_2-t_1)\mathrm{e}^{-\mathrm{i}\mathcal{L}(t_2-t_1)}\,T\,p(t_1)e^{-\mathrm{i}\mathcal{L}t_1}\rho(0)$$

- $F(t)\equiv\int_t^{\infty}\mathrm{d}\tau\;p(\tau)$: probability of no interaction during time $t$

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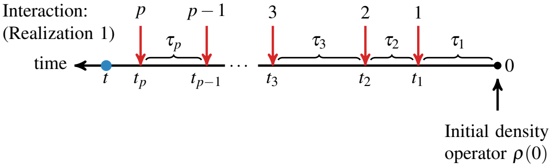


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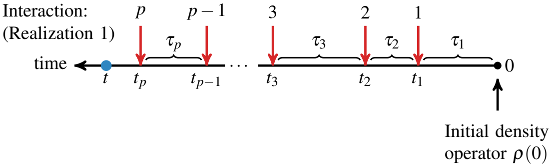


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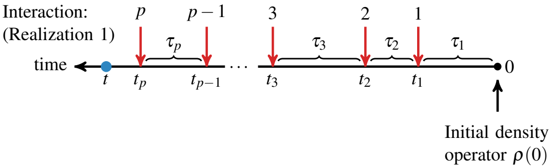


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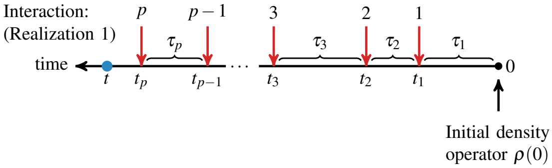
- $F(t) \equiv \int_t^{\infty}\mathrm{d}\tau\; p(\tau)$: probability of no interaction during time $t$
- $\overline{\rho}(t) = U(t)\rho(0)$; $U(t) \to$ superoperator

# Averaged density operator

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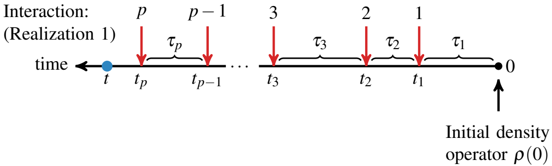


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- Laplace transform $\mathfrak{L}$ of a convolution $g_1 * g_2 \equiv \int_0^t \mathrm{d}\tau\ g_1(\tau)g_2(t-\tau)$:
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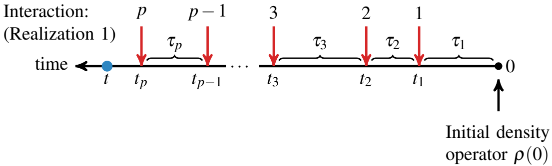


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- $\widetilde{\overline{\rho}}(s) \equiv \mathfrak{L}(\overline{\rho}(t)) = \widetilde{U}(s)\rho(0)$
- $\widetilde{U}(s) = \mathfrak{L}\left(F(t)\mathrm{e}^{-\mathrm{i}\mathcal{L}t}\right) \sum_{p=0}^{\infty} \left[T\mathfrak{L}\left(p(t)\mathrm{e}^{-\mathrm{i}\mathcal{L}t}\right)\right]^p = \frac{\mathfrak{L}(F(t)\mathrm{e}^{-\mathrm{i}\mathcal{L}t})}{\mathbb{I}-T\mathfrak{L}(p(t)\mathrm{e}^{-\mathrm{i}\mathcal{L}t})}$

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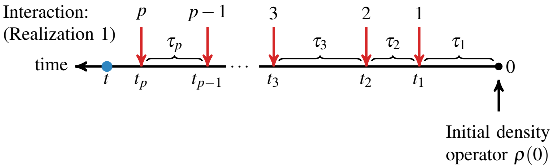


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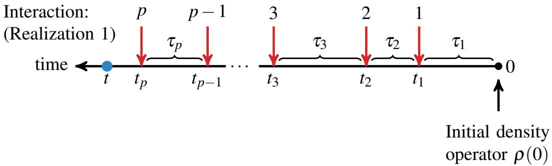


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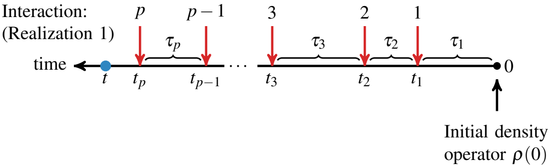
- Hamiltonian $H$
- Interaction operator $T$
- Distribution of time gaps $p(\tau)$

# Averaged density operator for the case of exponential $p(\tau)$

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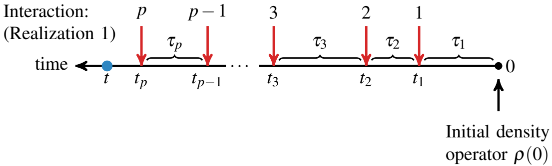


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- Interactions happen at exponentially-distributed time intervals with constant rate $\lambda$: $p(\tau) = \lambda\mathrm{e}^{-\lambda\tau}$
- $F(t) = \exp(-\lambda t)$
- $\widetilde{U}(s) = [(s+\lambda)\mathbb{I} + \mathrm{i}\mathcal{L} - T\lambda]^{-1}$ $\left(\mathfrak{L}(t^n\mathrm{e}^{-at}) = \frac{n!}{(s+a)^{n+1}};\ t > 0,\ n \in [0, 1, \ldots, \infty)\right)$
- $\widetilde{U}(s) = \widetilde{U}_0(s) + \underbrace{\lambda\widetilde{U}_0(s)T\widetilde{U}_0(s)}_{\text{One interaction}} + \underbrace{\lambda^2\widetilde{U}_0(s)T\widetilde{U}_0(s)T\widetilde{U}_0(s)}_{\text{Two interactions}} + \ldots$
- No-interaction term $\widetilde{U}_0(s) \equiv [(s+\lambda)\mathbb{I} + \mathrm{i}\mathcal{L}]^{-1}$

# Application to the tight-binding model

# The tight-binding model (TBM) in absence of interaction

- A quantum particle residing on the sites of a $1d$ lattice and because of quantum fluctuations undergoing tunnelling to nearest-neighbour sites
- $H = -\frac{\Delta}{2} \sum_{n=-\infty}^{\infty} \Big( |n\rangle\langle n+1| + |n+1\rangle\langle n| \Big)$
- $|n\rangle$: Wannier state of the particle on site $n$;
  $\langle m|n\rangle = \delta_{mn}$, $\sum_{m=-\infty}^{\infty} |m\rangle\langle m| = \mathbb{I}$

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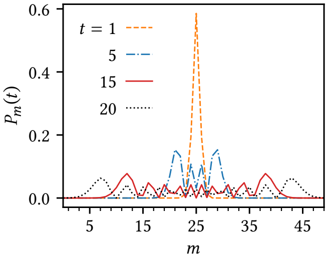
- **Probability $P_m(t)$ to be on site $m$ at time $t > 0$, while starting from site $n_0$ at time $t = 0$??** $P_m(t) = J^2_{m-n_0}(\Delta t)$




$(n_0 = 25,\ \Delta = 1.0)$

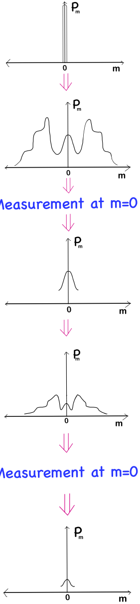
**Spreading with time (Delocalization)**

1. Average displacement from $n_0$:
   $\mu \equiv \sum_{m=-\infty}^{\infty}(m-n_0)P_m(t) = 0$
2. Mean-squared displacement:
   $S(t) \equiv \sum_{m=-\infty}^{\infty}(m-n_0)^2 P_m(t)) = \frac{\Delta^2 t^2}{2}$

**TBM subject to representative interactions:**

1. **Projective measurements at random times**
2. **Stochastic resets at random times**

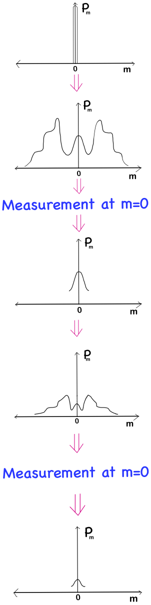
# TBM in presence of Projective Measurements at Random Times



## TBM + Projective Measurements at Random Times: Earlier work

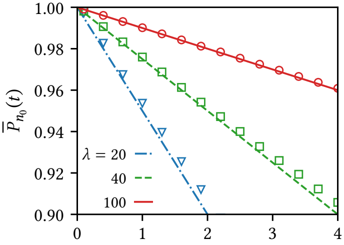
1. **Earlier work** *(Dhar, Dasgupta, Dhar (2015), Dhar, Dasgupta, Dhar, Sen (2015), Friedman, Kessler, Barkai (2017), Friedman, Kessler, Barkai (2017), Thiel, Barkai, Kessler (2018), Thiel, Mualem, Kessler, Barkai (2019), Lahiri, Dhar (2019), Meidan, Barkai, Kessler (2019), Yin, Ziegler, Thiel, Barkai (2019), Thiel, Mualem, Meidan, Barkai, Kessler (2020), Thiel, Mualem, Kessler, Barkai (2020), Dubey, Bernardin, Dhar (2021), Thiel, Mualem, Kessler, Barkai (2021), Liu, Ziegler, Kessler, Barkai (2022), Kessler, Barkai, Ziegler (2021),...)*: Evolution following projective measurements (operator $P$) continued with the **leftover component**:
   1. $\rho_-(t_1) = e^{-i\mathcal{L}t_1}\rho(0)$
   2. $\rho_+(t_1) = (\mathbb{I}-P)\rho_-(t_1)(\mathbb{I}-P)^\dagger$
   3. $\rho_-(t_2) = e^{-i\mathcal{L}(t_2-t_1)}\rho_+(t_1)$
   4. $\rho_+(t_2) = (\mathbb{I}-P)\rho_-(t_2)(\mathbb{I}-P)^\dagger$
   5. ...
   6. A representative result: Start at $n_0$ and perform projective measurements to $n_0$ at regular time intervals $\tau$: Survival probability $P_{n_0}(t)$ decays as a power law $P_{n_0}(t) \sim t^{-3/2}$ *(Dhar, Dasgupta, Dhar (2015))*
2. **Our work:** Evolution with the **projected component** (Zeno effect set-up):
   1. $\rho_+(t_1) = P\rho_-(t_1)P^\dagger$
   2. $\rho_-(t_2) = e^{-i\mathcal{L}(t_2-t_1)}\rho_+(t_1)$
   3. $\rho_+(t_2) = P\rho_-(t_2)P^\dagger$
   4. ...

## TBM + Projective Measurements at Random Times: Results




1. Start at $n_0$ and perform projective measurements to $n_0$ at random time intervals $\tau$ distributed as $p(\tau) = \lambda e^{-\lambda\tau}$

2. Zeno limit of frequent-enough measurements: $\lambda \to \infty$ at fixed $t$

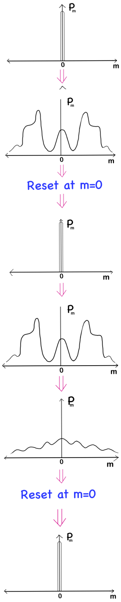



($n_0$ = 10, $\Delta$ = 1.0)

3. $\overline{P}_{n_0}(t) \approx 1 - \frac{\Delta^2 t}{\lambda}$
   **Comparable suppression in conventional Zeno effect**

4. Measurements at random times much more feasible than at regular intervals

# TBM in presence of Stochastic Resets at Random Times



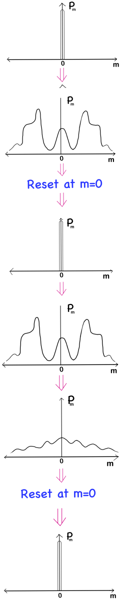
## Stochastic Resets at Random Times: Earlier work

1. **Classical systems:** Introduced in the context of Brownian motion *(Evans, Majumdar (2011))*; many interesting static and dynamic effects in single and many-body systems (review: *(Evans, Majumdar, Schehr (2020))*)

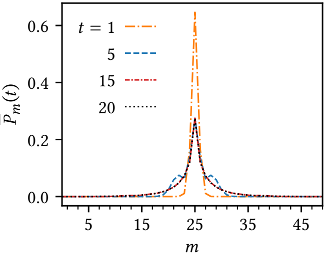
2. **Quantum systems:**

    1. Integrable and non-integrable systems *(Mukherjee, Sengupta, Majumdar (2018))*
    2. Purity, fidelity in closed quantum systems *(Sevilla, Valdé s-Hernández (2023))*
    3. Dynamics of a qubit in presence of detectors *(Dubey, Chetrite, Dhar (2023))*
    4. Quantum-search processes *(Yin, Barkai (2023))*
    5. Eigenvalue spectrum of a Markovian generator *(Rose, Touchette, Lesanovsky, Garrahan (2018))*
    6. Entanglement in many-body systems *(Turkeshi, Dalmonte, Fazio, Schirò (2022))*
    7. von Neumann entropy, fidelity, and concurrence *(Kulkarni, Majumdar (2023))*
    8. Long-range correlations *(Magoni, Carollo, Perfetto, Lesanovsky (2022))*
    9. Quantum-jump trajectories *(Perfetto, Carollo, Lesanovsky (2022))*
    10. Quantum collapse *(Riera-Campeny, Ollé, Masó-Puigdellosas (2021))*
    11. Ground state preparation from frustration-free Hamiltonians *(Puente, Motzoi, Calarco, Morigi, Rizzi (2023))*

# TBM + Stochastic Resets at Random Times: Results



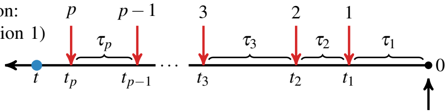

1. Start at $n_0$ and perform stochastic resets to $n_0$ at random time intervals $\tau$ distributed as $p(\tau) = \lambda e^{-\lambda\tau}$




($n_0 = 25$, $\Delta = 1.0$, $\lambda = 0.25$)

2. $t \to \infty$:
**Time-independent probabilities to be on different sites (Localization)**

# Notion of a superoperator




- $\overline{\rho}(t) = \sum_{p=0}^{\infty} \int_0^t \mathrm{d}t_p \int_0^{t_p} \mathrm{d}t_{p-1} \ldots \int_0^{t_3} \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_1$

$$F(t-t_p)\mathrm{e}^{-\mathrm{i}\mathcal{L}(t-t_p)}\,T\,p(t_p-t_{p-1})\mathrm{e}^{-\mathrm{i}\mathcal{L}(t_p-t_{p-1})}\,T..p(t_2-t_1)\mathrm{e}^{-\mathrm{i}\mathcal{L}(t_2-t_1)}\,T\,p(t_1)e^{-\mathrm{i}\mathcal{L}t_1}\rho(0)$$
$$= U(t)\rho(0)$$

- $\mathcal{L}$ and $\mathcal{T}$ are superoperators: act on operators to yield operators

$$TA = B;\ A, B \rightarrow \text{ordinary operators}$$

- If $A, B$ are defined in Hilbert space $\mathcal{H}$ with complete basis set $\{|m\rangle\}$, superoperator lives in a product Hilbert space $\{|mn) \equiv |m\rangle \otimes |n\rangle\}$; $\sum_{m,\,n} |mn)(mn| = \mathbb{I}$
- "Matrix elements" of $T$ labeled by four indices: $\langle m|B|n\rangle = \langle m|TA|n\rangle = \sum_{m',n'} (mn|T|m'n')\langle m'|A|n'\rangle$

# TBM + Projective Measurements at Random Times: Analysis

- $\widetilde{\overline{\rho}}(s) = \widetilde{U}(s)\rho(0)$;
  $$\widetilde{U}(s) = \widetilde{U}_0(s) + \underbrace{\lambda \widetilde{U}_0(s) T \widetilde{U}_0(s)}_{\text{One interaction}} + \underbrace{\lambda^2 \widetilde{U}_0(s) T \widetilde{U}_0(s) T \widetilde{U}_0(s)}_{\text{Two interactions}} + \ldots$$
- Start at $n_0$, project to $\mathcal{N}$, obtain the probability at $m$
- Density operators before and after an interaction must satisfy
  $\rho_+(t) = T\rho_-(t) = P\rho_-(t)P^\dagger$; $P = |\mathcal{N}\rangle\langle\mathcal{N}|$
  $$\Rightarrow (n_1 n_1' | T | n_2 n_2') = \delta_{n_1\mathcal{N}}\delta_{n_1'\mathcal{N}}\delta_{n_2\mathcal{N}}\delta_{n_2'\mathcal{N}}$$
- $\overline{P}_m(t) = \langle m|\overline{\rho}(t)|m\rangle$

$$\widetilde{\overline{P}}_m(s) = \langle m|\widetilde{U}_0(s)\rho(0) + \lambda \widetilde{U}_0(s) T \widetilde{U}_0(s)\rho(0) + \lambda^2 \widetilde{U}_0(s) T \widetilde{U}_0(s) T \widetilde{U}_0(s)\rho(0) + \ldots |m\rangle$$
$$\equiv \sum_{p=0}^{\infty} \widetilde{\overline{P}}_m^{(p)}(s)$$

- $\overline{P}_m^{(0)}(s) = (mm|\widetilde{U}_0(s)|n_0 n_0)$
  $\overline{P}_m^{(1)}(s) = \lambda(mm|\widetilde{U}_0(s)|\mathcal{N}\mathcal{N})(\mathcal{N}\mathcal{N}|\widetilde{U}_0(s)|n_0 n_0)$
  $\overline{P}_m^{(2)}(s) = \lambda^2(mm|\widetilde{U}_0(s)|\mathcal{N}\mathcal{N})(\mathcal{N}\mathcal{N}|\widetilde{U}_0(s)|\mathcal{N}\mathcal{N})(\mathcal{N}\mathcal{N}|\widetilde{U}_0(s)|n_0 n_0)$

# TBM + Projective Measurements at Random Times: Analysis

- $\widetilde{\overline{\rho}}(s) = \widetilde{U}(s)\rho(0)$;
  $$\widetilde{U}(s) = \widetilde{U}_0(s) + \underbrace{\lambda \widetilde{U}_0(s) T \widetilde{U}_0(s)}_{\text{One interaction}} + \underbrace{\lambda^2 \widetilde{U}_0(s) T \widetilde{U}_0(s) T \widetilde{U}_0(s)}_{\text{Two interactions}} + \ldots$$

- Start at $n_0$, project to $\mathcal{N}$, obtain the probability at $m$

- Density operators before and after an interaction must satisfy
  $\rho_+(t) = T\rho_-(t) = P\rho_-(t)P^\dagger$; $P = |\mathcal{N}\rangle\langle\mathcal{N}|$
  $$\Rightarrow (n_1 n_1' | T | n_2 n_2') = \delta_{n_1 \mathcal{N}} \delta_{n_1' \mathcal{N}} \delta_{n_2 \mathcal{N}} \delta_{n_2' \mathcal{N}}$$

- $\overline{P}_m(t) = \langle m | \overline{\rho}(t) | m \rangle$

$$\widetilde{\overline{P}}_m(s) = \langle m | \widetilde{U}_0(s)\rho(0) + \lambda \widetilde{U}_0(s) T \widetilde{U}_0(s)\rho(0) + \lambda^2 \widetilde{U}_0(s) T \widetilde{U}_0(s) T \widetilde{U}_0(s)\rho(0) + \ldots | m \rangle$$
$$\equiv \sum_{p=0}^{\infty} \widetilde{\overline{P}}_m^{(p)}(s)$$

- $$\overline{P}_m^{(p)}(t) = \lambda^p \int_0^t \mathrm{d}t_p \int_0^{t_p} \mathrm{d}t_{p-1} \ldots \int_0^{t_3} \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_1$$
  $$\times \left[\mathrm{e}^{-\lambda(t-t_p)} J_{m-\mathcal{N}}^2(\Delta(t-t_p))\right] \left[\mathrm{e}^{-\lambda(t_p-t_{p-1})} J_{\mathcal{N}-\mathcal{N}}^2(\Delta(t_p-t_{p-1}))\right] \ldots$$
  $$\times \left[\mathrm{e}^{-\lambda(t_2-t_1)} J_{\mathcal{N}-\mathcal{N}}^2(\Delta(t_2-t_1))\right] \left[\mathrm{e}^{-\lambda t_1} J_{\mathcal{N}-n_0}^2(\Delta t_1)\right]$$

# TBM + Stochastic Resets at Random Times: Analysis

- $\widetilde{\overline{\rho}}(s) = \widetilde{U}(s)\rho(0)$;
  $\widetilde{U}(s) = \widetilde{U}_0(s) + \underbrace{\lambda \widetilde{U}_0(s) T \widetilde{U}_0(s)}_{\text{One interaction}} + \underbrace{\lambda^2 \widetilde{U}_0(s) T \widetilde{U}_0(s) T \widetilde{U}_0(s)}_{\text{Two interactions}} + \ldots$
- Start at $n_0$, reset to $\mathcal{N}$, obtain the probability at $m$
- Density operators before and after an interaction must satisfy
  $\rho_+(t) = T\rho_-(t) = |\mathcal{N}\rangle\langle\mathcal{N}|$; $\mathrm{Tr}[\rho_+(t)] = \mathrm{Tr}[\rho_-(t)] = 1$

$$\Rightarrow (n_1 n_1' | T | n_2 n_2') = \delta_{n_1 n_1'} \delta_{n_2 n_2'} \delta_{n_1 \mathcal{N}}$$

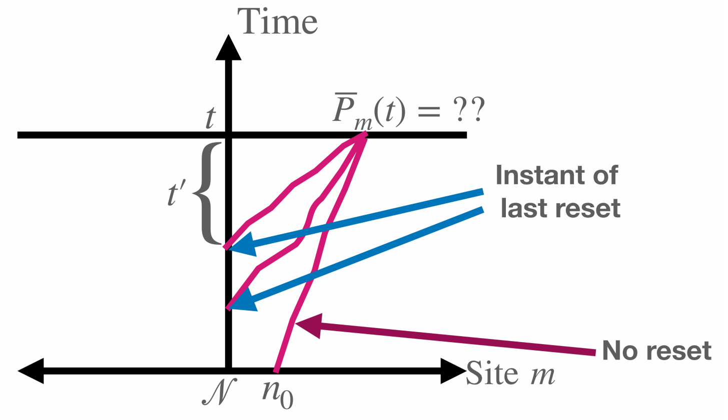
- $\overline{P}_m(t) = \langle m | \overline{\rho}(t) | m \rangle$

$$\widetilde{\overline{P}}_m(s) = \langle m | \widetilde{U}_0(s)\rho(0) + \lambda \widetilde{U}_0(s) T \widetilde{U}_0(s)\rho(0) + \lambda^2 \widetilde{U}_0(s) T \widetilde{U}_0(s) T \widetilde{U}_0(s)\rho(0) + \ldots | m \rangle$$
$$\equiv \sum_{p=0}^{\infty} \widetilde{\overline{P}}_m^{(p)}(s)$$

- $\overline{P}_m^{(p)}(t) = \lambda^p \, \mathrm{e}^{-\lambda t} \int_0^t \mathrm{d}t' \, \frac{(t-t')^{p-1}}{(p-1)!} \, J_{m-\mathcal{N}}^2(\Delta t')$; $\quad p \in [1, \infty)$
- $\overline{P}_m(t) = \mathrm{e}^{-\lambda t} J_{m-n_0}^2(\Delta t) + \lambda \int_0^t \mathrm{d}t' \, \mathrm{e}^{-\lambda t'} J_{m-\mathcal{N}}^2(\Delta t')$

# TBM + Stochastic Resets at Random Times: Analysis

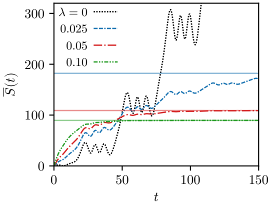
$$\overline{P}_m(t) = \mathrm{e}^{-\lambda t} J^2_{m-n_0}(\Delta t) + \lambda \int_0^t \mathrm{d}t' \; \mathrm{e}^{-\lambda t'} J^2_{m-\mathcal{N}}(\Delta t')$$




*(Evans, Majumdar (2011), Mukherjee, Sengupta, Majumdar (2018), Das, Dattagupta, Gupta (2022))*

# Conclusions

- **TBM + projective measurements at random times: Freezing** of the system in the initial state, akin to the Zeno effect
- **TBM + stochastic resets at random times: Localization**
- **TBM subject to external forcing field that is periodic in time + stochastic resets at random times:**
  $H(t) = -\frac{\Delta}{2}(K + K^\dagger) + F_0 \cos(\omega t) \sum_{n=-\infty}^{\infty} n|n\rangle\langle n|$: **Localization**




  Coherence-to-Decoherence cross-over: **Delocalization/dynamic localization** crosses over to **Localization in presence of stochastic resets**
- **Future: Unitary evolution + continuous monitoring** *(Lami, Santini, Collura (2023))*
  **Many-body interacting quantum systems**

**Thank you for your attention**

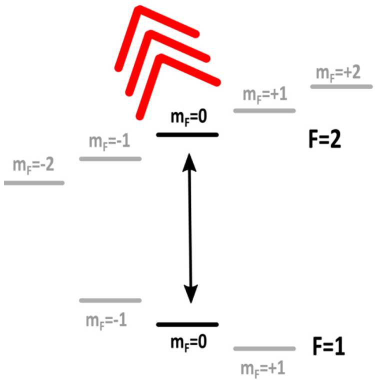
References:


- **DD** *and S. Gupta, J. Stat. Mech. 033212 (2022)*
- **DD**, *S. Dattagupta, and S. Gupta, J. Stat. Mech. 053101 (2022)*
- *S. Dattagupta,* **DD**, *and S. Gupta, J. Stat. Mech. 103210 (2022)*

## The tight-binding model (TBM) in absence of interaction: Analysis

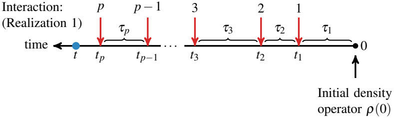
- A quantum particle mostly localized on the sites of a $1d$ lattice but because of spontaneous quantum fluctuations makes occasional tunnelling to nearest-neighbour sites
- $H = -\frac{\Delta}{2}\sum_{n=-\infty}^{\infty}\Big(|n\rangle\langle n+1| + |n+1\rangle\langle n|\Big)$
- $|n\rangle$: Wannier state of the particle when on site $n$;
  $\langle m|n\rangle = \delta_{mn}$, $\sum_{m=-\infty}^{\infty}|m\rangle\langle m| = \mathbb{I}$
- $H = -\frac{\Delta}{2}(K + K^\dagger)$; $K \equiv \sum_{n=-\infty}^{\infty}|n\rangle\langle n+1|$, $[K, K^\dagger] = \mathbb{I}$
- Bloch state $|k\rangle \equiv \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}nk}|n\rangle$; $K|k\rangle = \mathrm{e}^{-\mathrm{i}k}$
- $\mathrm{e}^{\mathrm{i}Ht}|m'\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\mathrm{d}k\ \mathrm{e}^{\mathrm{i}km'}\mathrm{e}^{\mathrm{i}Ht}|k\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\mathrm{d}k\ \mathrm{e}^{\mathrm{i}km'}\mathrm{e}^{-\mathrm{i}\Delta t\cos k}|k\rangle$
- **Probability $P_m(t)$ to be on site $m$ at time $t > 0$, given that the particle was on site $n_0$ at time $t = 0$??**
- $P_m(t) = \langle m|\rho(t)|m\rangle$; $\rho(t) = \mathrm{e}^{-\mathrm{i}Ht}\rho(0)\mathrm{e}^{\mathrm{i}Ht} = \mathrm{e}^{-\mathrm{i}Ht}|n_0\rangle\langle n_0|\mathrm{e}^{\mathrm{i}Ht}$
- $P_m(t) = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\mathrm{d}k\int_{-\pi}^{\pi}\mathrm{d}k'\ \mathrm{e}^{\mathrm{i}(m-n_0)(k-k')}\ \mathrm{e}^{-\mathrm{i}\Delta(\cos k'-\cos k)t} = J^2_{m-n_0}(\Delta t)$

## The TBM in presence of Projective Measurements at Random Times




Ground state of $^{87}$Rb in presence of a magnetic field: two hyperfine levels ($F = 1$ and $F = 2$). A laser induced Raman transition couples the sub-levels $|F = 1, m_F = 0\rangle$ and $|F = 2, m_F = 0\rangle$, while a laser resonant with the transition $|F = 2\rangle \rightarrow |F' = 3\rangle$ (red arrows) depletes the population of the former *(Gherardini, Lovecchio, Müller, Lombardi, Caruso and Cataliotti (2017))*

# Averaged density operator for the case of exponential $p(\tau)$: Generalization




$$\overline{\rho}(t) = \sum_{p=0}^{\infty} \int_0^t \mathrm{d}t_p \int_0^{t_p} \mathrm{d}t_{p-1} \ldots \int_0^{t_3} \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_1$$

$$\times F(t-t_p)\mathrm{e}^{-\mathrm{i}\mathcal{L}(t-t_p)} T p(t_p - t_{p-1})\mathrm{e}^{-\mathrm{i}\mathcal{L}(t_p - t_{p-1})} T .. p(t_2 - t_1)\mathrm{e}^{-\mathrm{i}\mathcal{L}(t_2-t_1)} T p(t_1) e^{-\mathrm{i}\mathcal{L}t_1}\rho(0)$$

$$= U(t)\rho(0)$$

- Formalism very general:
  1. Works for any $H$ and any $T$
  2. Works even for time-dependent Hamiltonian:

$$\overline{\rho}(t) = \sum_{p=0}^{\infty} \int_0^t \mathrm{d}t_p \int_0^{t_p} \mathrm{d}t_{p-1} \ldots \int_0^{t_3} \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_1$$

$$\times F(t-t_p)\mathrm{e}_+^{-\mathrm{i}\int_{t_p}^{t} \mathrm{d}t' \ \mathcal{L}(t')} T p(t_p - t_{p-1})\mathrm{e}_+^{-\mathrm{i}\int_{t_{p-1}}^{t_p} \mathrm{d}t' \mathcal{L}(t')} T .. p(t_2 - t_1)\mathrm{e}_+^{-\mathrm{i}\int_{t_1}^{t_2} \mathrm{d}t' \ \mathcal{L}(t')} \mathcal{T}$$

$$\times p(t_1)\mathrm{e}_+^{-\mathrm{i}\int_0^{t_1} \mathrm{d}t' \ \mathcal{L}(t')}\rho(0)$$

- $$\widetilde{U}(s) = \widetilde{U}_0(s) + \underbrace{\lambda \widetilde{U}_0(s) T \widetilde{U}_0(s)}_{\text{One interaction}} + \underbrace{\lambda^2 \widetilde{U}_0(s) T \widetilde{U}_0(s) T \widetilde{U}_0(s)}_{\text{Two interactions}} + \ldots;$$

$$\widetilde{U}_0(s) \equiv \int_0^{\infty} \mathrm{d}t \ \mathrm{e}_+^{-(s+\lambda)t - \mathrm{i}\int_0^t \mathrm{d}t' \ \mathcal{L}(t')}$$