Kinks and Domain Walls

an introduction

Tanmay Vachaspati, Arizona State University

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Reviews:

- Solitons and Instantons, R. Rajaraman, 1982.
- Solitons and Particles, C. Rebbi & G. Soliani, 1985.
- Aspects of Symmetry: Classical Lumps and Their Quantum Descendants, S. Coleman, 1985.
- Cosmic Strings and Other Topological Defects, A. Vilenkin & E.P.S. Shellard, 2000.
- Kinks and Domain Walls, T. Vachaspati, 2006.

(Please see Reviews for further references.)

Most of the lecture drawn from:

- Kinks and Domain Walls, T. Vachaspati, 2006.

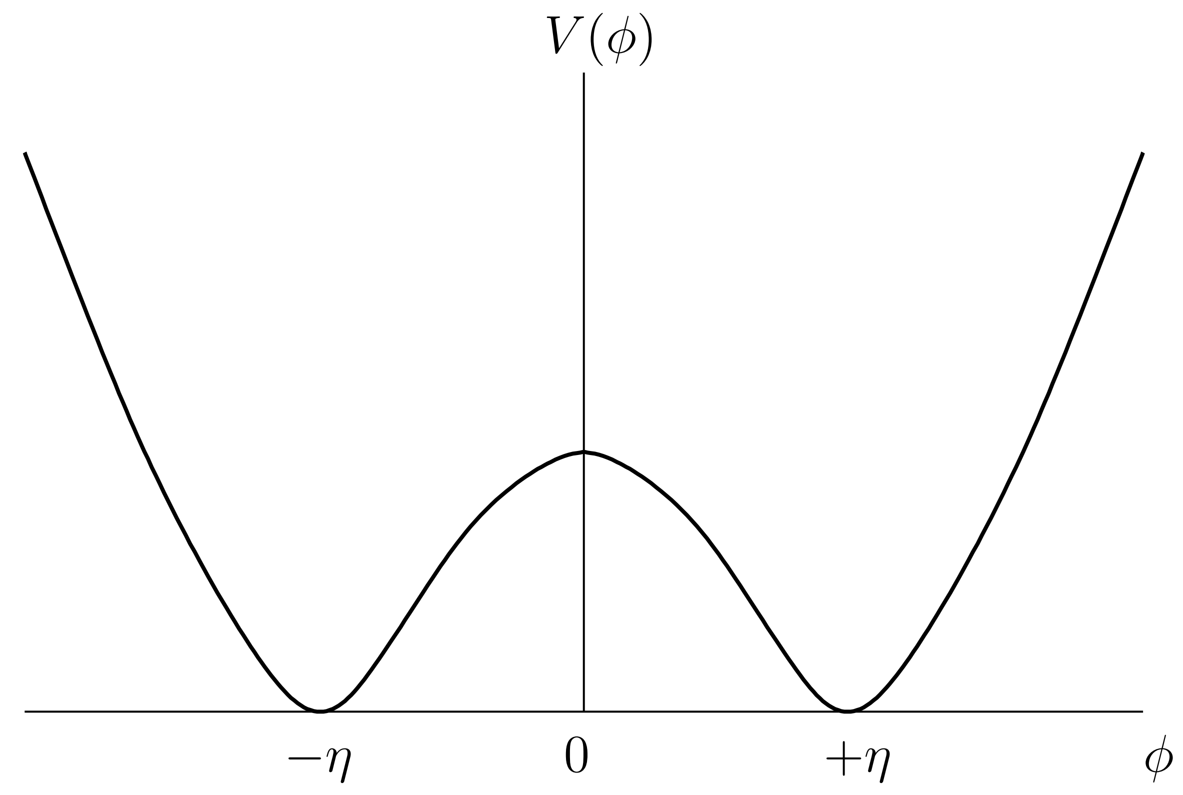
Focus — topics related to cosmology.

Missing — quantum properties.

Lectures intended for younger audiences but hope there will be something new for everyone.

Kinks in elementary models

$$\lambda\phi^4: \quad L = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{\lambda}{4}(\phi^2 - \eta^2)^2 \qquad V(\phi) = \frac{\lambda}{4}(\phi^2 - \eta^2)^2$$

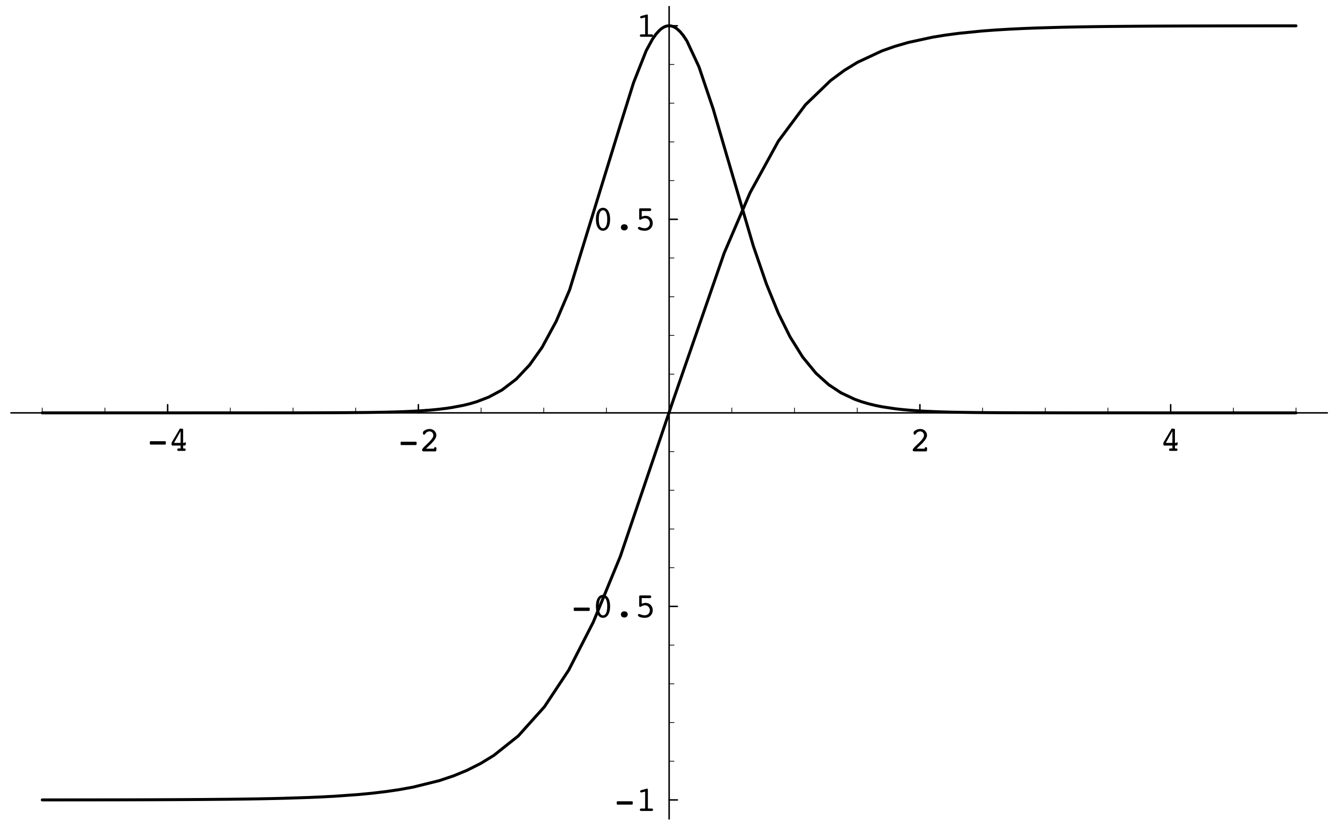


$$\partial_t^2\phi - \partial_x^2\phi + \lambda(\phi^2 - \eta^2)\phi = 0$$

$$\phi = \eta + \psi \qquad m_\psi = \sqrt{2}m$$

define m

$$\phi_k(t,x) = \eta \tanh\left(\sqrt{\frac{\lambda}{2}}\eta X\right) \qquad X \equiv \frac{x - vt}{\sqrt{1 - v^2}}$$



$$E = \int dx\, \mathcal{E} = \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda} \qquad w = \sqrt{\frac{2}{\lambda}}\frac{1}{\eta} = \frac{\sqrt{2}}{m} = \frac{2}{m_\psi}$$

Other models: e.g. sine-Gordon $V(\phi) = \frac{\alpha}{\beta^2}(1 - \cos(\beta\phi))$

Bogomolnyi method

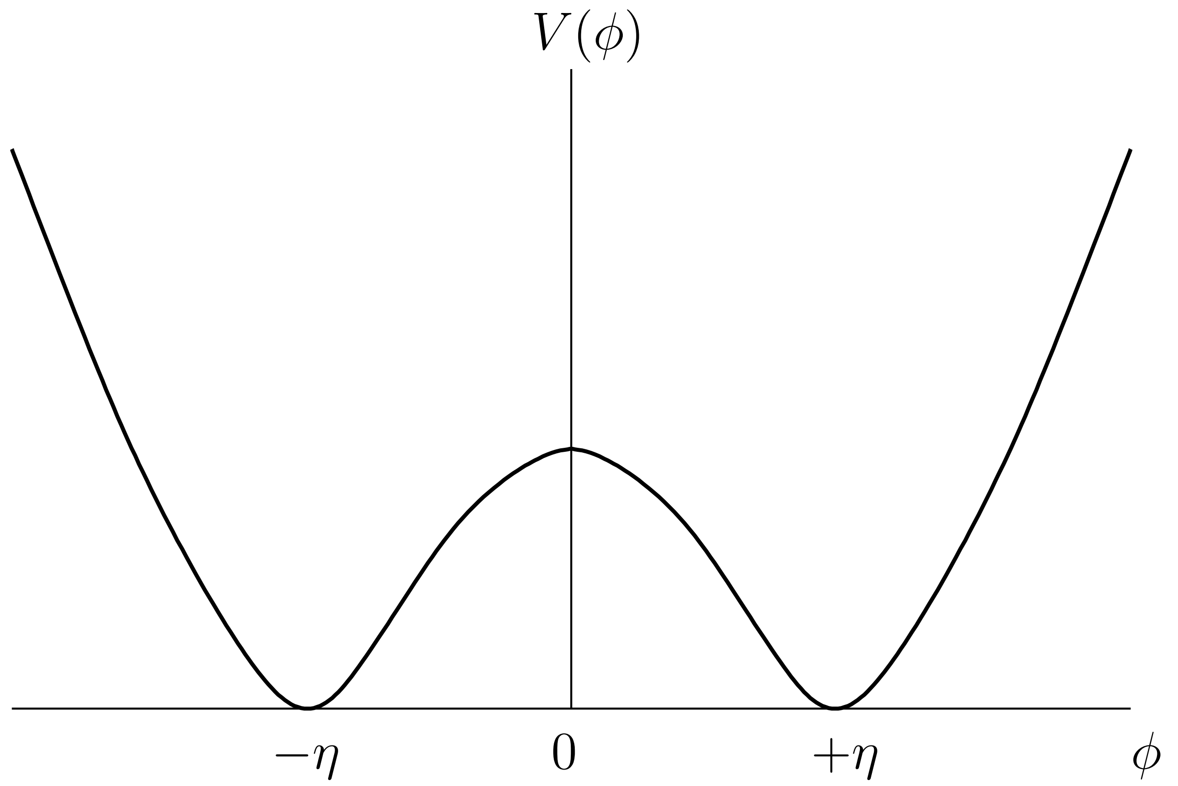
$$
\begin{aligned}
E &= \int dx \; [\frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}(\partial_x\phi)^2 + V(\phi)] \\
&= \int dx \; [\frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}(\partial_x\phi \mp \sqrt{2V(\phi)}\;)^2 \pm \sqrt{2V(\phi)}\partial_x\phi] \\
&= \int dx \; [\frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}(\partial_x\phi \mp \sqrt{2V(\phi)}\;)^2] \pm \int_{\phi(-\infty)}^{\phi(+\infty)} d\phi' \sqrt{2V(\phi')}
\end{aligned}
$$

$$
\partial_t\phi = 0 \qquad\qquad \partial_x\phi = \pm\sqrt{2V(\phi)}
$$

$$
E_{\min} = \pm \int_{\phi(-\infty)}^{\phi(+\infty)} d\phi' \sqrt{2V(\phi')} = \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda}
$$

General considerations

Domain wall solutions exist if the model has multiple disconnected degenerate vacua.



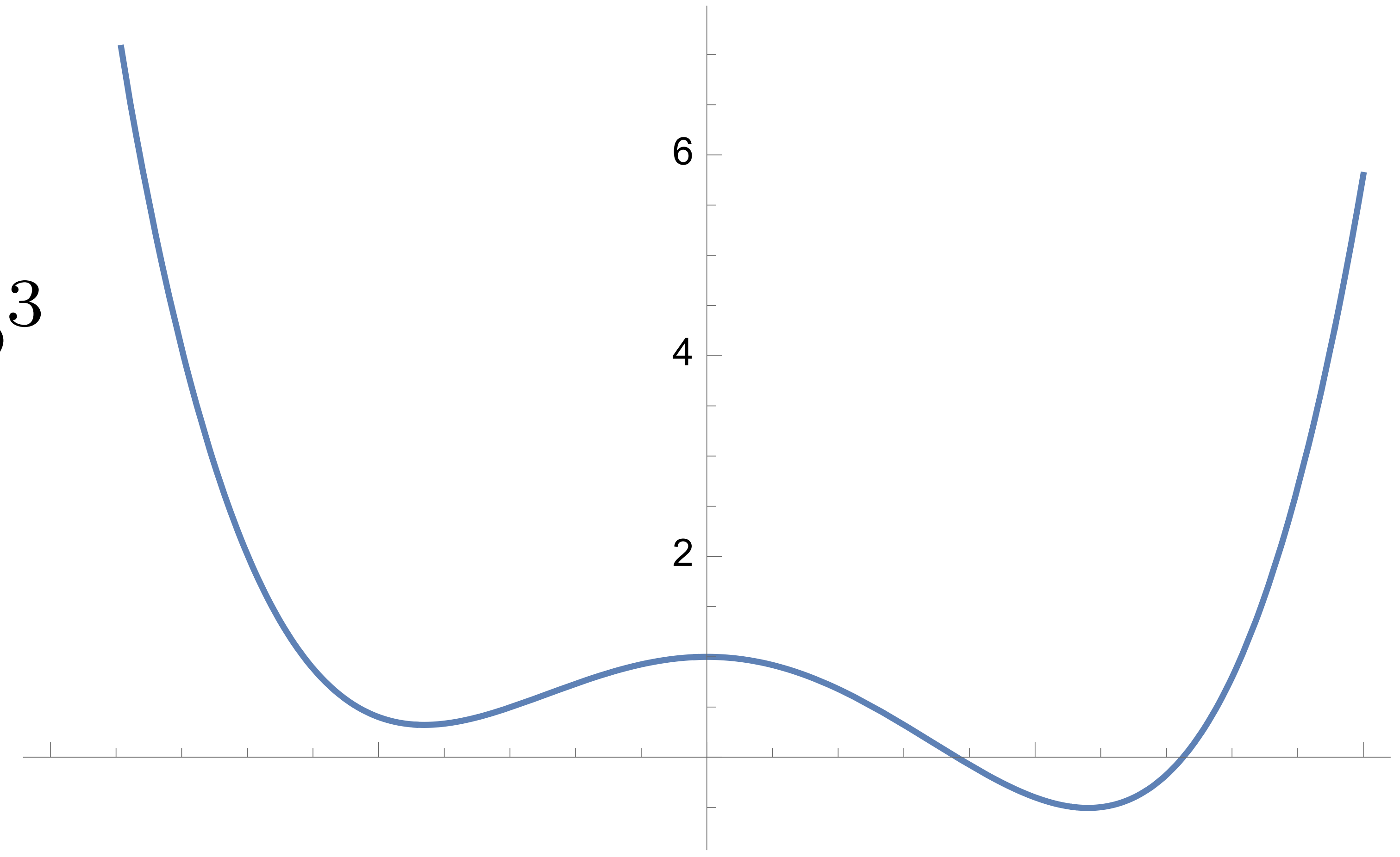
Generically, “degeneracy” is due to a discrete symmetry.

Spontaneously broken discrete symmetry may imply domain wall solution.

E.g. $Z_2 \to 1$

Approximate symmetries

$$L = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\lambda}{4}(\phi^2 - \eta^2)^2 + \epsilon\phi^3 \qquad V(\phi) = \frac{\lambda}{4}(\phi^2 - \eta^2)^2 - \epsilon\phi^3$$



No static kink solution exists but approximate kink-like objects, “biased kinks”, are still present in the model and can be relevant in cosmology (discussed later).

More generally, any potential with several disconnected minima (that need not be degenerate) will have biased kinks that may play a cosmological role.

(Biased) domain walls should be very common in unified models of particle physics.

Kinks in more complicated models (SU(5))

(Relevant to Grand Unified Theories.)

Pogosian & TV, 2000 & follow-up.

$$\{\Phi,\ X_\mu^a\} \qquad SU(5) \to [SU(3)\times SU(2)\times U(1)]/Z_3 \times Z_2$$

Φ is in the adjoint representation of SU(5): 5x5 traceless Hermitian matrix.

$$L = \mathrm{Tr}(D_\mu\Phi)^2 - \frac{1}{2}\mathrm{Tr}(X_{\mu\nu}X^{\mu\nu}) - V(\Phi)$$

$$V(\Phi) = -m^2\mathrm{Tr}(\Phi^2) + h[\mathrm{Tr}(\Phi^2)]^2 + \lambda\mathrm{Tr}(\Phi^4) + \gamma\mathrm{Tr}(\Phi^3) - V_0$$

Assume cubic term is absent (γ=0).

Then there is an additional Z_2 symmetry (**Φ** to -**Φ**) that gets broken.

Simplest example of kink boundary conditions:

$$\Phi(-\infty) \propto -\mathrm{diag}(2,2,2,-3,-3) \qquad \Phi(+\infty) = -\Phi(-\infty) \propto \mathrm{diag}(2,2,2,-3,-3)$$

More kink solutions

$$\Phi_- = \frac{\eta}{2\sqrt{15}}\mathrm{diag}(2,2,2,-3,-3)$$

Energy is minimized only if: $[\Phi_+, \Phi_-] = 0$

Three classes of solutions:

$$\Phi_+^{(0)} = -\frac{\eta}{2\sqrt{15}}\mathrm{diag}(2,2,2,-3,-3)$$
$$\Phi_+^{(1)} = -\frac{\eta}{2\sqrt{15}}\mathrm{diag}(2,2,-3,2,-3)$$
$$\Phi_+^{(2)} = -\frac{\eta}{2\sqrt{15}}\mathrm{diag}(2,-3,-3,2,2)$$

Form of kink solutions

$$\Phi_- = \frac{\eta}{2\sqrt{15}}\mathrm{diag}(2,2,2,-3,-3)$$

Three classes of solutions:

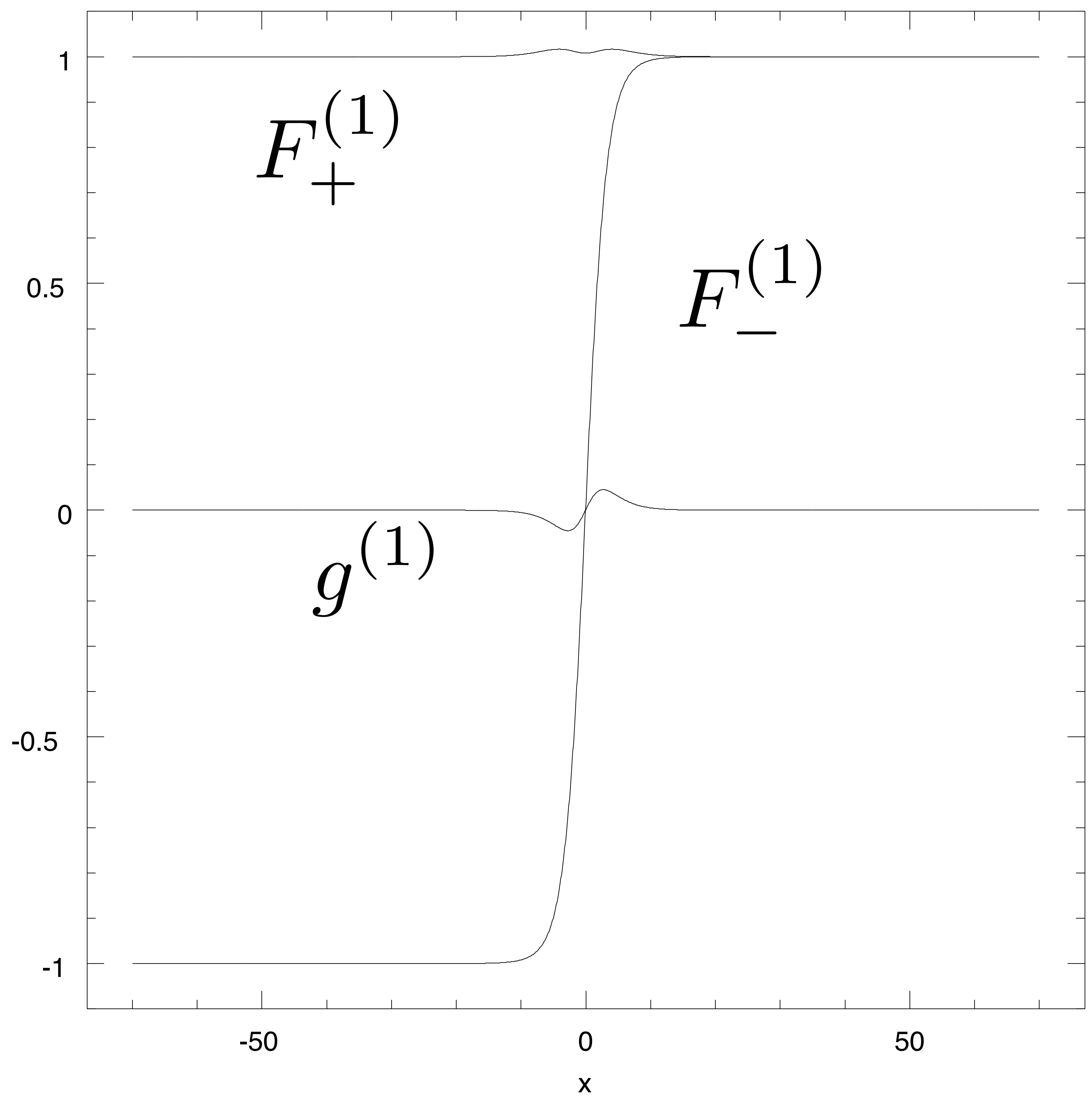
$$\Phi_k^{(q)} = F_+^{(q)}(x)\mathbf{M}_+^{(q)} + F_-^{(q)}(x)\mathbf{M}_-^{(q)} + g^{(q)}(x)\mathbf{M}^{(q)} \ ,$$

$$\mathbf{M}_+^{(q)} = \frac{\Phi_+^{(q)} + \Phi_-^{(q)}}{2} \ , \ \ \mathbf{M}_-^{(q)} = \frac{\Phi_+^{(q)} - \Phi_-^{(q)}}{2}$$

$$\mathbf{M}^{(q)} = 0 \text{ for } q = 0, 2$$

$$\mathbf{M}^{(1)} = \frac{\eta}{2\sqrt{7}}\mathrm{diag}(1,1,-2,-2,2)$$

$$F_-^{(q)}(\mp\infty) = \mp 1 \ , \ \ F_+^{(q)}(\mp\infty) = +1 \ , \ \ g^{(q)}(\mp\infty) = 0$$



Non-topological kink solutions

$$\Phi_- = \frac{\eta}{2\sqrt{15}}\mathrm{diag}(2,2,2,-3,-3)$$

Three classes of solutions:

$$\Phi^{(q)}_{NTk} = F^{(q)}_-(x)\mathbf{M}^{(q)}_+ + F^{(q)}_+(x)\mathbf{M}^{(q)}_- + g^{(q)}(x)\mathbf{M}^{(q)} \ ,$$

$$\mathrm{M}^{(q)}_+ = \frac{\Phi^{(q)}_+ + \Phi^{(q)}_-}{2} \ , \quad \mathrm{M}^{(q)}_- = \frac{\Phi^{(q)}_+ - \Phi^{(q)}_-}{2}$$

M$^{(q)}$= (as before)

(Note signs compared to topological solutions.)

$$F^{(q)}_+(\mp\infty) = \mp 1 \ , \quad F^{(q)}_-(\mp\infty) = +1 \ , \quad g^{(q)}(\mp\infty) = 0$$

These non-topological solutions are generally unstable.

Space of (topological) kink solutions

q=1 solution results from a 6 dimensional set of boundary conditions.

q=2 solution results from a 4 dimensional set of boundary conditions.

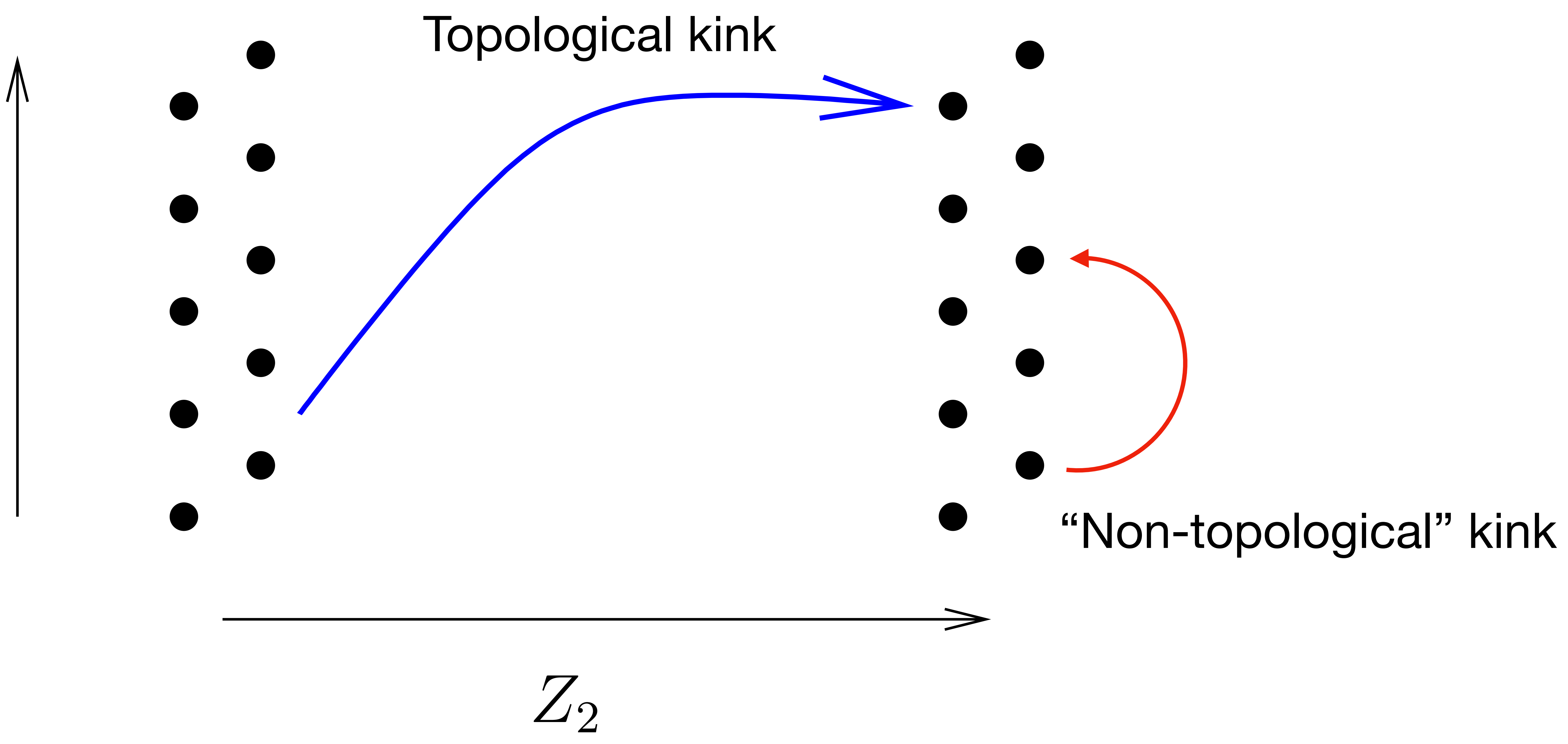
q=2 solution has least energy.

q=1 wall most likely to form but should decay into a q=2 wall.

General picture

$S_5/S_3 \times S_2$

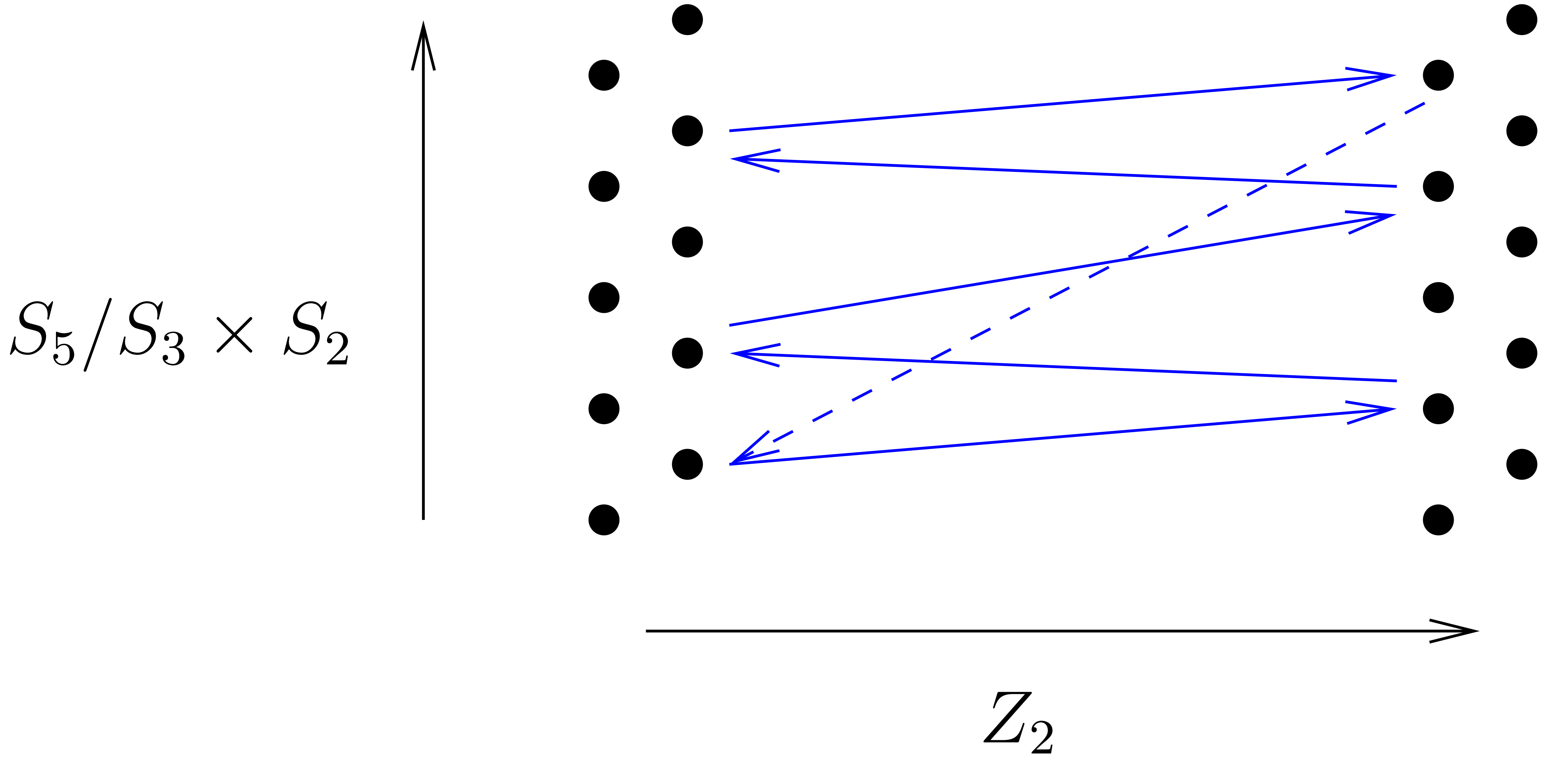
“discretized symmetries” restrict Phi to diagonal entries



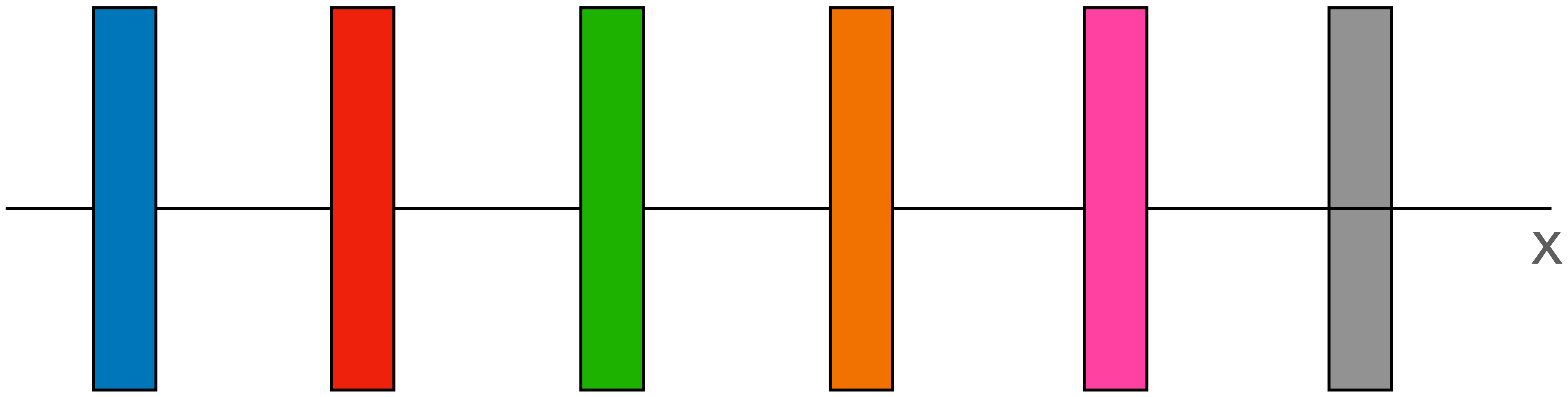
$$\Phi \to f_1\lambda_3 + f_2\lambda_8 + f_3\tau_3 + f_4 Y$$

Kink lattices

$$
\begin{aligned}
\ldots \to -(2,2,2,-3,-3) &\to +(2,-3,-3,2,2) \\
&\to -(-3,2,2,-3,2) \\
&\to +(2,-3,2,2,-3) \\
&\to -(2,2,-3,-3,2) \\
&\to +(-3,-3,2,2,2) \\
&\to -(2,2,2,-3,-3) \to \ldots
\end{aligned}
$$

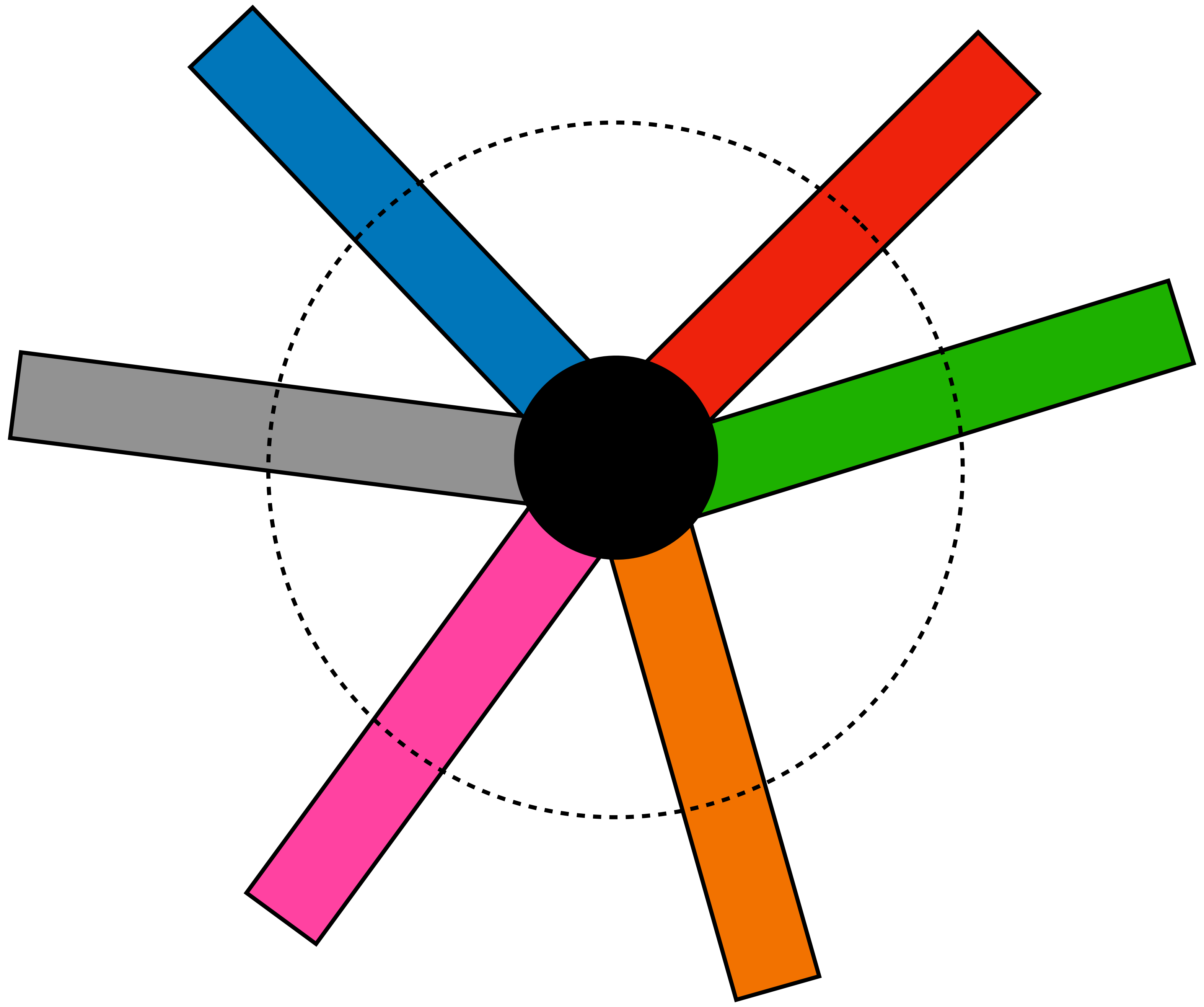


Kink-antikink can repel, depending on internal space orientation.



Lattice is stable in S_n model but not in gauged SU(5).

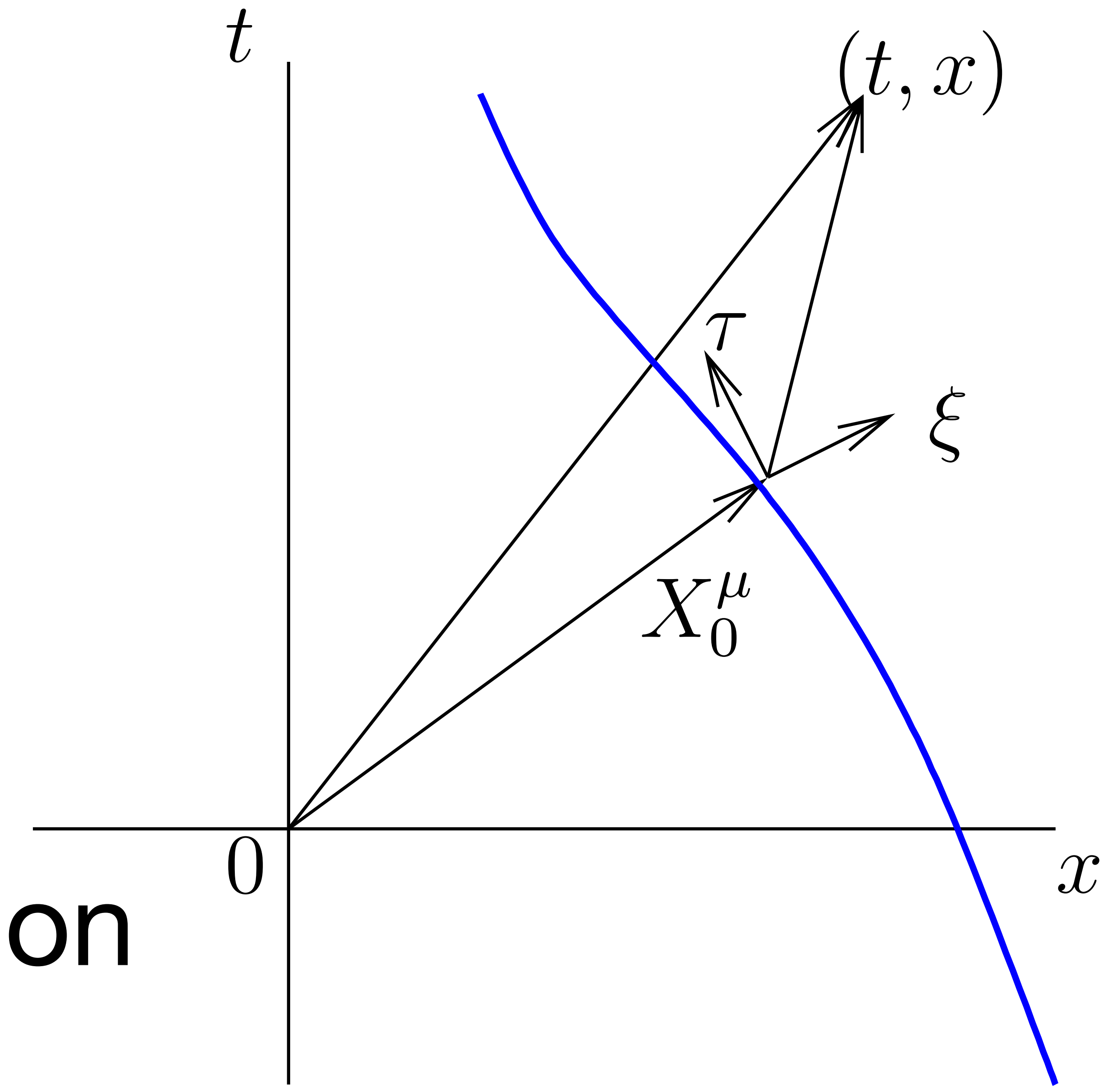
Kink nodes



Kink lattice on a circle.

Dynamics of domain walls

Full dynamics — wall motion plus excitations plus radiation — is given by the field theory. This may to too detailed for certain applications and an effective description might suffice.



Position of wall: $X^\mu(\tau,\zeta,\chi)$

$$x^\mu = X^\mu(\tau,\zeta,\chi) + \xi N^\mu(\tau,\zeta,\chi)$$

N^μ=normal to wall

"Nambu-Goto action"

$$S = \int d^4x\,\sqrt{-g}\,L_{\text{fields}} \to -\sigma \int d^3\rho\,\sqrt{|h|} + \ldots$$

σ = wall tension

$$h_{ab} = g_{\mu\nu}(X^\rho)\partial_a X^\mu \partial_b X^\nu$$

"induced metric"

Equations of motion

$$\frac{1}{\sqrt{|h|}}\partial_a(\sqrt{|h|}h^{ab}\partial_b X^\sigma) = \Gamma^\sigma_{\mu\nu}h^{ab}\partial_a X^\mu \partial_b X^\nu$$

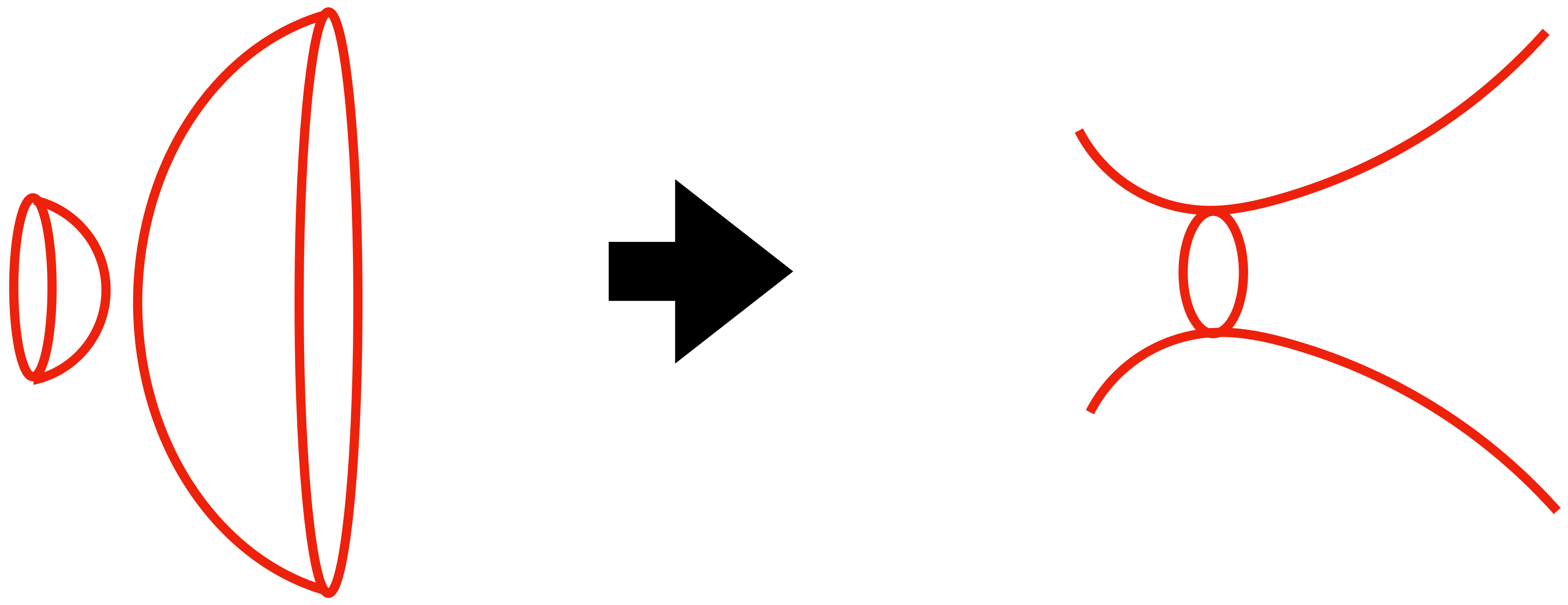
$$\Gamma^\sigma_{\mu\nu} = \frac{g^{\sigma\rho}}{2}(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu})$$

In Minkowski background ($\Gamma = 0$):

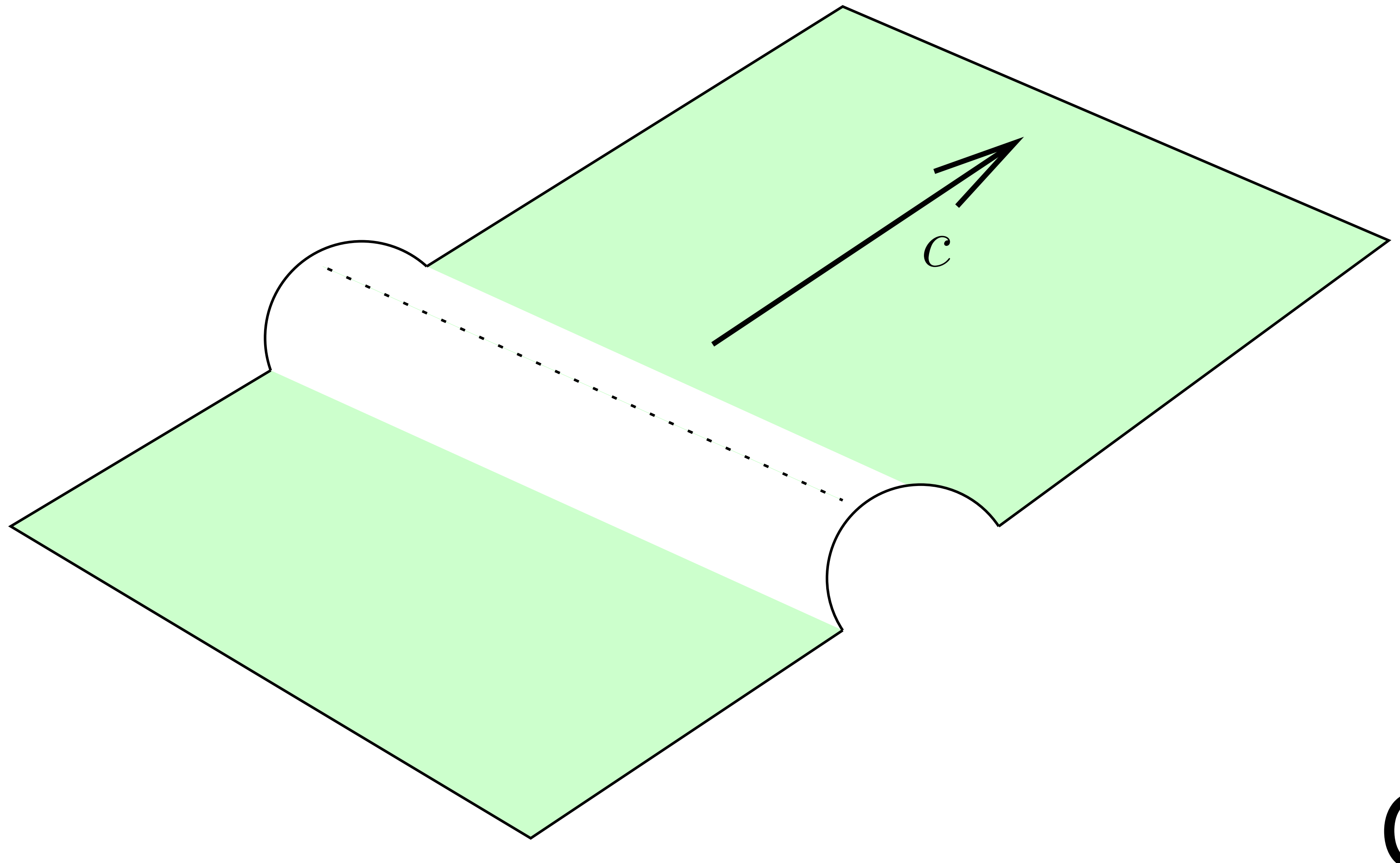
$$\partial_a(h^{ab}\partial_b X^\sigma) + \frac{1}{2}h^{cd}\partial_a h_{cd}\; h^{ab}\partial_b X^\sigma = 0$$

…but equation fails when there are self-intersections or regions of intense radiation (e.g. from singular points — kinks, cusps, on the wall).

Reconnections

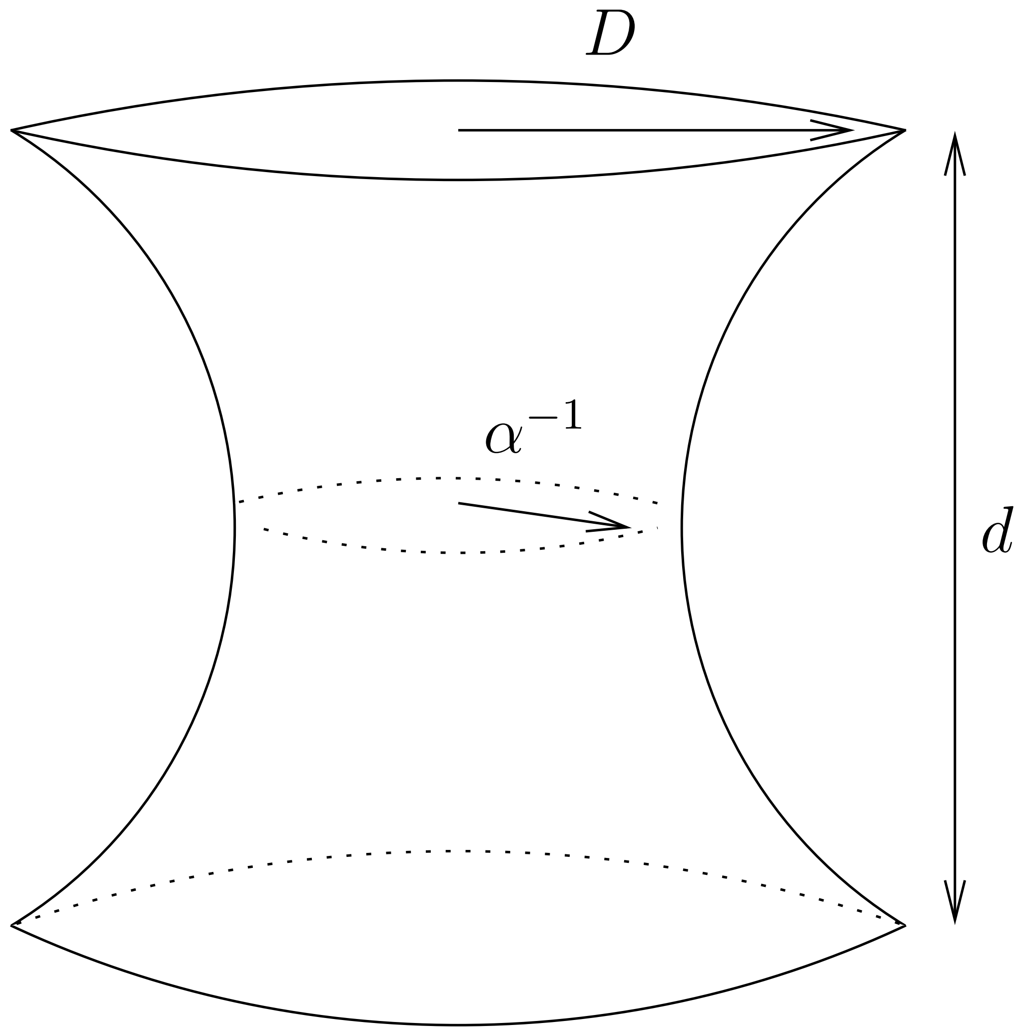


Some solutions of the Nambu-Goto equations



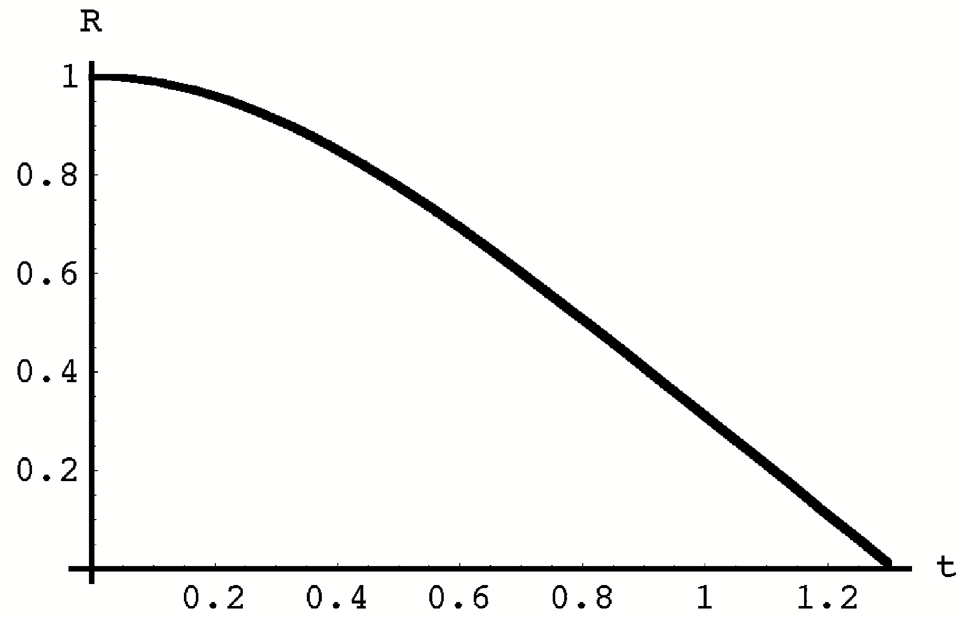
$$z = f(\tau \pm (n_1\zeta + n_2\chi)) \ , \quad n_1^2 + n_2^2 = 1$$

Solution even in field theory (probably with gravity included).



"catenoid" $R(z) = \frac{1}{\alpha}\cosh(\alpha z)$

Cylinder: $R(t) = R_0 \cos\left(\frac{t}{R_0}\right)$



Sphere: $\int_{x_*}^{x} \frac{dx}{\sqrt{1-x^4}} = \pm\frac{\tau-\tau_0}{R_0}$

Sphere in de Sitter background: $R = H^{-1}\sqrt{\frac{2}{3}}$

Gravity - thin planar wall

Vilenkin, 1983; Ipser & Sikivie, 1984

$$T^{\mu\nu}\Big|_{\mathrm{NG,plane}} = \sigma(1,0,-1,-1)\delta(x)$$

$$ds^2 = (1-\kappa|X|)^2 dt^2 - dX^2 - (1-\kappa|X|)^2 e^{2\kappa t}(dy^2+dz^2) \qquad |X| = \frac{1}{\kappa}(1-e^{-\kappa|x|})$$

$$\kappa = 2\pi G\sigma$$

de Sitter in constant x slices.

Rindler for constant y, z: $ds^2 = d\tau^2 - d\xi^2$

$$\tau = \frac{(1-\kappa|X|)}{2\kappa}(e^{\kappa t} - e^{-\kappa t})$$

$$\xi = \frac{(1-\kappa|X|)}{2\kappa}(e^{\kappa t} + e^{-\kappa t})$$

$$\xi^2 - \tau^2 = \left(\frac{1}{\kappa} - |X|\right)^2$$

Constant acceleration for particle fixed at constant X.

Particles are repelled from the wall with constant acceleration κ.

(typo in book — κ, not $1/\kappa$)

Gravity - thin planar wall (bouncing sphere)

$$ds^2 = (1-\kappa|X|)^2 dt^2 - dX^2 - (1-\kappa|X|)^2 e^{2\kappa t}(dy^2 + dz^2) \qquad |X| = \frac{1}{\kappa}(1 - e^{-\kappa|x|})$$

A coordinate transformation exists such that

$$ds^2 = dt_M^2 - dx_M^2 - dy_M^2 - dz_M^2 \quad \text{(Minkowski metric)}$$

but then the wall is located at

$$x_M^2 + y_M^2 + z_M^2 = t_M^2 + \frac{1}{\kappa^2}$$

which describes a bouncing spherical wall, i.e. one undergoing constant acceleration.

Gravity - thick walls

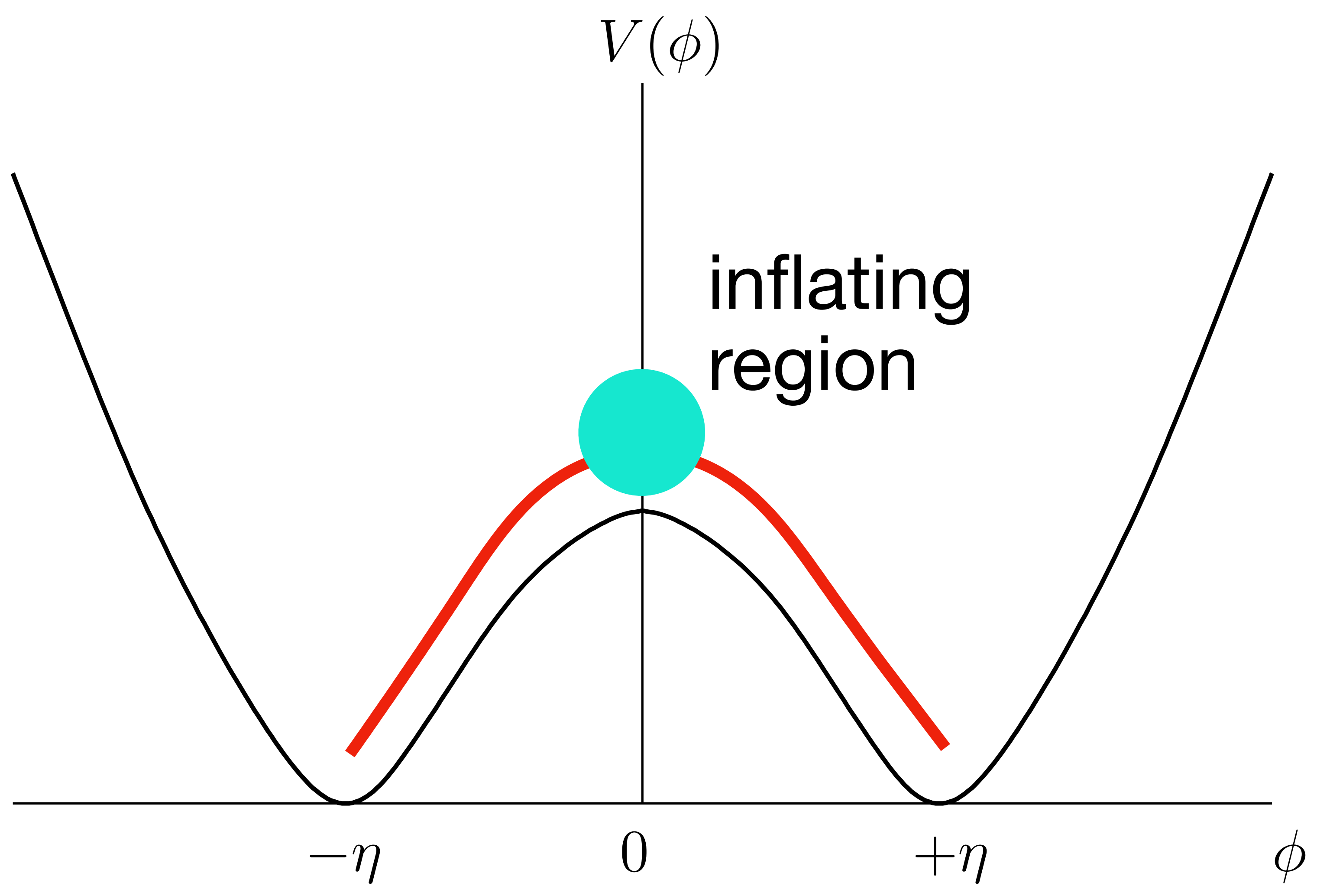
A. Vilenkin, 1994; A. Linde, 1994

Qualitatively not much changes with wall thickness taken into account unless the wall tension is very large.

Topological inflation if $16\pi G\eta^2 > 1$

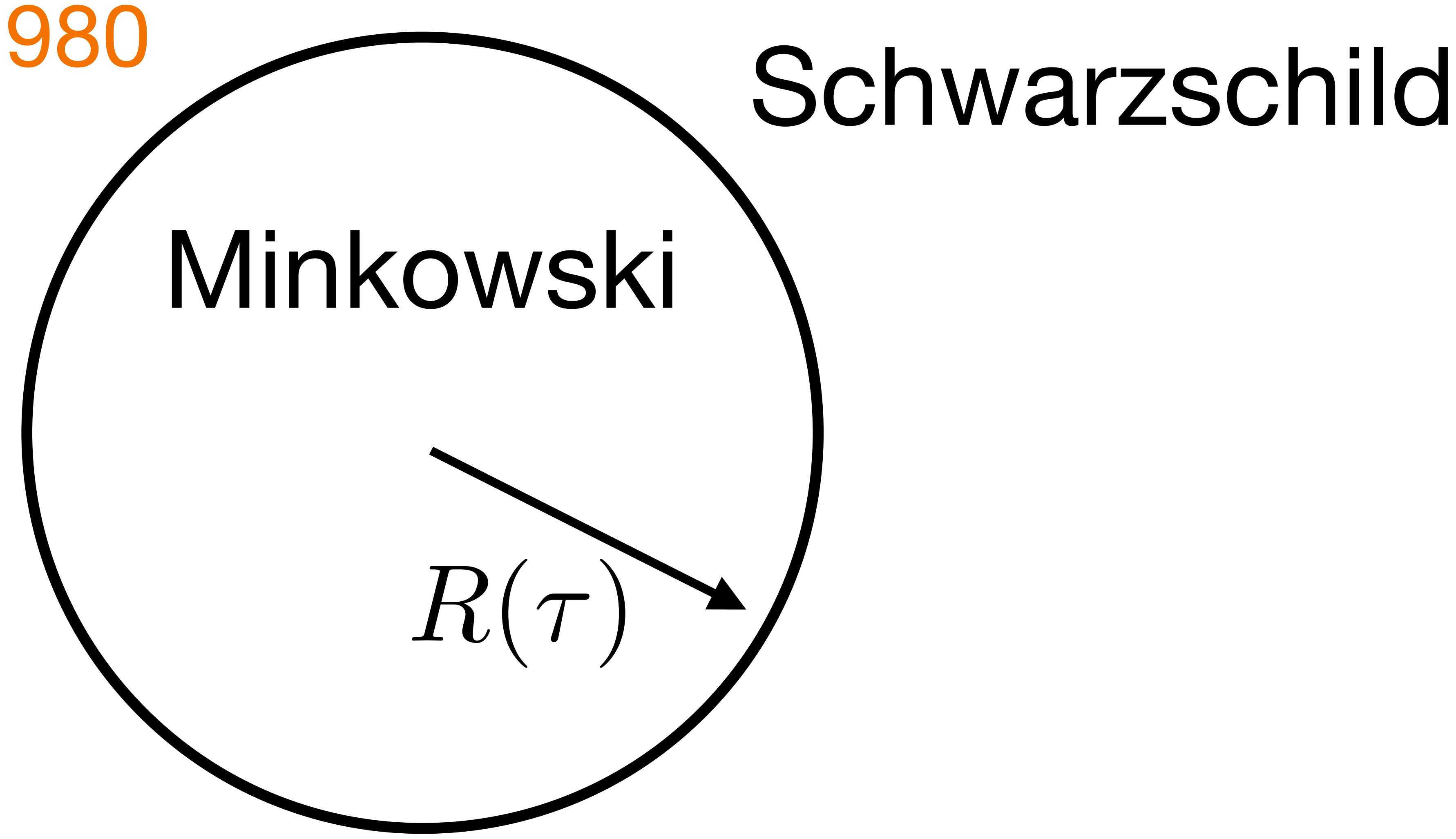
Some spatial regions are stuck at a maximum of the potential.

Graceful exit in a fixed region if the region slides off the potential.



Spherical wall

Ipser & Sikivie, 1980



$$ds^2 = dT^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \; r < R(\tau)$$

$$\dot{T} = (1 + \dot{R}^2)^{1/2} \qquad \left(\cdot = \frac{d}{d\tau}\right)$$

$(\tau = \text{proper time of wall})$

$$ds^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \;, \quad r > R(\tau)$$

$$\left(1 - \frac{2GM}{R}\right)\dot{t} = \left(1 - \frac{2GM}{R} + \dot{R}^2\right)^{1/2}$$

$$M = 4\pi\sigma R_m^2(1 - 2\pi G R_m) \qquad R_m = \text{ max. radius}$$

Cosmology - phase transitions

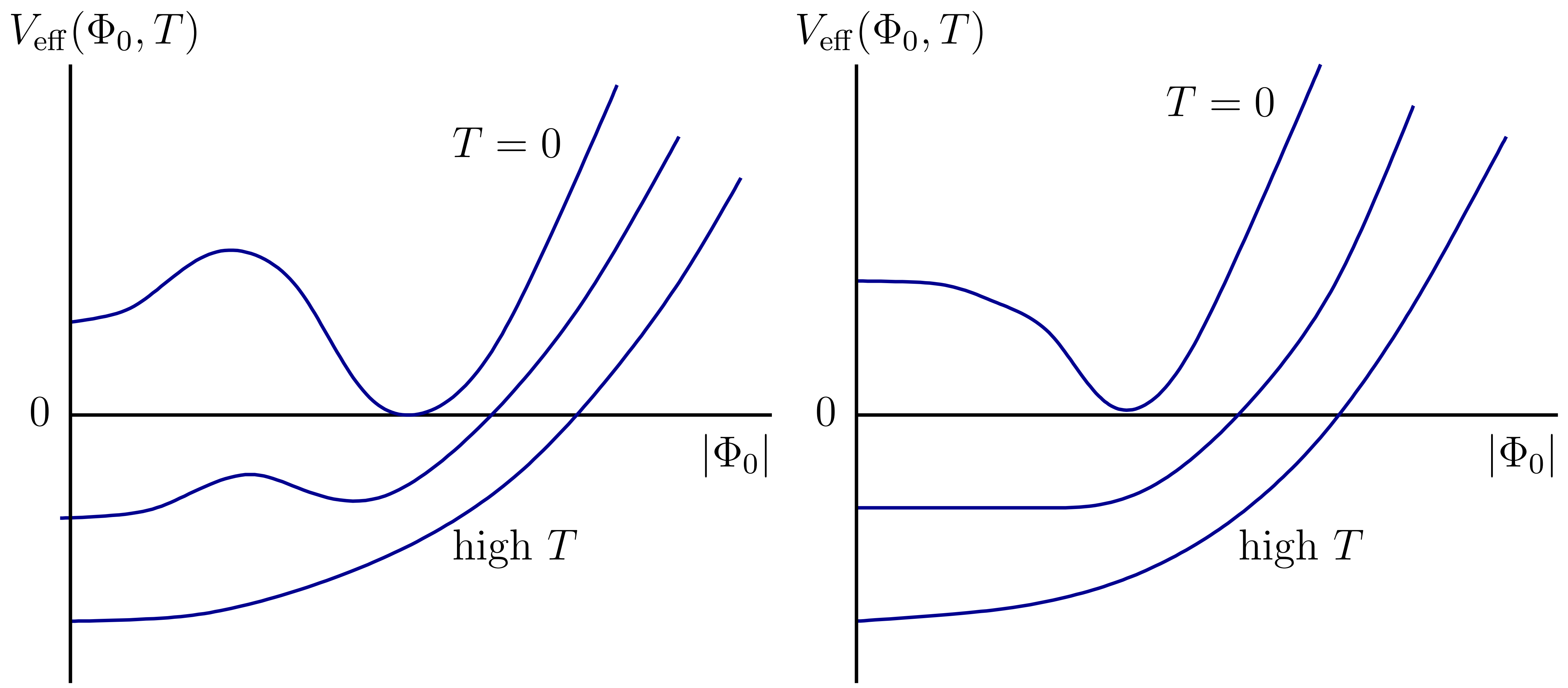
Cosmological expansion implies a cooling universe, so one has to deal with thermal corrections in quantum field theory.

Mass matrices: $\mu_{ij}^2 = \left.\frac{\partial^2 V}{\partial \Phi_i \partial \Phi_j}\right|_{\Phi=\Phi_0}$, scalar fields

$m = \Gamma_i \Phi_{0i}$, spinor fields $\qquad M_{ab}^2 = e^2(\lambda_a \lambda_b)_{ij}\Phi_{0i}\Phi_{0j}$, vector fields

$$V_{\text{eff}}(\Phi_0, T) = V(\Phi_0) + \frac{\mathcal{M}^2}{24}T^2 - \frac{\pi^2}{90}\mathcal{N}T^4$$

$$\mathcal{M}^2 = \text{Tr}(\mu^2) + 3\text{Tr}(M^2) + \frac{1}{2}\text{Tr}(\gamma^0 m \gamma^0 m)$$



…but the details of the phase transition are not that important for us.

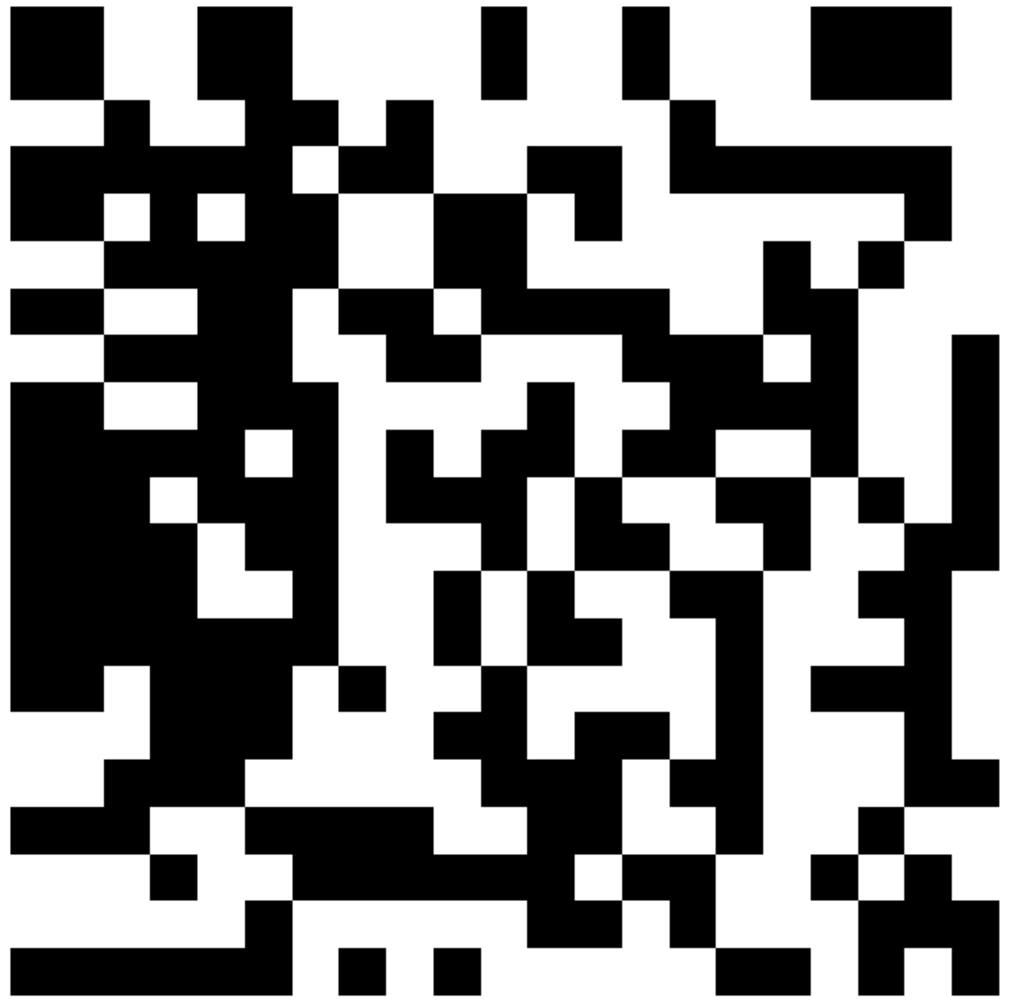
Cosmology: formation of kinks

Symmetry is all we need.

With symmetry, all degenerate vacua are equally likely.

Therefore populate space with random choices of vacua.

Z_2 case:



In 3 spatial dimensions:

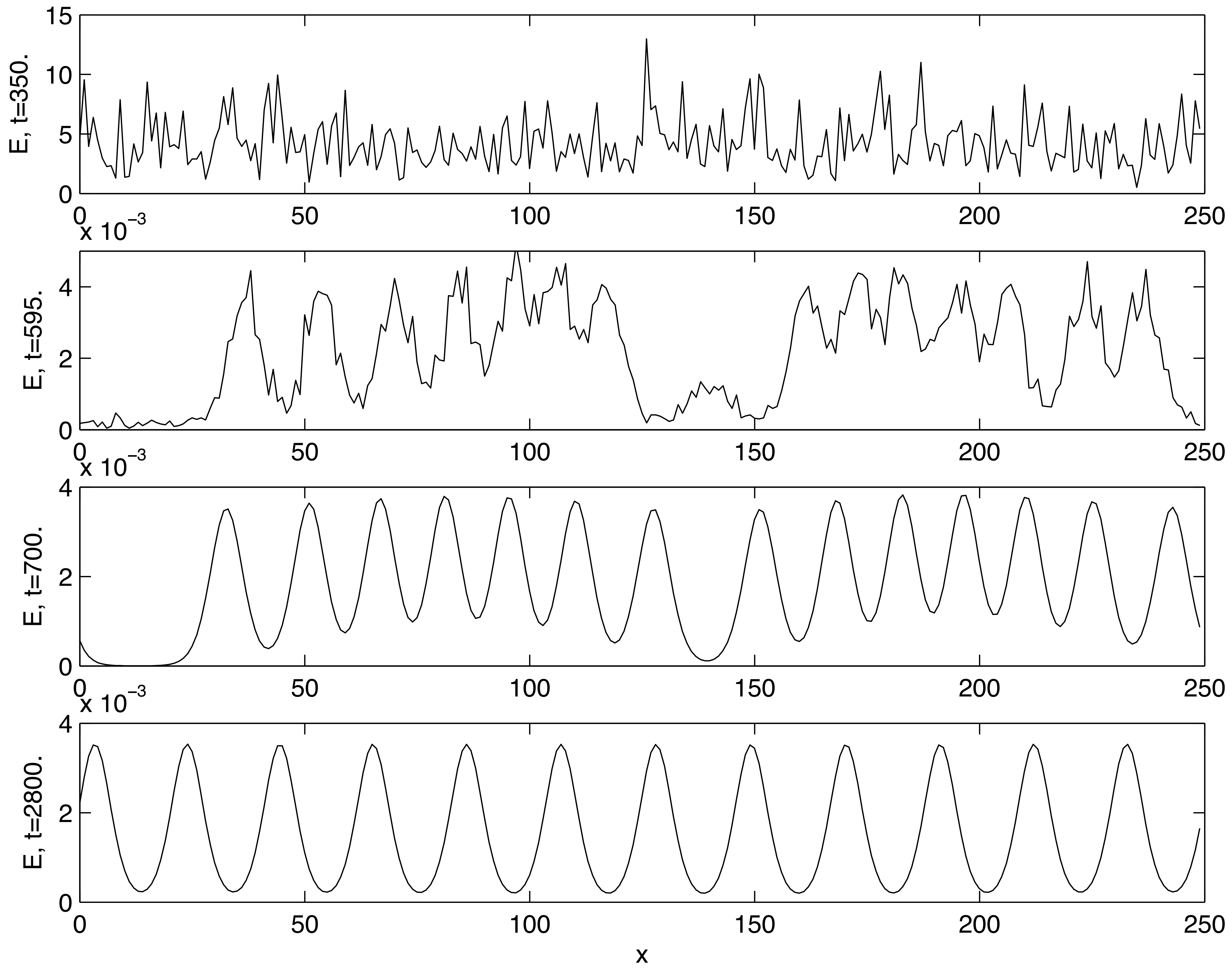
Cluster size	1	2	3	4	6	10	31082
Number	462	84	14	13	1	1	1

Formation of S_5 kinks

In 1 spatial dimension.

Antunes, Pogosian & TV, 2004

Kink lattice:



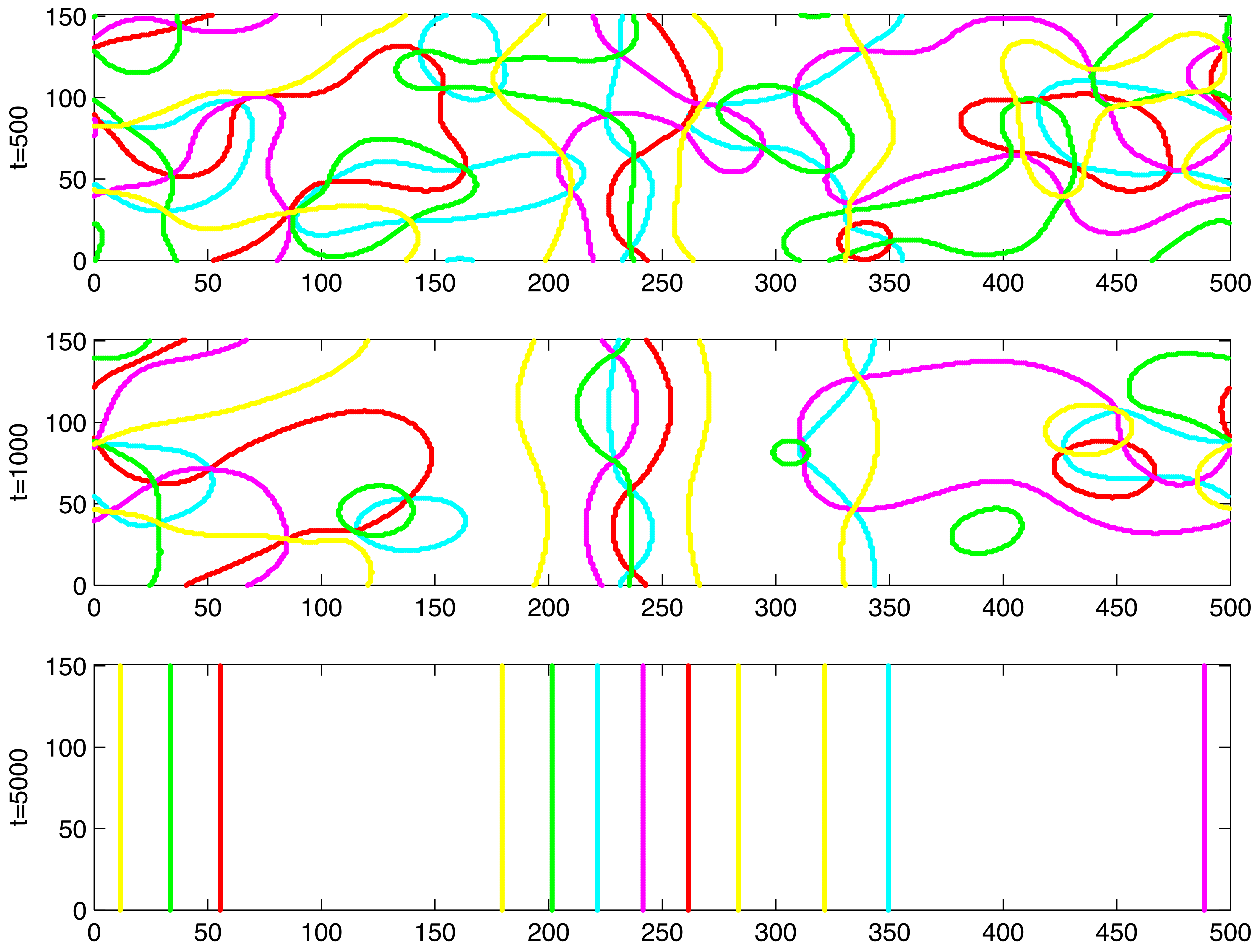
Formation of S_5 kinks

In 2 spatial dimensions.

Antunes & TV, 2004

Formation of S_5 kinks

In 2 spatial toroidal dimensions.



Cosmological constraints

A single domain wall in our cosmological volume would contribute an energy density,

$$\Omega_{\text{walls}} \equiv \frac{\rho_{\text{walls}}}{\rho_{\text{crit}}} \sim \frac{\sigma t^2/t^3}{3H^2/(8\pi G)} \sim G\sigma t \qquad \sigma \sim \sqrt{\lambda}\,\eta^3$$

Assuming coupling constants of O(1),

$$\Omega_{\text{walls}} \sim \frac{\eta^3 t_0}{m_P^2} = \left(\frac{\eta}{m_P}\right)^3 \left(\frac{10^{17}s}{10^{-43}s}\right) < 1$$

$$\eta \lesssim 100\,\text{MeV}$$

(Other constraints — CMB, BBN, expansion rate — may be tighter.)

Cosmological scenarios

How can we reconcile the cosmological constraints with the earlier finding that domain walls are quite generic in complicated but realistic models?

If the discrete symmetry is approximate, walls will be biased and will only survive for a finite duration. If the duration is short enough, cosmological domain walls can still be viable.

Biased wall network decays at,

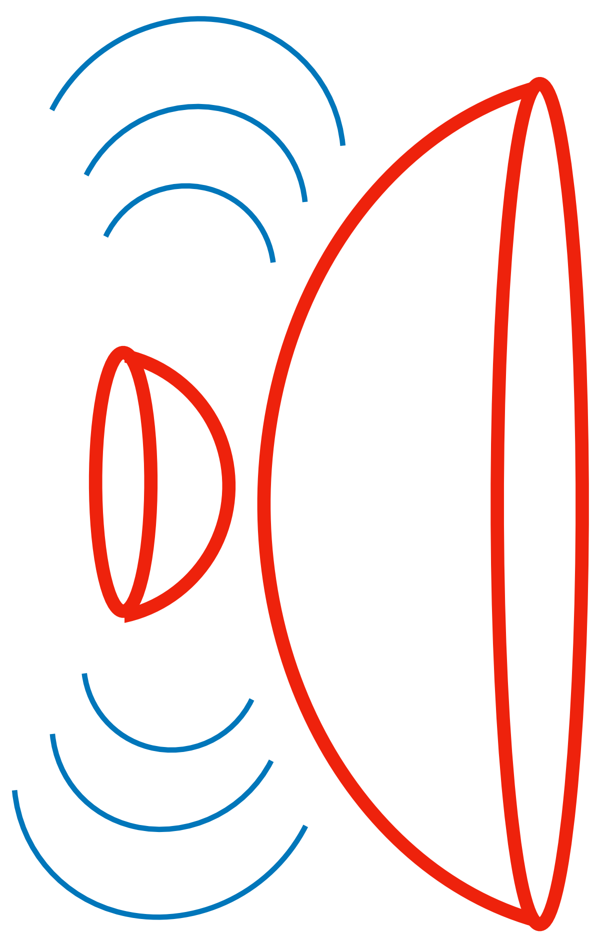
$$t_{\rm decay} \sim \frac{t_{\rm form}}{(\gamma/m)^2}$$

where γ is the symmetry breaking coupling constant.

Gravitational waves

Ferreira et al, 2023; 2024

Evolving domain wall network will emit gravitational waves and produce a stochastic background.



colliding domain walls

Quadrupole approximation (power radiated):

$$P_g = \sum_{i,j} \frac{G}{5} \left(\frac{d^3 I_{ij}^T}{dt^3} \right)^2 \sim G \left(\frac{MR^2}{R^3} \right)^2 \sim (4\pi\sigma)^2 G R^2$$

Assumes time scale is R but wall thickness may also play a role. (See Inomata et al, 2412.17912.)

Hope to hear more from the experts at this workshop….

Summary

- (Biased) domain walls occur very naturally in particle physics, implying their existence in cosmology.
- The network and evolution could be quite involved in realistic particle physics models.
- Domain walls could erase other defects, e.g. magnetic monopoles. (Discussed in Lecture 3.)
- Domain walls result in a stochastic background of gravitational waves that could be of interest to observations.