Quantifying Three-Manifolds (Physicists' and Mathematicians' Perspectives)

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Women at the Intersection of Mathematics &Physics
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Based on work with Swatee Naik (arXiv:9901061)
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Quantum Field Theory

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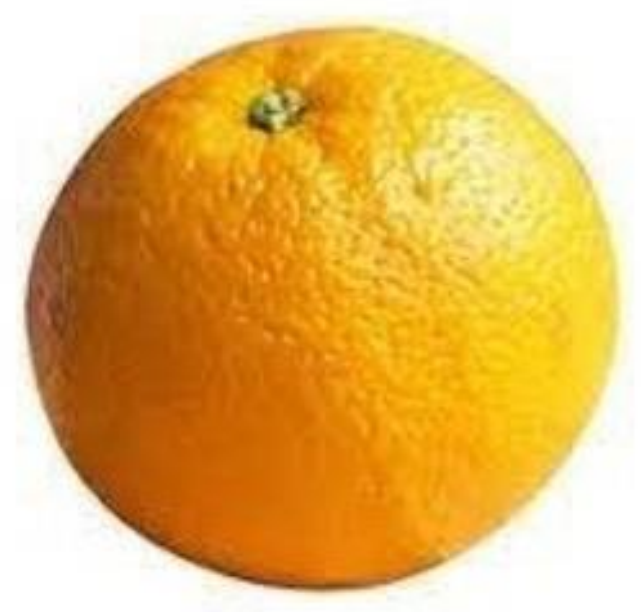
Topology in low dimension

Quantum Topology

Meeting ground for mathematicians and physicists

Challenging open problems- Machine learning/neural network approach

Two-dimensional topological spaces



$S^2 \not\equiv T^2$

Orange surface is S^2 (two-sphere) **Doughnut surface is T^2 (torus)**

Euler characteristic $\chi = V - E + F = 2 - 2g$

Such a mathematical quantity for 3-dimensional manifolds?

$$S^3, S^2 \times S^1, S^1 \times S^1 \times S^1, S^3/\Gamma$$

Fundamental Theorem of Lickorish-Wallace

- Any connected, closed , orientable **3-manifold** can be obtained by **surgery** on a **framed knot/link** in S^3

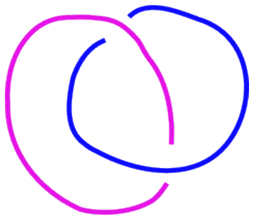
- Can we quantify this theorem?

Definition of a knot/Link

- **Knot** is an embedding of a curve inside 3-manifold:

Trefoil

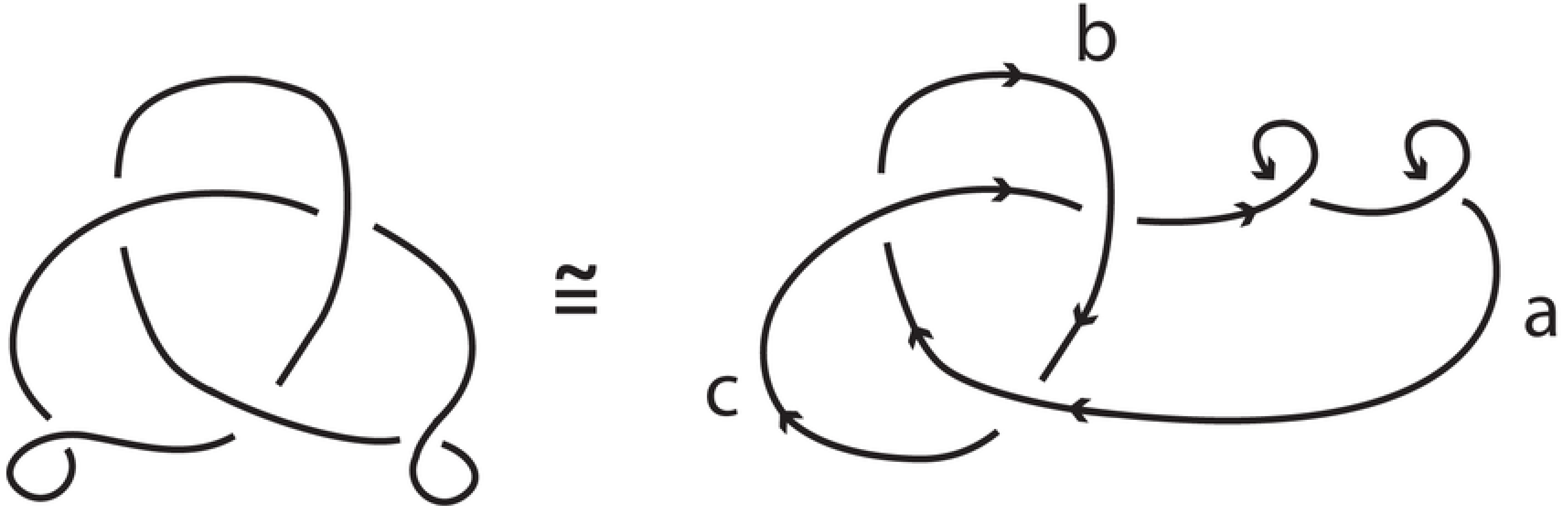
- **Link** is a collection of knots entangled



2-component Hopf-Link

What is framed knot/link?

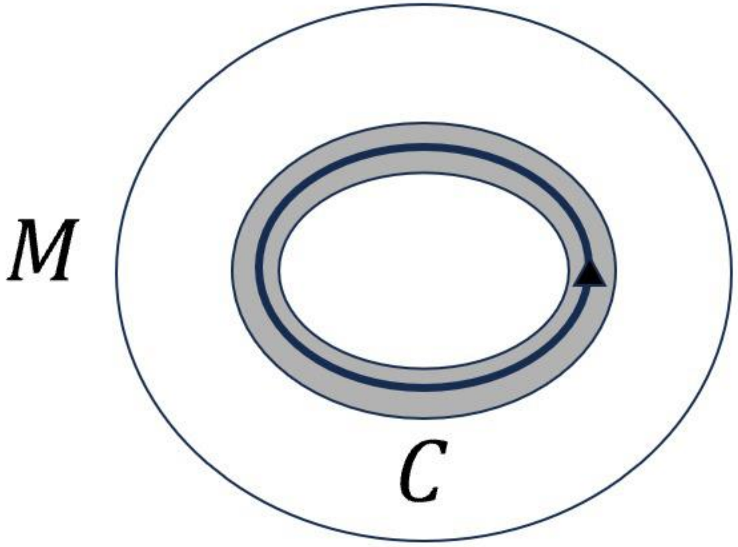
- **Framed Knot/Link** projected on a black-board frame

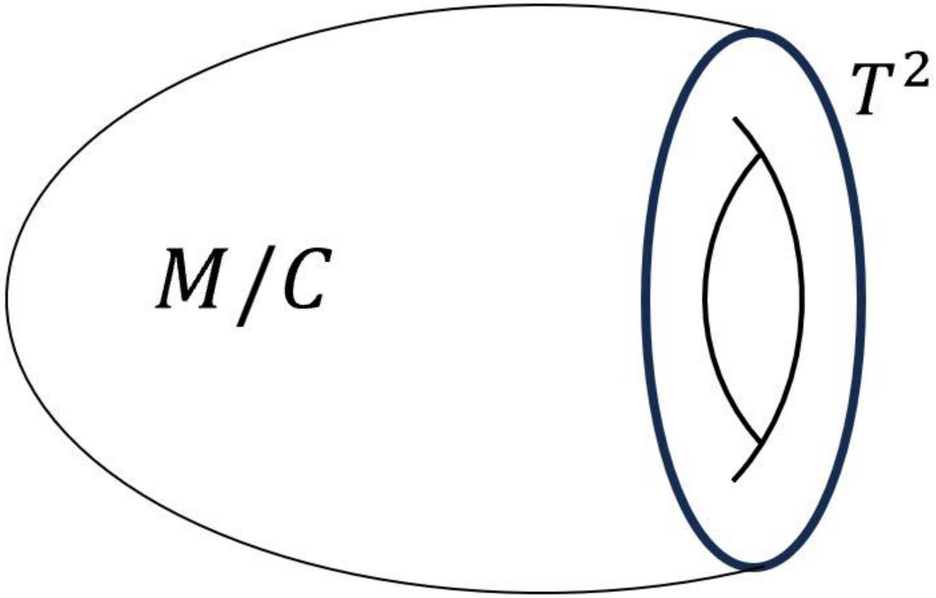


- The above knot is **framed trefoil** with framing +1 : [T,f=1]
- Similar framing **f** on every component of a link L: [L,**f**]

Surgery of framed knot/link in M

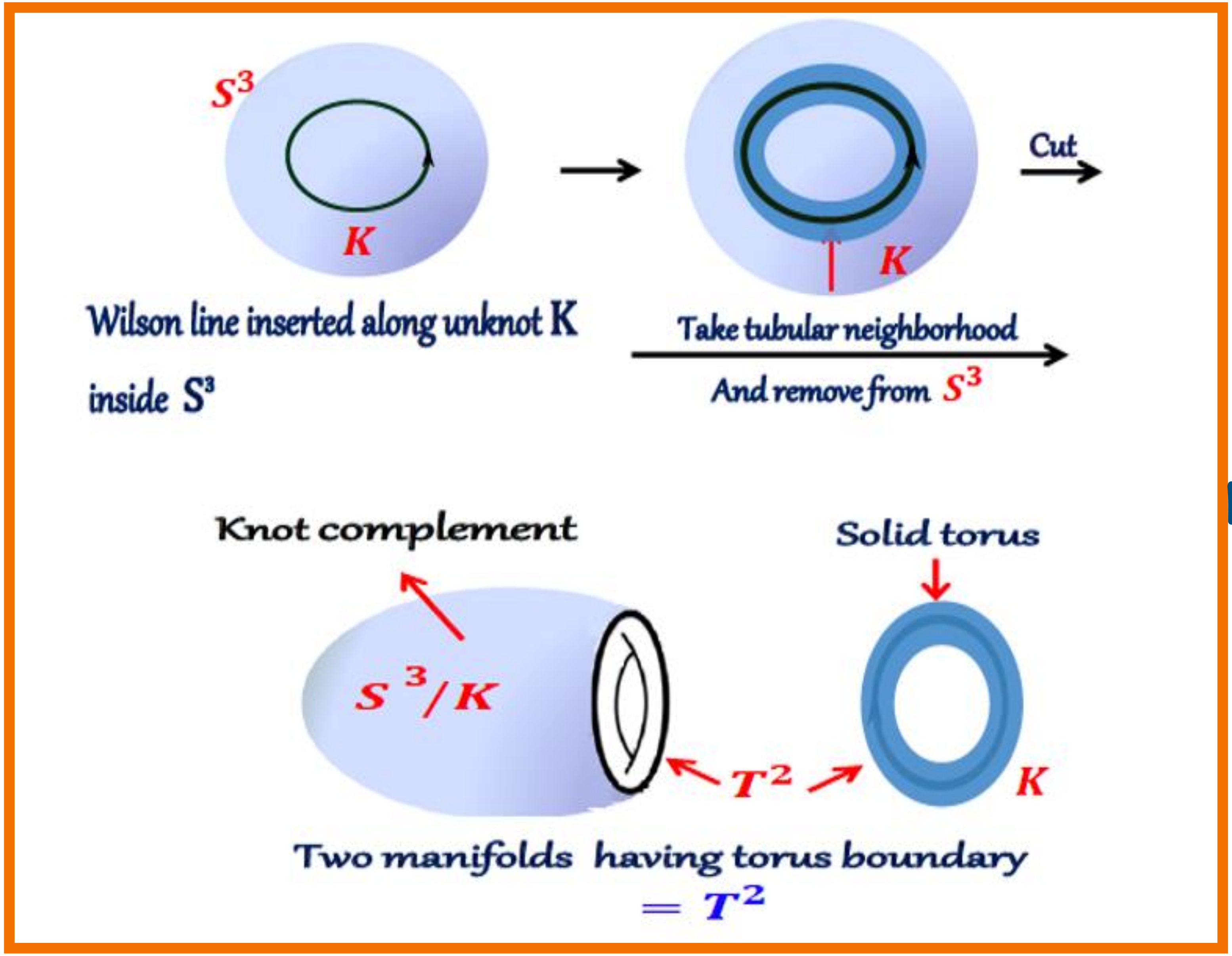
- Tubular neighbourhood of a knot C in a 3-manifold





- Remove this tubular neighbourhood- we call this knot complement M/C

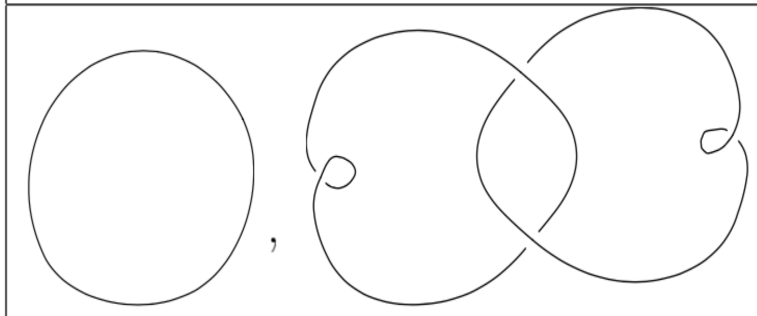
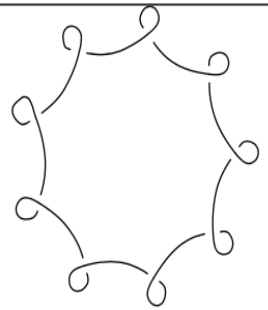
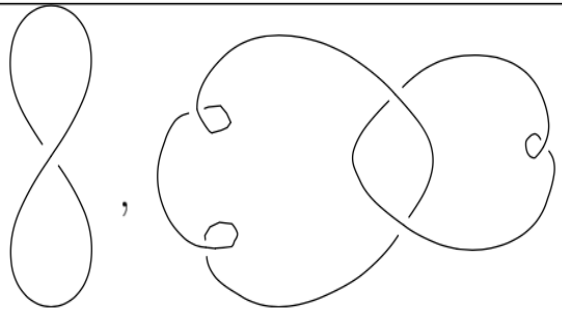
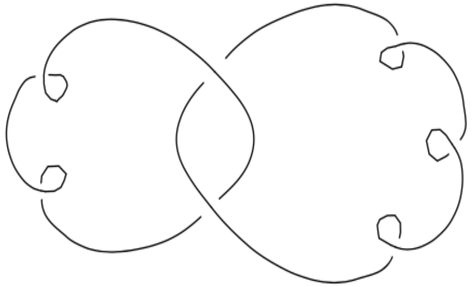
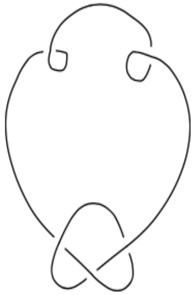
Three-Manifolds – from Surgery of knots in S^3



$S : \partial(S^3/K) \rightarrow T^2$

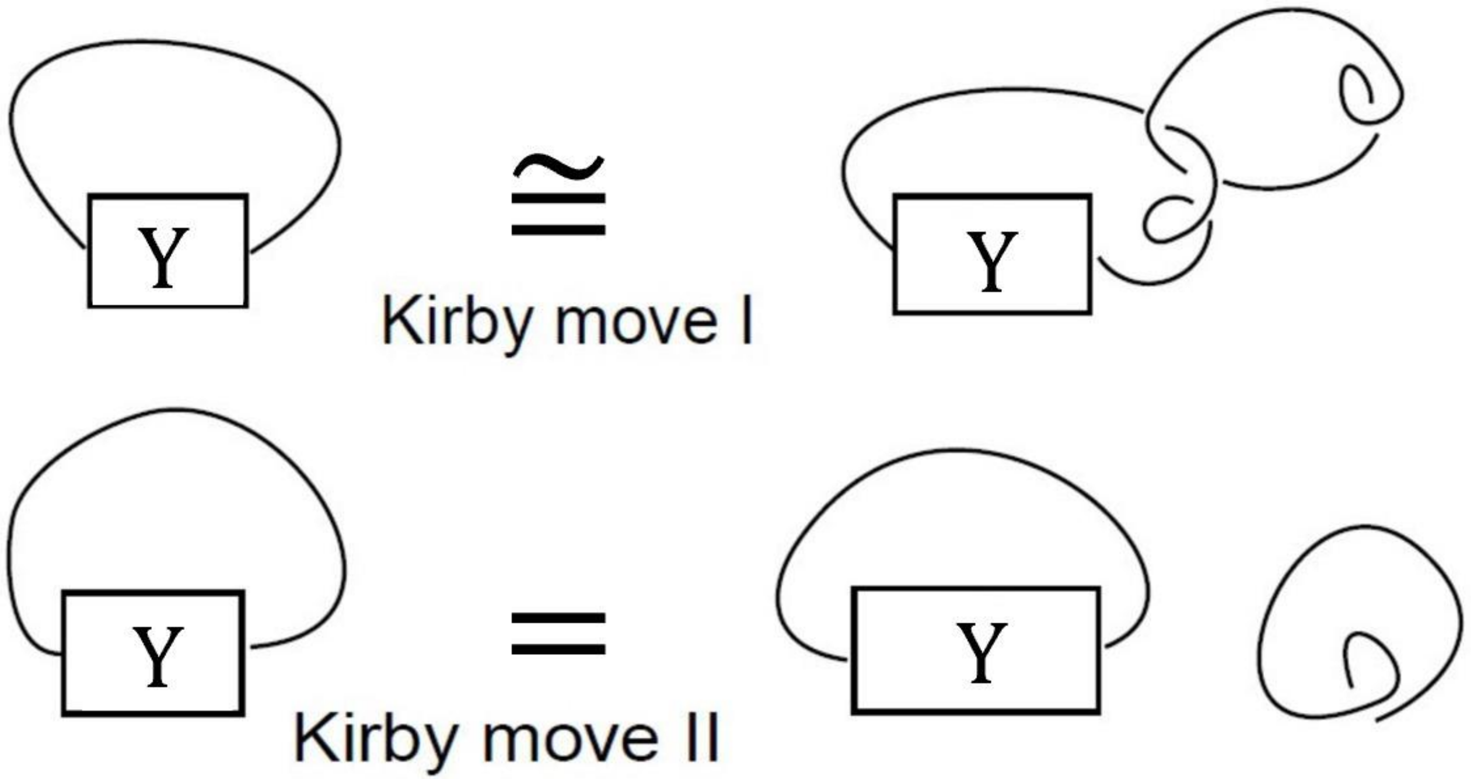
p/r surgery : $S^3_{\frac{p}{r}}(K)$

Some examples of 3-manifolds

Framed Link Diagram	3-manifold	Framed Link Diagram	3-manifold
	$S^2 \times S^1$		$L(9,1)$
	S^3		$L(5,1)$
	$\mathbb{RP}^3$		$\mathbb{P}^3$

framed link corresponding to M is not unique

3-manifolds are same for framed links related by Kirby moves



Strategy to quantify 3-manifolds

- An algebraic expression involving
  (i) Invariants of the framed links
  (ii) Homeomorphism map
- Such that the expression is unchanged under Kirby moves
- Then the expression qualifies to be a **three-manifold invariant**

Invariants of framed links

- Bracket polynomials $\langle D_L \rangle$ through a recursive relation

$$\langle \times \rangle = A \langle)(\rangle + A^{-1} \langle \asymp \rangle$$

- Proportional to the Jones polynomial

$$\langle D_L \rangle = q^{\left(\frac{3\omega}{4}\right)} J[L,q] |_{A=-q^{1/4}}$$

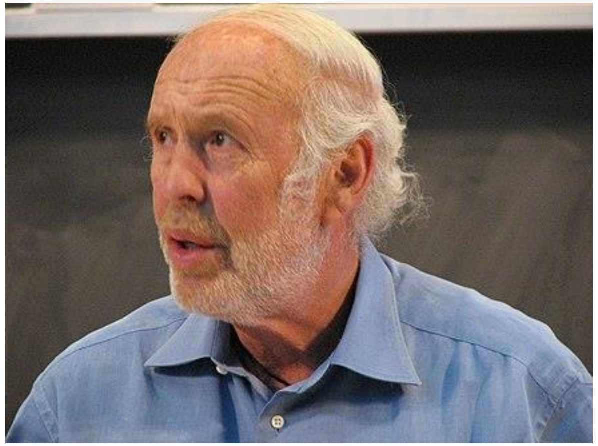
Invariants of framed links(contd)

- Invariants from Chern-Simons field theory
- Includes Jones polynomials and the new pool of generalized invariants
- Brief overview of Chern-Simons theory

Chern-Simons Theory

(Historical remarks)

- This theory was discovered first by Albert Schwarz
- Named after the two mathematicians **Shiing-Shen Chern** and **James Harris Simons** introduced the 3-form term.



Founder of Simons Foundation

April 25, 1938- May 10, 2024

Chern-Simons Theory- Schwarz type theory

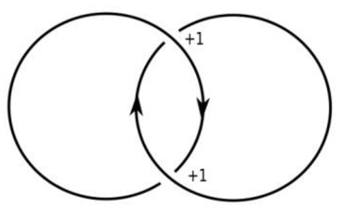
Edward Witten

Three-dimensional gauge theory whose partition function

$$Z[M^3] = \int \mathcal{D}A \exp\left[i \int_{M^3} S(A)\right]$$

Metric independent

Trefoil knot



Hopf Link

The expectation value of Wilson loop operators

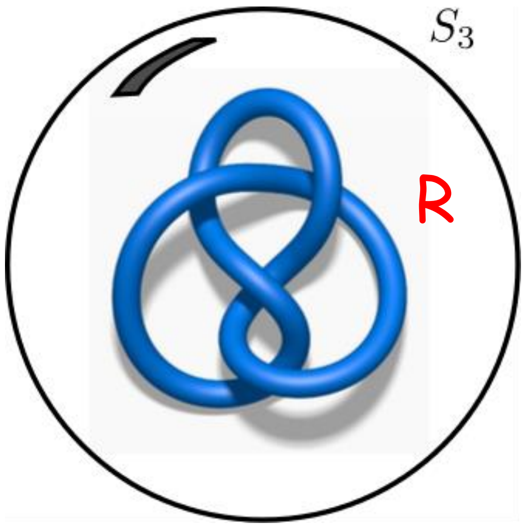
$Z[M^3]$: Topological invariant of the three-manifold M^3

Knot and Link invariants

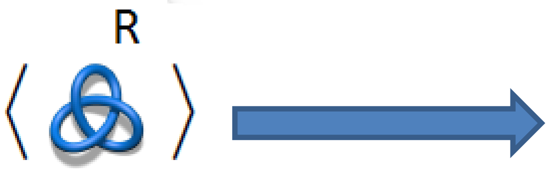
$\propto$ Witten-Reshitikhin-Turaev (WRT) invariant

Wilson loop operator

$$W_{R}[K] = \mathrm{Tr}_{\mathrm{R}} exp \oint_{K} dx^{u} A_{\mu}^{a} T_{R}^{a}$$



T_R^a=Generators for representation R of SU(N)

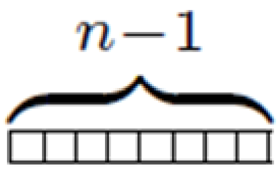


Knot invariant

	R= □ (fundamental)	Higher rank representation
SU(2)	Jones Polynomial	Colored Jones
SU(N)	HOMFLY-PT Polynomial	Colored HOMFLY-PT

J ( , q) = q + q³ − q⁴

$$J_n(K,q)$$

$n \equiv$ 

<u>Variables:</u> $q = e^{\frac{2\pi i}{k+N}}, \quad a = q^N$

Colored framed link invariants

$$P_{n_1,n_2,\dots}[L,f] = q^{3\omega/4}\, J_{n1,n2\dots}[L,f]$$

Colored framed link invariants

$P_{n_1,n_2}\,\dots\,[L,f] = q^{3\omega/4}\, J_{n1,n2\dots}\,[L,f]$

Bracket polynomial

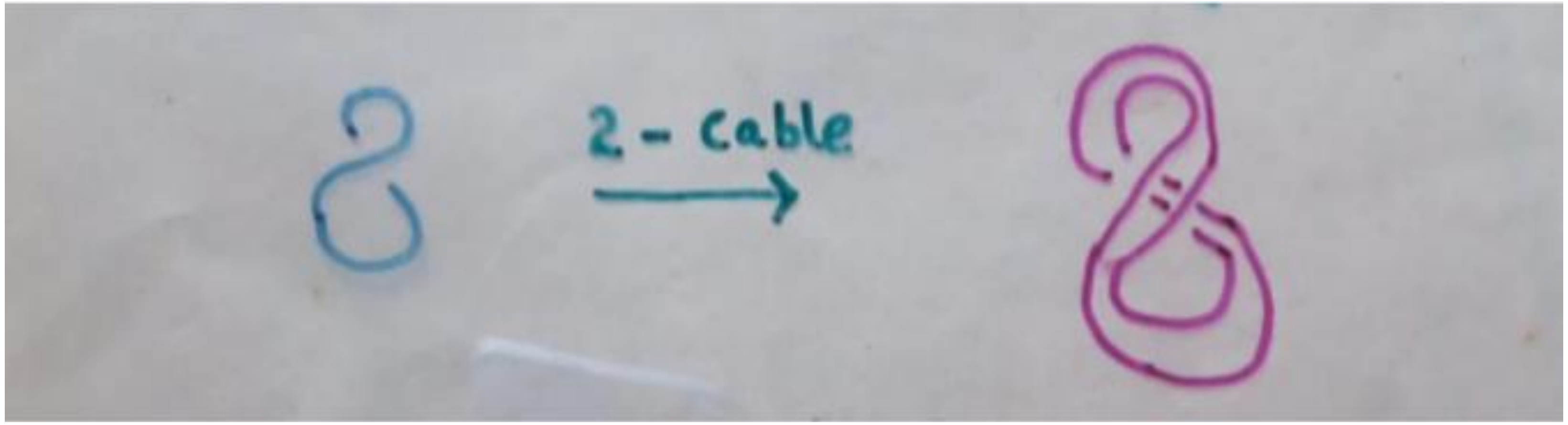
$\langle D_L \rangle |\, A = -q^{\frac{1}{4}} = P_{2,2,,\dots}\,[L,f]$

Two different looking 3-manifold invariants

- I) Lickorish invariant $F_\ell[M]$ involve

(i) bracket polynomials of the framed links

(ii) forces the introduction of **c-cables** of framed knot/link $\langle\, c * D_L \rangle$



Two different looking 3-manifold invariants(contd)

- I) Lickorish invariant

$$F_{\ell}[M] \propto \sum_c \lambda_c \langle \boldsymbol{c} * D_K \rangle$$

where A is $4rth$ root of unity &

$c = [1, r-2]$

Two different looking 3-manifold invariants (contd)

- II) Kaul invariant $\mathrm{F}_k[M]$ involves

(i)colored framed link invariants in SU(2) Chern-Simons theory:

- $\boldsymbol{F_k[M] \propto \sum_n \mu_n \; P_n[K, f]}$
- **Here $\boldsymbol{n \leq k+1}$**

Two different looking 3-manifold invariants(contd)

- We validated that

  Cabling =tensor product of spins $\frac{1}{2}$

- So $\langle c * DL \rangle$ **can be converted as a linear combination of colored framed link invariants**
- **Indeed we verified** in the paper **[with Swatee Naik (arXiv:9901061)]**
- $$\mathrm{F}_k[M] \propto F_\ell[M]$$

Thank You