Thermodynamic uncertainty relation in nondegenerate and degenerate maser heat engines

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Plan of the talk

- Carnot cycle
- Entropy production and power-efficiency trade-off
- Thermodynamic uncertainty relation (TUR)
- TUR in three-level maser heat engines
- TUR in degenerate maser heat engine

Carnot cycle

Two adiabatic and two isothermal branches [1] :

(1) No heat exchange takes place during the adiabatic branches.

(2) Temperature of the working medium remains constant during the isothermal branches.

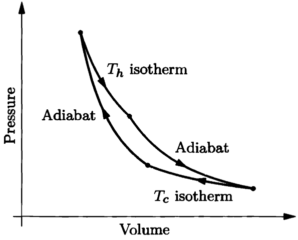


Figure: Carnot cycle for an ideal gas

- Carnot efficiency, $\eta_C = 1 - \frac{T_c}{T_h}$, as theoretical upper bound.
- Maximum efficiency, vanishing power output $(P = W/\tau)$.

[1] D. Kondepudi and I. Prigogine, *Modern thermodynamics*, John Wiley & Sons, UK (2014).

Effect of entropy production on the efficiency of a engine

Finite-time Thermodynamics

- Ideal heat engines; vanishing power output.
- Irreversible heat engines (finite entropy production); finite power output.

Steady state heat engine coupled to two thermal baths:

$$\begin{aligned}\frac{dU}{dt} &= \dot{Q}_h + \dot{Q}_c + P = 0,\\ \frac{dS}{dt} &= \sigma + \frac{\dot{Q}_h}{T_h} + \frac{\dot{Q}_c}{T_c} = 0.\end{aligned}$$

Efficiency of the engine:

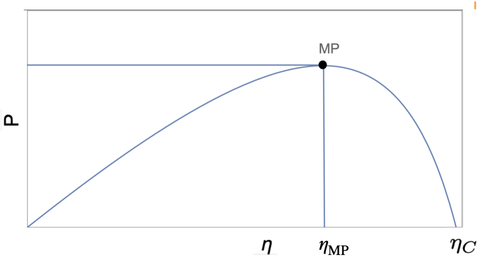
$$\eta = -\frac{P}{\dot{Q}_h} = 1 + \frac{\dot{Q}_c}{\dot{Q}_h} = 1 - \frac{T_c}{T_h} - \frac{T_c}{\dot{Q}_h}\sigma$$

$$\Rightarrow \eta = \eta_C - \frac{T_c}{\dot{Q}_h}\sigma \leq \eta_C .$$

Due to positive entropy production, efficiency of the engine is smaller than Carnot efficiency.

Desirable features of a heat engine

Trade-off between efficiency and power



Most desirable features of a heat engine:

- Beating trade-off between efficiency and power.
- Operation at finite power with maximum (Carnot) efficiency.

- Can an engine beat the trade-off between efficiency and power?

Thermodynamic uncertainty relation

- Nanoscale heat engines are subjected to strong fluctuations which affect their precision.
- At the same time, operation of any engine is associated with entropy production rate, which quantifies the thermodynamics cost.
- **Thermodynamic uncertainty relation (TUR)** represents a trade-off between the rate of entropy production (σ) and relative fluctuations (precision) in power output of the heat engines [2].

$$\frac{\sigma}{k_B}\frac{\Delta P}{\langle P\rangle^2} \geq 2,$$

where $\Delta P = \lim_{t\to\infty}(\langle P^2\rangle - \langle P\rangle^2)t$.

- TUR provides more information than the second law of thermodynamics, which states $\sigma \geq 0$.
- Applicable to systems in nonequilibrium steady states obeying continuous time Markovian dynamics with explicit time independent drive.

[2] Phys. Rev. Lett. **114**, 158101 (2015), Phys. Rev. Lett. **116**, 120601 (2016).

Thermodynamic uncertainty relation

Thermodynamic uncertainty relation (TUR)

$$\frac{\sigma}{k_B}\frac{\Delta P}{\langle P\rangle^2} \geq 2.$$

can be translated to [3]

$$\langle P\rangle \leq \frac{\Delta P}{2k_B T_c}\left(\frac{\eta_C}{\eta} - 1\right).$$

> Above inequality implies that engines can operate at finite power with Carnot efficiency at the cost of diverging power fluctuations.

[3] Phys. Rev. Lett. **120**, 190602 (2018).

TUR in maser heat engines

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Thermodynamic uncertainty relation in nondegenerate and degenerate maser heat engines

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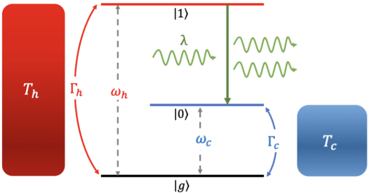
Model of Three-Level Laser Heat Engine [4,5]

The very first model of a quantum heat engine



Hamiltonian and Interaction term:

$$H_0 = \hbar \sum \omega_k |k\rangle\langle k|$$

$$V(t) = \hbar\lambda\big(e^{-i\omega t}|1\rangle\langle 0| + e^{i\omega t}|0\rangle\langle 1|\big)$$

Master equation

$$\dot{\rho} = -\frac{i}{\hbar}[H_0 + V(t), \rho] + \mathcal{L}_h[\rho] + \mathcal{L}_c[\rho],$$

$$\mathcal{L}_h[\rho] = \Gamma_h(n_h+1)\Big(A_k\rho A_k^\dagger - \frac{1}{2}\{A_k^\dagger A_k, \rho\}\Big) + \Gamma_h n_h\Big(A_k^\dagger \rho A_k - \frac{1}{2}\{A_k A_k^\dagger, \rho\}\Big).[A_k = \sigma_{g1}]$$

[4] H. E. D. Scovil and E. O. Schulz-DuBois, Phys. Rev. Lett. **2**, 262 (1959).

[5] K. Dorfman, D. Xu and J. Cao, Phys. Rev. E **97**, 042120 (2018).

Equations of motion in the rotating frame

Master equation in the rotating frame:

$$\dot{\rho} = -\frac{i}{\hbar}[V_R, \rho] + \mathcal{L}_h[\rho] + \mathcal{L}_c[\rho], \qquad V_R = \lambda(|1\rangle\langle 0| + |0\rangle\langle 1|).$$

Density matrix equations:

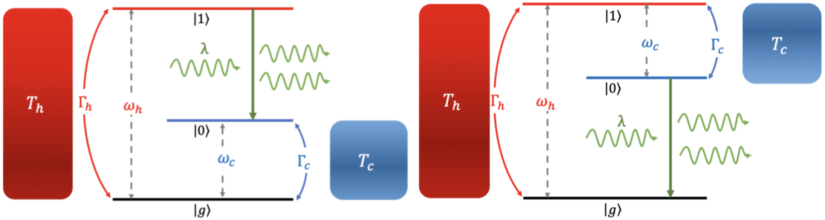
$$\begin{aligned}
\dot{\rho}_{11} &= i\lambda(\rho_{10} - \rho_{01}) - 2\Gamma_h[(n_h+1)\rho_{11} - n_h\rho_{gg}], \\
\dot{\rho}_{00} &= -i\lambda(\rho_{10} - \rho_{01}) - 2\Gamma_c[(n_c+1)\rho_{00} - n_c\rho_{gg}], \\
\dot{\rho}_{10} &= -[\Gamma_h(n_h+1) + \Gamma_c(n_c+1)]\rho_{10} + i\lambda(\rho_{11} - \rho_{00}), \\
\rho_{11} &= 1 - \rho_{00} - \rho_{gg}, \\
\dot{\rho}_{01} &= \dot{\rho}_{10}^*.
\end{aligned}$$

Populations and coherences are decoupled for $\lambda = 0$.

In the steady-state:

$$\boxed{\dot{\rho}_{mn} = 0 \qquad (m, n = 0, 1)}$$

Comparison of TUR in two different models of three-level maser engine



Comparison of TUR in two different models of three-level maser engine

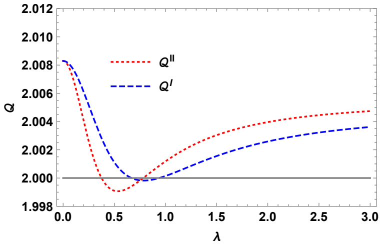


Figure: TUR quantifier $\mathcal{Q}$ versus matter-field coupling parameter λ. Here, $\Gamma_h = 0.1$, $\Gamma_c = 2$, $n_h = 5$, $n_c = 0.027$.

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Phys. Rev. E 104, L012103 (2021)

- TUR relation

$$\mathcal{Q} \equiv \frac{\sigma}{k_B} \frac{\Delta P}{\langle P \rangle^2} \geq 2.$$

- $\mathcal{Q}^{\mathrm{I}} \neq \mathcal{Q}^{\mathrm{II}}$.
- We observe TUR violations in both versions of the SSD model.
- $\mathcal{Q}^{\mathrm{I}}_{\mathrm{min}} > \mathcal{Q}^{\mathrm{II}}_{\mathrm{min}}$

- **In the high-temperature limit, both models yield the same TUR relation.**

$$\mathcal{Q}_{HT} = 2 - \frac{16(n_h - n_c)^2 \Gamma_h \Gamma_c \lambda^2 \left(\Gamma_c^2 n_c^2 + \Gamma_h^2 n_h^2 + 5\Gamma_c \Gamma_h n_h n_c + \lambda^2\right)}{9 n_h n_c (\Gamma_c n_c + \Gamma_h n_h)^2 (4\lambda^2 + \Gamma_h \Gamma_c n_h n_c)^2}.$$

- **In the high-temperature limit, $\mathcal{Q}_{HT}$ is always smaller than 2.**
- **This might be due to neglecting contributions from spontaneous emission.**

TUR in degenerate four-level maser heat engine

Degenerate four-level maser heat engine [6]

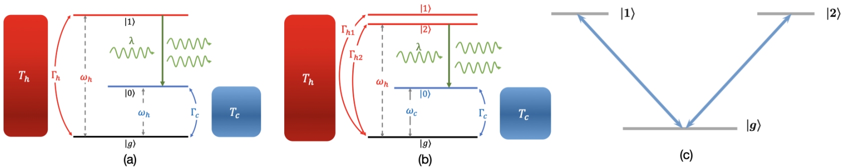

Figure: (a) Three-level nondegenerate and (b) four-level degenerate maser heat engines. (c) Physical interpretation of phenomenon of noise-induced coherence.

Dissipator

$$\mathcal{L}_h[\rho] = \sum_k \Gamma_{hk}\Big[(n_h+1)\Big(A_k\rho A_k^\dagger - \frac{1}{2}\{A_k^\dagger A_k,\rho\}\Big) + n_h\Big(A_k^\dagger\rho A_k - \frac{1}{2}\{A_k A_k^\dagger,\rho\}\Big)\Big]$$
$$+\Gamma\cos\theta\Big[(n_h+1)\Big(A_1\rho A_2^\dagger - \frac{1}{2}\{A_2^\dagger A_1,\rho\}\Big) + n_h\Big(A_1^\dagger\rho A_2 - \frac{1}{2}\{A_2 A_1^\dagger,\rho\}\Big)\Big]$$
$$+\Gamma\cos\theta\Big[(n_h+1)\Big(A_2\rho A_1^\dagger - \frac{1}{2}\{A_1^\dagger A_2,\rho\}\Big) + n_h\Big(A_2^\dagger\rho A_1 - \frac{1}{2}\{A_1 A_2^\dagger,\rho\}\Big)\Big]$$

where

$$A_k = \sigma_{gk} = |g\rangle\langle k|, \quad \Gamma = \sqrt{\Gamma_{h1}\Gamma_{h2}}, \quad p = \cos\theta = \frac{d_{g1}.d_{g2}}{|d_{g1}|\,|d_{g2}|}. \quad [d_{gk} = \langle g|\mathbf{d}|k\rangle]$$

Coherence induced between degenerate states $|1\rangle$ and $|2\rangle$ can be quantified by parameter p.

[6] Proc. Natl. Acad. Sci. USA **108**, 15097 (2011).

Density matrix equations

The time evolution of the density matrix equations:

$$
\begin{aligned}
\dot{\rho}_{11} &= i\lambda(\rho_{10}-\rho_{01}) - \Gamma_h[(n_h+1)\rho_{11} - n_h\rho_{gg}] - \frac{1}{2}p\Gamma_h(n_h+1)(\rho_{12}+\rho_{21}),\\
\dot{\rho}_{22} &= i\lambda(\rho_{20}-\rho_{02}) - \Gamma_h[(n_h+1)\rho_{22} - n_h\rho_{gg}] - \frac{1}{2}p\Gamma_h(n_h+1)(\rho_{12}+\rho_{21}),\\
\dot{\rho}_{00} &= i\lambda(\rho_{01}+\rho_{02}-\rho_{10}-\rho_{20}) - \Gamma_c[(n_c+1)\rho_{00} - n_c\rho_{gg}],\\
\rho_{gg} &= 1-\rho_{11}-\rho_{22}-\rho_{00},\\
\dot{\rho}_{12} &= i\lambda(\rho_{10}-\rho_{02}) - \frac{1}{2}[\Gamma_h(n_h+1)+\Gamma_h(n_h+1)]\rho_{12}\\
&\quad -\frac{1}{2}p\Gamma_h[(n_h+1)\rho_{11}+(n_h+1)\rho_{22}-(n_h+n_h)\rho_{gg}],\\
\dot{\rho}_{10} &= i\lambda(\rho_{11}-\rho_{00}+\rho_{12}) - \frac{1}{2}[\Gamma_c(n_c+1)+\Gamma_h(n_h+1)]\rho_{10} - \frac{1}{2}p\Gamma_h(n_h+1)\rho_{20},\\
\dot{\rho}_{20} &= i\lambda(\rho_{22}-\rho_{00}+\rho_{21}) - \frac{1}{2}[\Gamma_c(n_c+1)+\Gamma_h(n_h+1)]\rho_{20} - \frac{1}{2}p\Gamma_h(n_h+1)\rho_{10}.
\end{aligned}
$$

Due to the noise-induced coherence ($p \neq 0$), populations and coherences are coupled even for $\lambda = 0$.

Effect of the noise-induced coherence on TUR

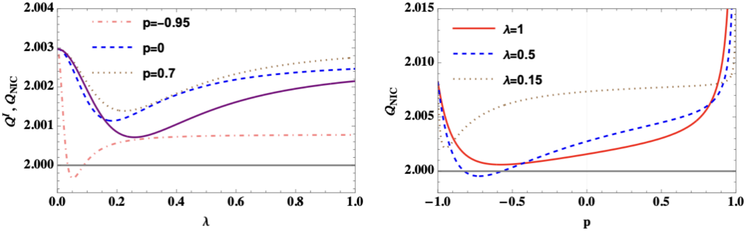

Figure: **(a)** Left panel: $\mathcal{Q}_{\rm NIC}$ as a function of λ for different values of noise-induced coherence parameter p. Solid brown curve represents the TUR ratio for the three-level engine, Model I. Here, $\Gamma_h = 0.3$, $\Gamma_c = 0.03$, $n_h = 6$ and $n_c = 3$. **(b)** Right panel: $\mathcal{Q}_{\rm NIC}$ as a function of p for different values of λ. Here, $\Gamma_h = 0.6$, $\Gamma_c = 0.4$, $n_h = 5$ and $n_c = 2$.

A few observations

$$\mathcal{Q}_{\rm NIC}(p=-1) = \ln\left[\frac{n_h(n_c+1)}{n_c(n_h+1)}\right]\frac{n_h+n_c+2n_hn_c}{n_h-n_c} \geq 2.$$

$$\mathcal{Q}_{\rm NIC}^{\rm HT} = 2.$$

$\mathcal{Q}_{\rm NIC}$ diverges for completely constructive interference ($p=1$) [Proc. Natl. Acad. Sci. USA **108**, 15097 (2011).]

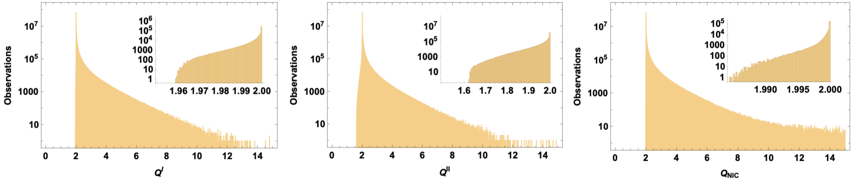

Figure: Histograms of sampled values of $\mathcal{Q}^{\mathrm{I}}$, $\mathcal{Q}^{\mathrm{II}}$ and $\mathcal{Q}_{\mathrm{NIC}}$ for random sampling over a region of the parametric space. The insets show the subset of the sampled data for which STUR violations are happening. The parameters are sampled over the uniform distributions $\Gamma_{h,c} \in [10^{-4}, 5]$, $n_{h,c} \in [0, 10]$ and $\lambda \in [10^{-4}, 1]$. For plotting the histograms, we choose a bin width of 0.01 to arrange 10^8 data points.

Minimum value of TUR ratio $\mathcal{Q}$:

$$\mathcal{Q}^{\mathrm{I}}_{\mathrm{min}} = 1.957, \qquad \mathcal{Q}^{\mathrm{NIC}}_{\mathrm{min}} = 1.985, \qquad \mathcal{Q}^{\mathrm{I}}_{\mathrm{min}} < \mathcal{Q}^{\mathrm{NIC}}_{\mathrm{min}}.$$

Conclusions and future outlook

Conclusions

- Classical TUR relation can be violated in the three-level maser heat engine.
- Spontaneous emission plays important role in the degree of violation of classical TUR as in the absent of spontaneous emission, Model I and Model II discussed here yield the same TUR.
- In the high-temperature limit, classical TUR is always violated for three-level maser heat engines.
- Depending on the parametric regime of operation, the phenomenon of noise-induced coherence can either amplify or suppress the relative power fluctuations.

Future outlook

- To calculate the contribution of coherences in the violation of standard TUR [7].
- This can be done by going beyond steady state and quantum trajectory approach may be employed to this end.

THANK YOU

[7] T. V. Vu and K. Saito, Phys. Rev. Lett. **128**, 140602 (2022)