Magnetic Monopoles

an introduction

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Full Field Theory + General Relativity
Evolution of Cosmic String Loop
by Josu Aurrekoetxea

Reviews:

- Solitons and Particles, C. Rebbi & G. Soliani, 1985.
- Magnetic Monopole: 50 Years Later, S. Coleman, 1985.
- Vortices and Monopoles, J. Preskill, Les Houches Lectures, 1986.
- Cosmic Strings and Other Topological Defects, A. Vilenkin & E.P.S. Shellard, 2000.
- Topological Solitons, N. Manton & P. Sutcliffe, 2004.
- Magnetic Monopoles, Y. Shnir, 2005.

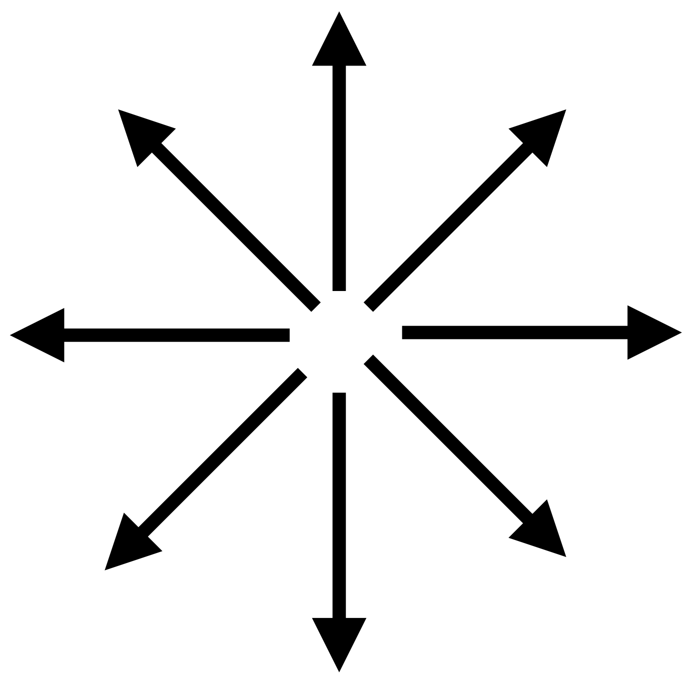
(Please see Reviews for further references.)

Global monopoles

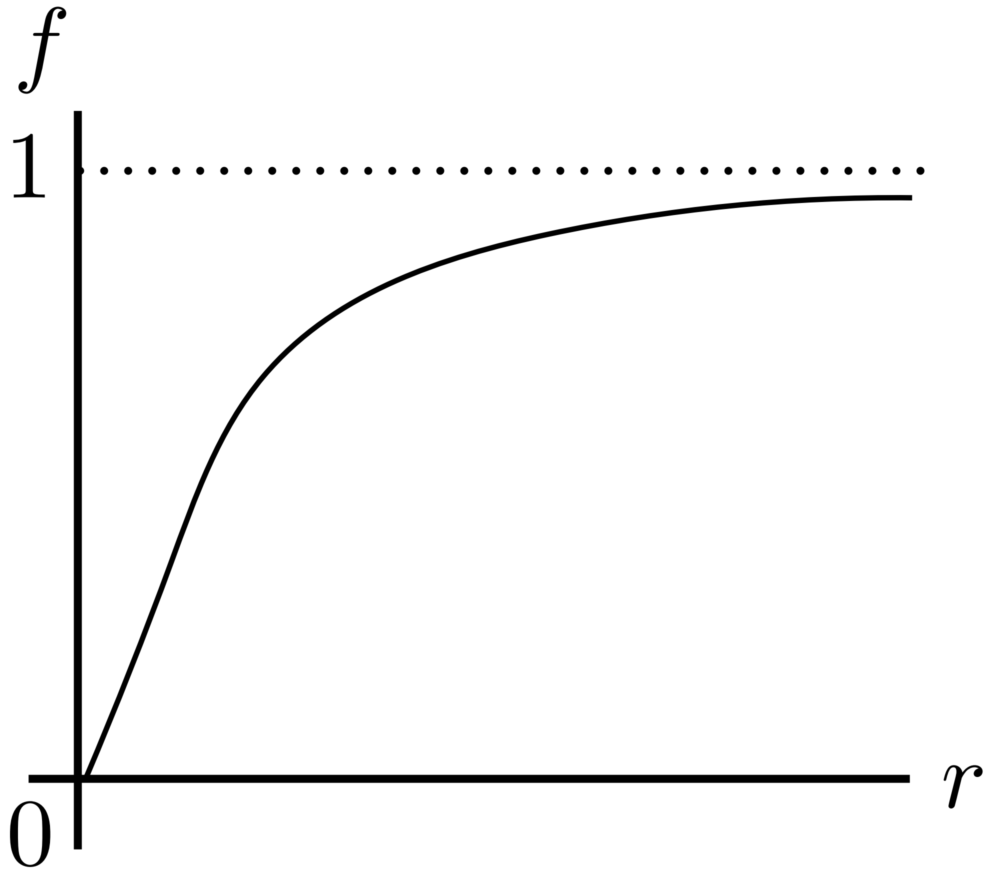
$$\vec{\phi} = (\phi_1, \phi_2, \phi_3)$$

$$L = \frac{1}{2}(\partial_\mu \vec{\phi})^2 - \frac{\lambda}{4}\left(\vec{\phi}^{\,2} - \eta^2\right)^2$$

"hedgehog"



$$\vec{\phi} = \eta f(r)\hat{r} \qquad f(0) = 0, \quad f(\infty) = 1$$



Exercise: derive and (numerically) solve the differential equation for f.

Topology

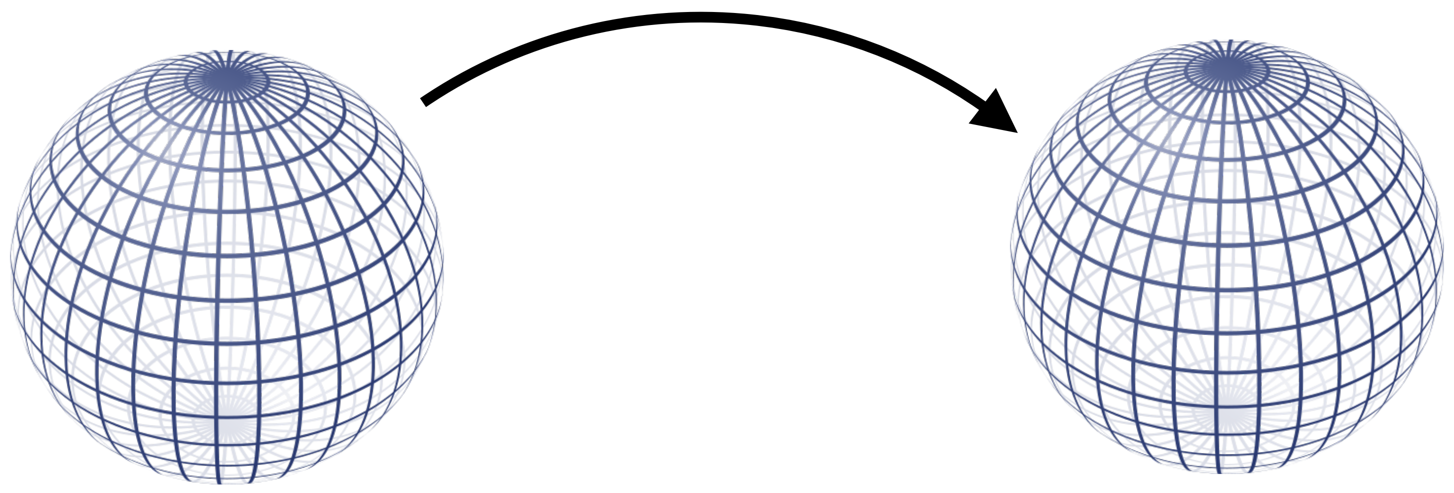
$$\vec{\phi} = (\phi_1, \phi_2, \phi_3) \qquad\qquad L = \frac{1}{2}(\partial_\mu \vec{\phi})^2 - \frac{\lambda}{4}\left(\vec{\phi}^2 - \eta^2\right)^2$$

Vacuum manifold: $S^2 = \{|\vec{\phi}| = \eta\}$

O(3) global symmetry

Space at infinity: $S^2 = (r = \infty, \theta, \phi)$

$$\vec{\phi}(\infty, \theta, \phi) : \; S^2_{\text{space}} \to S^2_{\text{vac mnfld}}$$



e.g. $\vec{\phi}(\infty, \theta, \phi) = \eta(0, 0, 1)$ winding=0

$$\vec{\phi}(\infty, \theta, \phi) = \eta(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

winding=1

Second homotopy group: $\pi_2(S^2_{\text{vac mnfld}}) = \mathcal{Z}$

Topology and solutions

Non-trivial topology implies singularities (regions where the order parameter is not in its vacuum manifold) and hence non-vanishing energy.

Non-trivial topology does not imply a static solution of the equations of motion.

Even if a static solution exists, topology does not guarantee its stability.

With gauge fields

$$L = \frac{1}{2}(D_\mu \vec{\phi})^2 - \frac{1}{4}W^a_{\mu\nu}W^{\mu\nu a} - \frac{\lambda}{4}\left(\vec{\phi}^{\,2} - \eta^2\right)^2$$

O(3) gauge symmetry

$$D_\mu \vec{\phi} = \partial_\mu \vec{\phi} - g\vec{W}_\mu \times \vec{\phi}$$

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\epsilon^{abc}W^b_\mu W^c_\nu$$

but topology discussion only involved the vacuum manifold and gauge fields played no role*.

Therefore there should be a gauged monopole configuration/solution.

**semilocal defects are an exception*.

BPS solution

$$L = \frac{1}{2}(D_\mu \vec{\phi})^2 - \frac{1}{4} W^a_{\mu\nu} W^{\mu\nu a} - \frac{\lambda}{4}\left(\vec{\phi}^{\,2} - \eta^2\right)^2$$

Let λ=0 but still require $|\vec{\phi}(r = \infty)| = \eta$

Write energy as sum of squares, similar to kink case.
Solve first order differential equations.

Exercise: write the energy as a sum of squares similar to the kink case.

$$\phi^a = P(r)\hat{x}^a \qquad P_{\mathrm{BPS}}(r) = \frac{1}{\tanh(r)} - \frac{1}{r}$$

$$W^a_i = \frac{(1 - K(r))}{r}\epsilon^{aij}\hat{x}^j \qquad K_{\mathrm{BPS}}(r) = \frac{r}{\sinh(r)}$$

Symmetry breaking pattern

To understand the properties of the solution, first discuss symmetries and symmetry breaking pattern.

Global symmetry: $\vec{\phi} \to \mathbf{R}(\hat{n}, \alpha)\vec{\phi} \qquad \mathbf{R} \in O(3)$

but some rotations don't change the order parameter:

$$\vec{\phi} \to \mathbf{R}(\hat{\phi}, \alpha)\vec{\phi} = \vec{\phi}$$

This O(2) sub-group of the full O(3) is unbroken: $O(3) \to O(2)$

Example: $\vec{\phi} = \eta(0, 0, 1) \quad \mathbf{R} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{R}\hat{\phi} = \hat{\phi}$

Spectrum of gauge particles

$$L = \frac{1}{2}(D_\mu \vec{\phi})^2 - \frac{1}{4} W^a_{\mu\nu} W^{\mu\nu a} - \frac{\lambda}{4}\left(\vec{\phi}^{\,2} - \eta^2\right)^2$$

$$(D_\mu \vec{\phi})^2 = (\partial_\mu \vec{\phi} - g\vec{W}_\mu \times \vec{\phi})^2 = ... + g^2(\vec{W}_\mu \times \vec{\phi})^2 = ... + g^2\eta^2 \left[(\vec{W}_\mu)^2 - (\hat{\phi}\cdot\vec{W}_\mu)^2\right]$$

The gauge fields get a mass except for the component in the unbroken symmetry direction. The massless gauge field is identified with the "electromagnetic" gauge field.

$$A_\mu \equiv \hat{\phi}\cdot\vec{W}_\mu$$

Example: $\vec{\phi} = \eta(0,0,1)$ $\qquad A_\mu \equiv W^3_\mu$

Electromagnetic field strength definition

$$
\begin{aligned}
A_{\mu\nu} &\stackrel{?}{=} \partial_\mu A_\nu - \partial_\nu A_\mu \\
&= \partial_\mu(\hat{\phi}\cdot\vec{W}_\nu) - \partial_\nu(\hat{\phi}\cdot\vec{W}_\mu) \\
&\to \partial_\mu(\hat{\phi}\cdot\vec{W}_\nu) - \partial_\nu(\hat{\phi}\cdot\vec{W}_\mu) + \partial_\mu(\hat{\phi}\cdot\partial_\nu\vec{\Lambda}) - \partial_\nu(\hat{\phi}\cdot\partial_\mu\vec{\Lambda}) \\
&\neq \partial_\mu A_\nu - \partial_\nu A_\mu
\end{aligned}
$$

Not an O(3) gauge invariant definition.

$$
\begin{aligned}
A_{\mu\nu} &\stackrel{?}{=} \hat{\phi}\cdot\vec{W}_{\mu\nu} \\
&= \hat{\phi}\cdot(\partial_\mu\vec{W}_\nu - \partial_\nu\vec{W}_\mu + g\vec{W}_\mu\times\vec{W}_\nu) \\
&\neq \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{even if } \hat{\phi} \text{ is constant}
\end{aligned}
$$

Electromagnetic field strength

$$A_{\mu\nu} = \hat{\phi} \cdot \vec{W}_{\mu\nu} + ...D_\mu \vec{\phi} \times D_\nu \vec{\phi} \;\; \text{gauge invariant}$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu + ...\partial_\mu \vec{\phi} \times \partial_\nu \vec{\phi} \;\; \text{reduces to Maxwell if } \hat{\phi} \text{ is constant}$$

Exercise: find the coefficient of the last term.

Why "magnetic" monopole?

$$A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ...\partial_\mu \vec{\phi} \times \partial_\nu \vec{\phi}$$

$$\phi^a = P(r)\hat{x}^a \qquad\qquad W_i^a = \frac{(1 - K(r))}{r}\epsilon^{aij}\hat{x}^j$$

$$A_i = \hat{\phi} \cdot \vec{W}_i \propto \epsilon^{aij}\hat{x}^a\hat{x}^j = 0$$

$$\phi^a = P(r)(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$\vec{B} = ...\hat{e}_\theta \times \hat{e}_\phi = \frac{1}{gr^2}\hat{r}$$

Grand Unification

Vacuum manifold can be written in terms of symmetry groups.

$$G \to H$$

Vacuum manifold is isomorphic to G/H.

Theorem: $\pi_2(G/H) = \pi_1(H) \;\text{ if }\; \pi_2(G) = 1 = \pi_1(G)$

$$H = [SU(3) \times U(1)]/Z_3$$

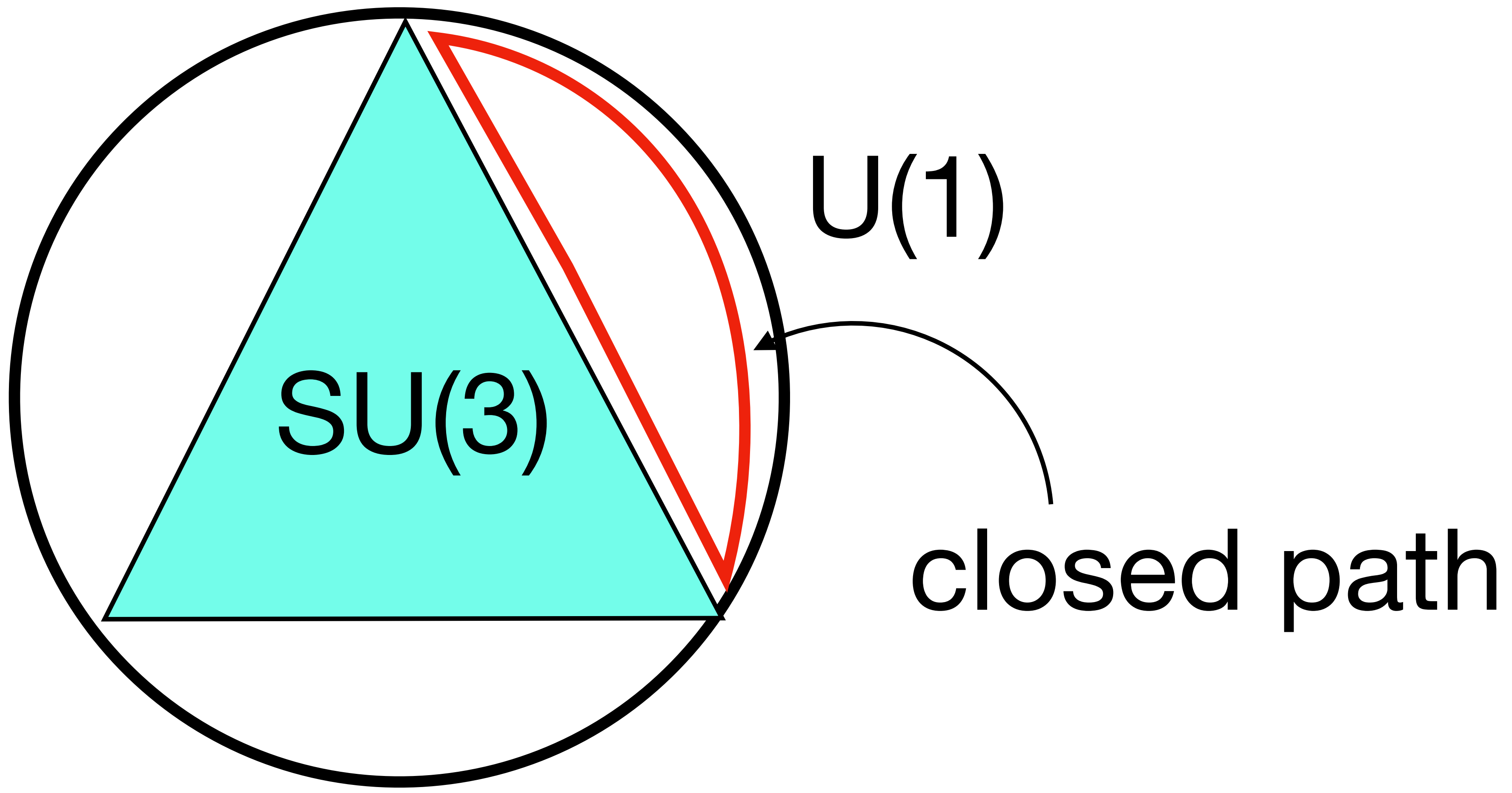
$$\pi_1(H) = \mathcal{Z}$$

Magnetic monopoles exist in all Grand Unified models!

Relevance of discrete factors

$$H = [SU(3) \times U(1)]/Z_3 \qquad \pi_1(H) = \mathcal{Z}$$

Center of SU(3) is also contained in U(1).



Fundamental monopole is charged under both SU(3) and U(1).

Cosmology

Magnetic monopoles formed due to random distribution of order parameter.

Magnetic monopoles are heavy and non-relativistic and redshift like matter.

Magnetic monopoles overwhelm the cosmic energy density.

$$\Omega_{\rm m} = \frac{\rho_{\rm m}}{\rho_\gamma} = \frac{\rho_{{\rm m},i}}{\rho_{\gamma,i}}\frac{T_{\rm GUT}}{T_0} \sim 10^{29}\Omega_{\rm m,i}$$

Several constraints — over-closure, Parker bound.

Inflation, if it occurs late enough, can dilute the monopole energy density.

Parker bound

Magnetic monopoles dissipate energy in magnetic fields.

Survival of galactic/cosmological magnetic fields places upper bounds on flux of magnetic monopoles.

$$\text{energy gain rate by monopole} = q_m B v$$

Dissipation time scale should be longer than Hubble time scale.

$$t_0 < \tau \sim \frac{B^2}{n_m q_m B v} \sim \frac{B}{q_m n_m v} \qquad B_{\rm gal} \sim 10^{-6}\,\mathrm{G} \sim 10^{-26}\,\mathrm{GeV}^2$$

$$\mathcal{F} \equiv n_m v < \frac{1}{(10^4\,\mathrm{km})^3} \sim \frac{1}{\text{Earth volume}} \qquad q_m \sim \frac{2\pi}{e} \sim 10$$

Electroweak magnetic monopoles

Standard electroweak model: $[SU(2)_L \times U(1)_Y]/Z_2 \to U(1)_{\mathrm{em}}$

Order parameter:

$$\Phi = \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}$$

Higgs field

Vacuum manifold:

$$\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 = \eta^2$$

Hopf parametrization:

$$\Phi = \eta \begin{pmatrix} \cos\alpha \ e^{i\beta} \\ \sin\alpha \ e^{i\gamma} \end{pmatrix}$$

“angular coordinates on a three-sphere”

Magnetic field definition

TV, 1991

$$A_{\mu\nu} = \sin\theta_w \hat{n}^a W^a_{\mu\nu} + \cos\theta_w Y_{\mu\nu} - i\frac{2\sin\theta_w}{g\eta^2}(D_\mu\Phi^\dagger D_\nu\Phi - D_\nu\Phi^\dagger D_\mu\Phi)$$

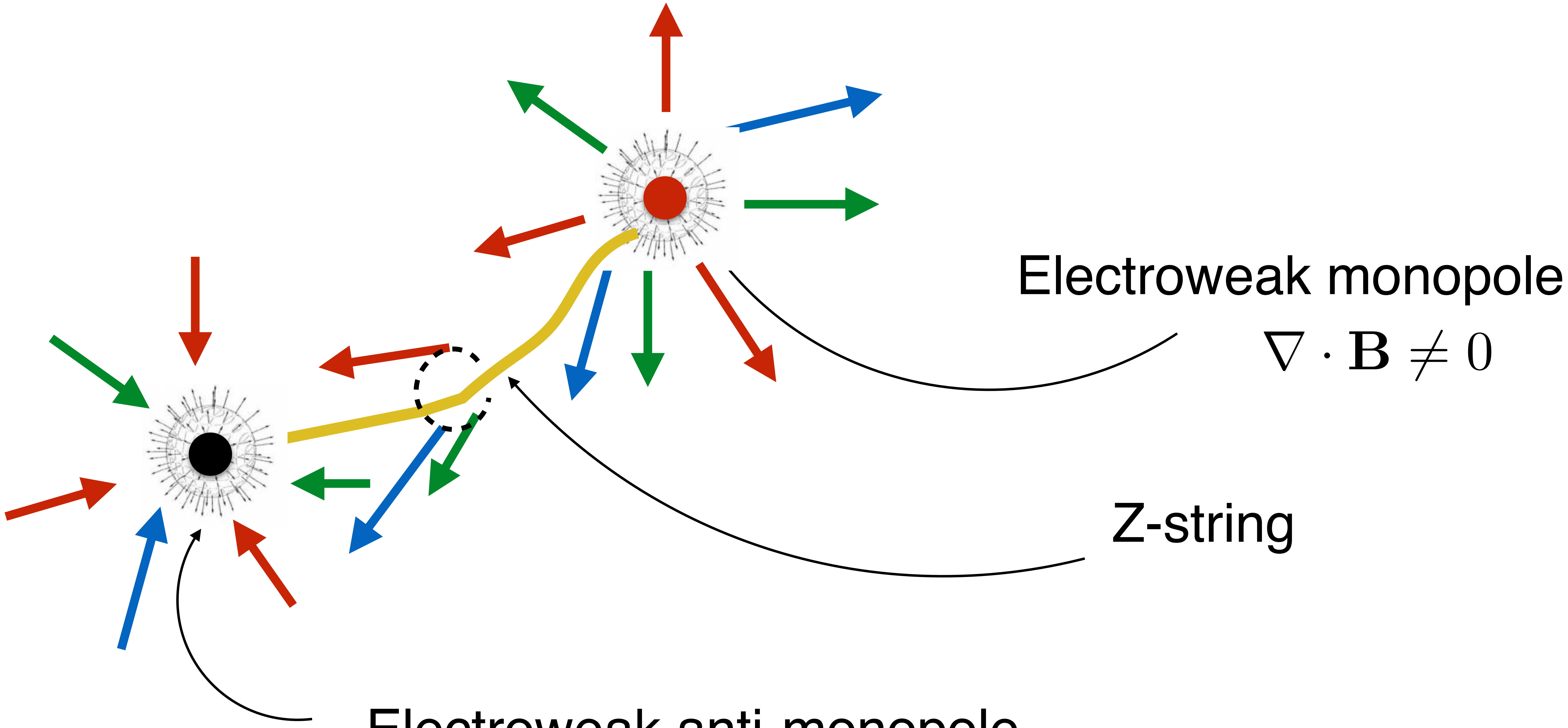
$$= \partial_\mu A_\nu - \partial_\nu A_\mu - i\frac{2\sin\theta_w}{g\eta^2}(\partial_\mu\Phi^\dagger\partial_\nu\Phi - \partial_\nu\Phi^\dagger\partial_\mu\Phi) \qquad (|\Phi| = \eta)$$

$$\mathbf{B} = \nabla\times\mathbf{A} - i\frac{2\sin\theta_w}{g\eta^2}\nabla\Phi^\dagger\times\nabla\Phi$$

Example: $\Phi = \eta\begin{pmatrix}\cos(\theta/2)\\ \sin(\theta/2)e^{i\phi}\end{pmatrix} \longrightarrow \mathbf{B}\sim\frac{\hat{r}}{r^2}$ (magnetic monopole)

Electroweak Dumbbells

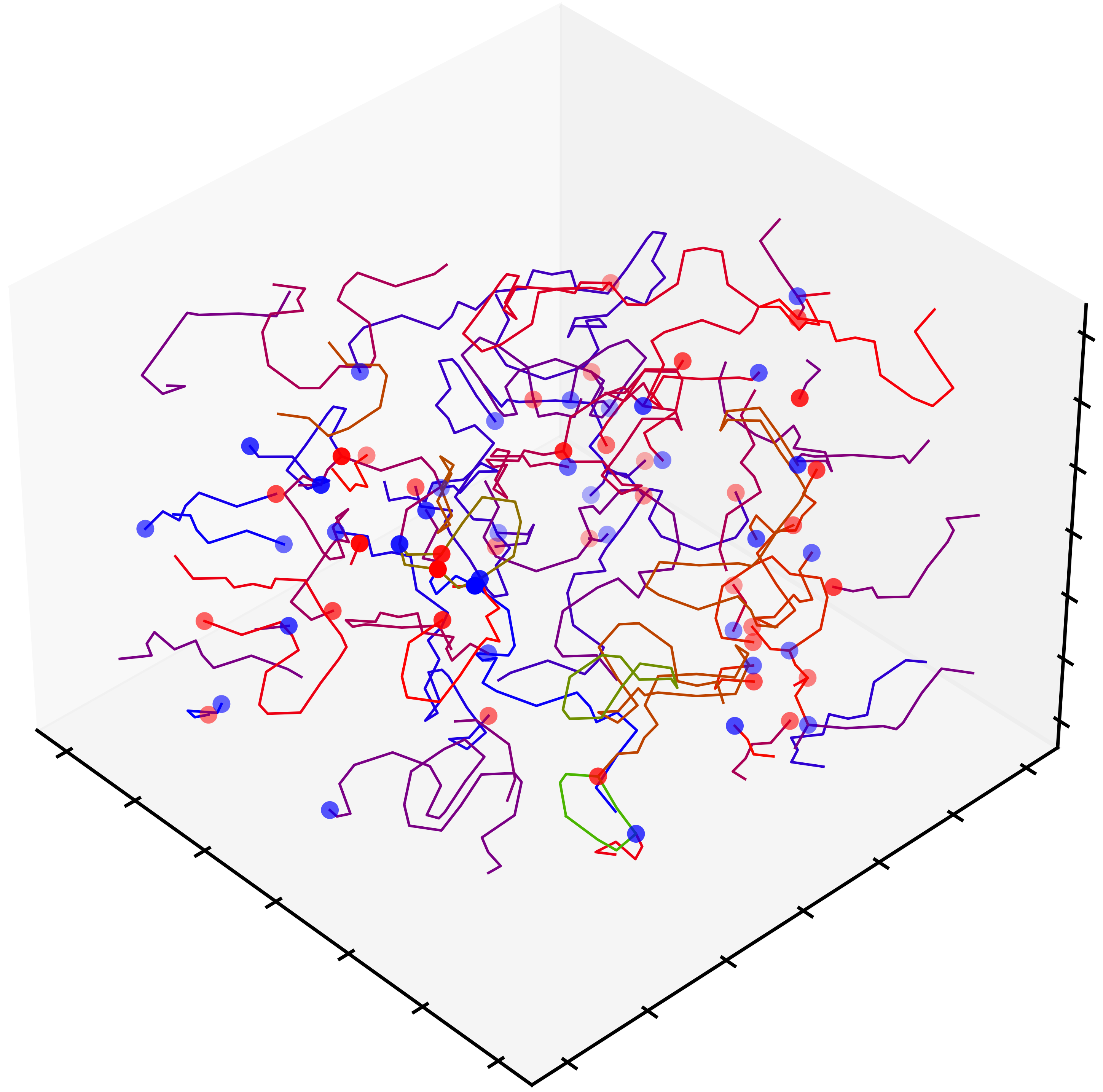
Nambu, 1977



Arrows indicate points on S^2, colors indicate points on S^1.

Simulation: singularities

T. Patel & TV, 2022



Where there are magnetic monopoles, there are magnetic fields….

An SU(5) transition

$$SU(5) \to [SU(3) \times SU(2) \times U(1)_Y]/(Z_3 \times Z_2) \to [SU(3) \times U(1)_{\rm em}]/Z_3$$

$$L = \mathrm{Tr}(D_\mu \Phi)^2 - \frac{1}{2}\mathrm{Tr}(X_{\mu\nu}X^{\mu\nu}) - V(\Phi)$$

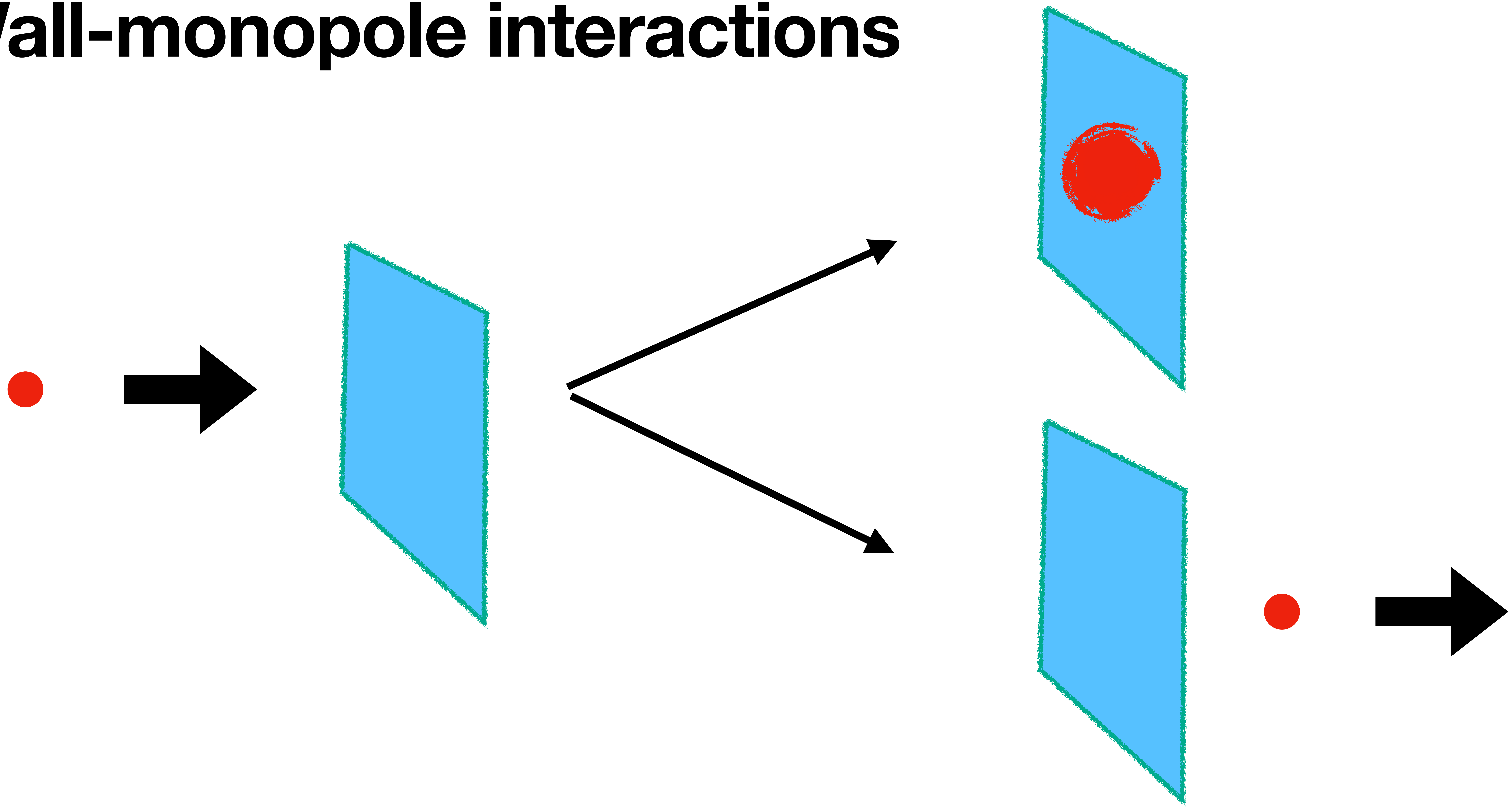
$$V(\Phi) = -m^2\mathrm{Tr}(\Phi^2) + h[\mathrm{Tr}(\Phi^2)]^2 + \lambda\mathrm{Tr}(\Phi^4) + \gamma\mathrm{Tr}(\Phi^3) - V_0$$

If cubic term is small,

$$SU(5) \times Z_2 \to [SU(3) \times SU(2) \times U(1)_Y]/(Z_3 \times Z_2) \to [SU(3) \times U(1)_{\rm em}]/Z_3$$

There are monopoles and (biased) domain walls after the first symmetry breaking.

Wall-monopole interactions



Outcome depends on wall-monopole orientations in field space.

SU(5) domain walls

Topology

$$\mathrm{diag}(2,2,2,-3,-3) \to -\mathrm{diag}(2,2,2,-3,-3)$$

$$\mathrm{diag}(2,2,2,-3,-3) \to -\mathrm{diag}(2,2,-3,2,-3)$$

★ $$\mathrm{diag}(2,2,2,-3,-3) \to -\mathrm{diag}(2,-3,-3,2,2)$$

SU(5) monopoles

Monopoles reside in a (2,-3) block.

Monopole & wall:

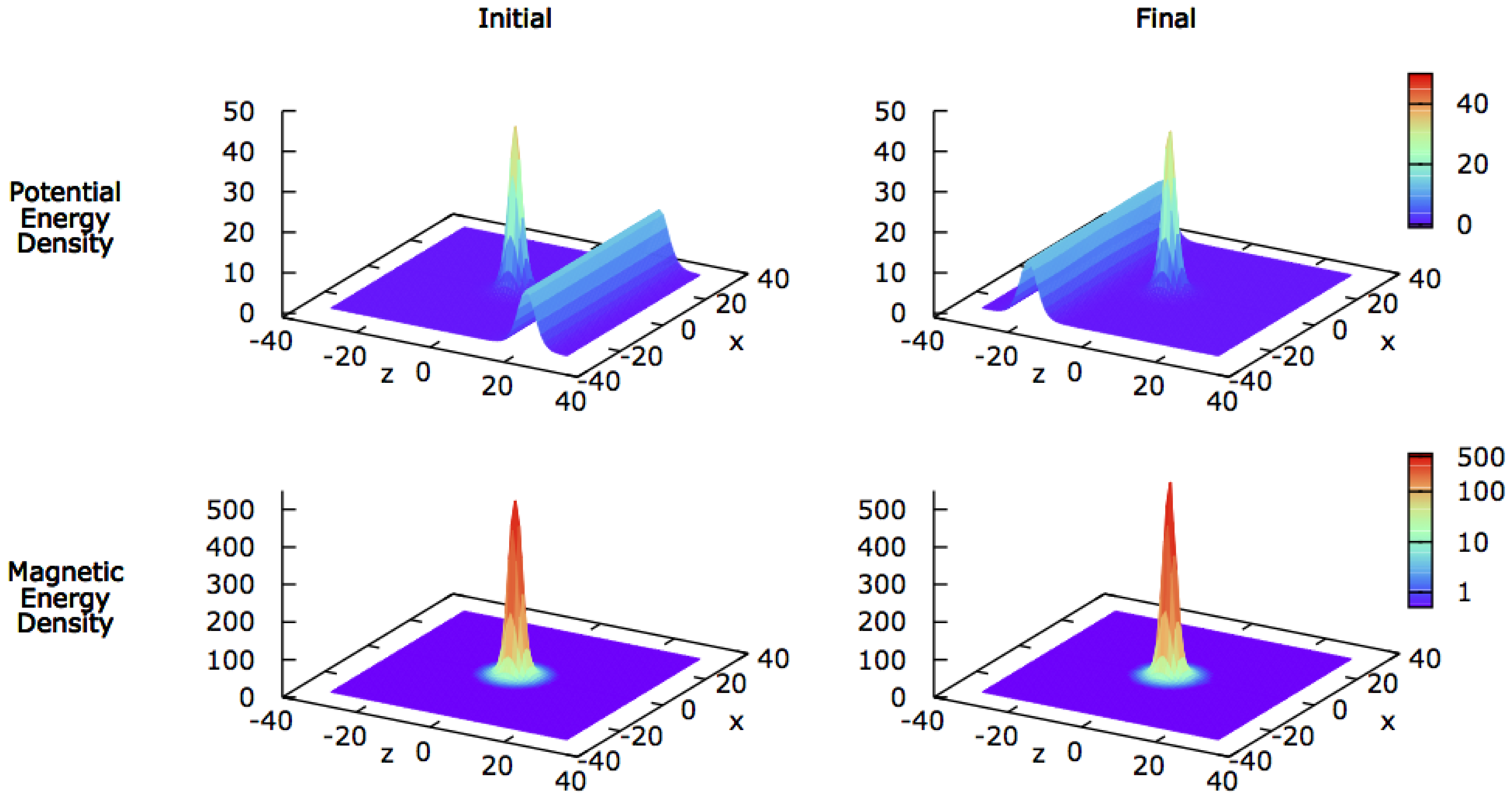
$$\mathrm{diag}(2,2,2,-3,-3) \to -\mathrm{diag}(2,-3,-3,2,2)$$

6 embeddings for monopole on left and 6 on right.

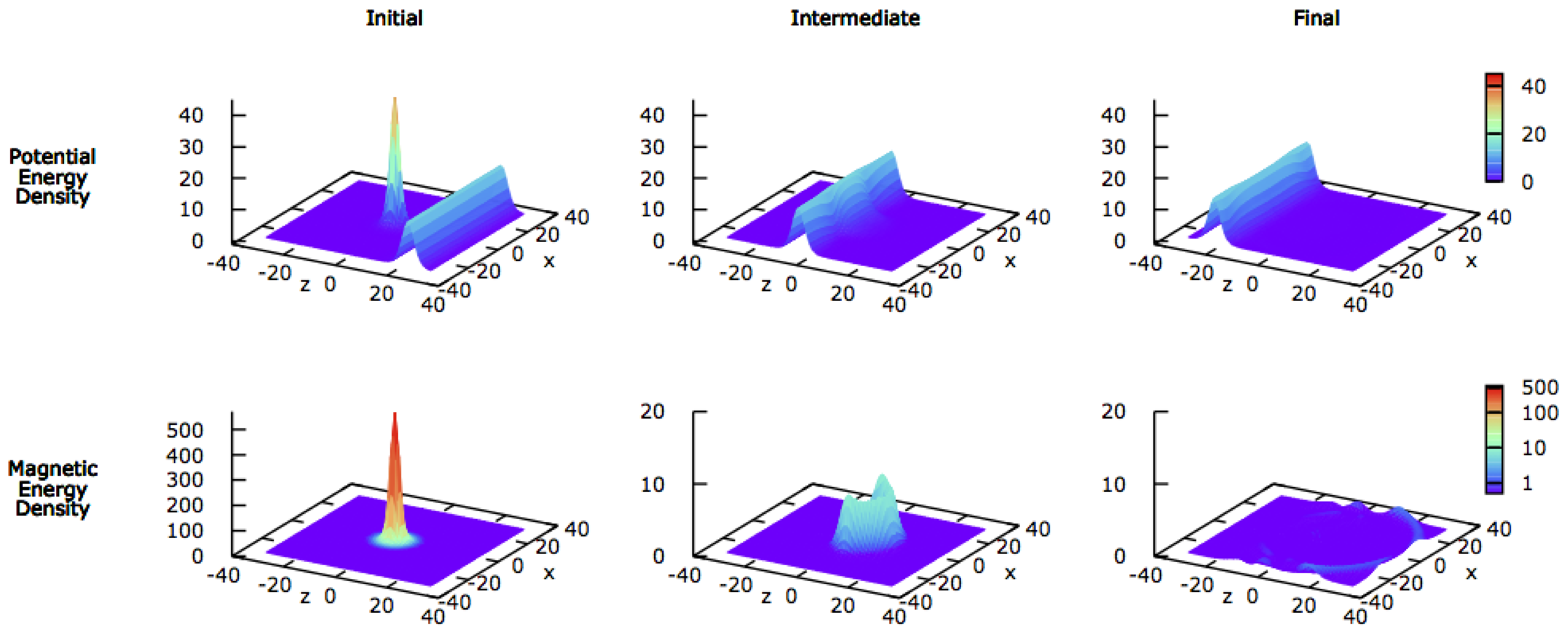
Only two distinct cases for monopole-domain wall collisions:

- Monopole in 34-block: (2,-3) on left goes to -(-3,2) on right.
- Monopole in 15-block: (2,-3) on left goes to -(2,2) on right.

(2,-3) to (3,-2) case


Initial
Final
Potential Energy Density
Magnetic Energy Density
z
x
0
20
40
1
10
100
500

(2,-3) to (-2,-2) case

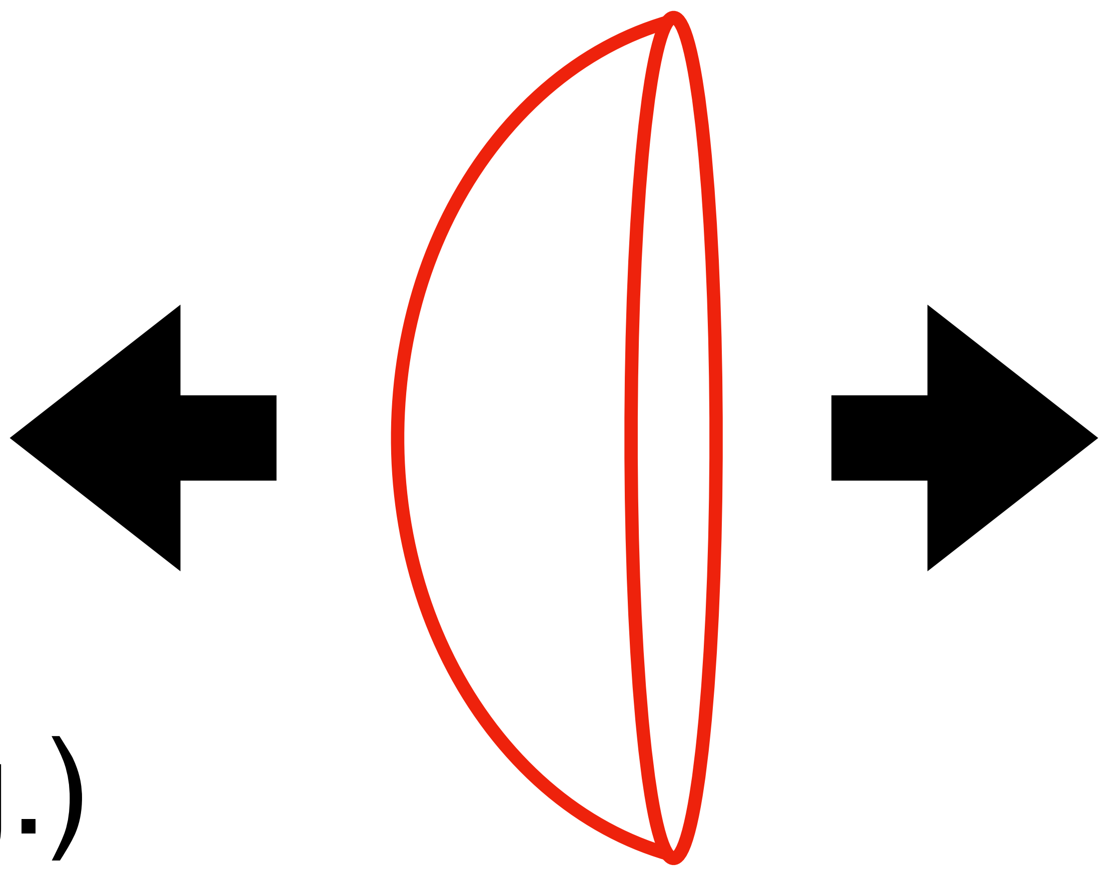


Possibility of “sweeping monopoles” (and strings?) in cosmology.
(Dvali, Liu & TV,1998).

Biased walls

Domain walls will survive for a duration that depends on the strength of the biasing.

$\text{pressure} \sim \gamma m(\text{VEV})^3$



$\text{tension} \sim \sigma/R$

(γ is the cubic coupling.)

Eventually, as the walls straighten out with cosmic expansion, pressure wins and the wall network annihilates.

Wall annihilation also implies monopole annihilation though this hasn't been tested.

Gravitational waves from defect interactions

If monopoles are Coulombic on one side of the wall but confined on the other, monopole-wall interactions can lead to gravitational wave emission.

Bachmaier, Dvali & Valbuena-Bermudez, 2024
https://www.youtube.com/watch?v=IPJAPjo3nSc
https://www.youtube.com/watch?v=JZaXUYikQbo

Summary

1. Magnetic monopoles are predicted in all grand unified models. Yet the abundance of heavy monopoles in the universe is highly constrained.
2. Inflation has been proposed as a solution but it is disconnected from the particle physics model.
3. Another solution involving defect interactions may also work — biased domain walls may be able to sweep away monopoles. Scenario needs further study, as well as the production of gravitational waves during defect interactions.
4. Electroweak magnetic monopoles can produce primordial magnetic fields that are probed by current observations.
5. If magnetic monopoles exist in the fundamental theory but have been erased in the universe, it should be possible (in principle) to create them in the lab.

Quiz: true or false

- Domain walls can be used to build a house.
- Bogomolnyi method evaluates the energy of a domain wall without solving any differential equations.
- Biased walls live forever.
- Biased walls do not lead to a gravitational wave background.
- A collapsing spherical domain wall emits gravitational waves.
- A collapsing spherical domain wall emits gravitons.
- Non-self intersecting cosmic string loops mostly look rectangular.
- Cosmic strings only produce a stochastic background of gravitational waves and no other signature.

Quiz: true or false (continued)

- The Nambu-Goto description of a string is valid under all circumstances.
- Cosmic strings can be infinite.
- Some strings can carry electric currents.
- A magnetic monopole is the same as a hedgehog (the animal).
- Different types of topological defects don’t interact.
- Magnetic monopoles can only carry Abelian magnetic charge.
- div(B)=0 in electroweak standard model. (B=e.m. magnetic field.)
- Experimentalists are looking for magnetic monopoles at CERN.
- There can be a universe inside a monopole.