Maximal surfaces, Born-Infeld solitons and Ramanujan's identities

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Lorentz-Minkowski space

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A smooth immersion $X : M \to \mathbb{L}^3$ of a 2-dimensional connected manifold is said to be a *spacelike surface* in $\mathbb{L}^3$ if the induced metric on M via X is a Riemannian metric.

Lorentzian catenoid

Lorentzian catenoid is a maximal surface, given by a map

$$F(x,y) = \left(\frac{x(x^2+y^2-1)}{2(x^2+y^2)}, \frac{y(x^2+y^2-1)}{2(x^2+y^2)}, \ln\sqrt{x^2+y^2} \right).$$

[1] Image courtesy of L. J. Alías, R. M. B. Chaves and P. Mira [?].

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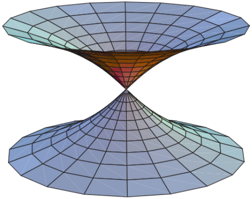

Figure: Elliptic Catenoid[1]

[1] Image courtesy of L. J. Alías, R. M. B. Chaves and P. Mira [?].

Maximal surface

Maximal surfaces are spacelike surfaces in the Lorentz-Minkowski space which arise as *solutions* of the variational problem of locally maximising the area among spacelike surfaces (i.e.,there exists neighbourhood of points on the surface such that it has maximum area for all possible variation (in the class of spacelike surfaces) which fixes the boundary of the neighbourhood).

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As a consequence of this locally they can be described as solutions of a certain nonlinear PDE

$$(1 - f_x^2)f_{yy} + 2f_x f_y f_{xy} + (1 - f_y^2)f_{xx} = 0. \qquad (1)$$

where $f_x^2 + f_y^2 < 1$. This equation is known as the *maximal surface equation*.

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Note: A maximal surface can have singularities (i.e., those points where the metric degenerates), then it is said to be a generalised maximal surface.

Weierstrass-Enneper representation

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Now using this definition of maximal surface, one can derive a complex representation formula for maximal surfaces, known as Weierestrass-Enneper representation, in Lorentz-Minkowski space $\mathbb{L}^3$.

$$\Psi(\tau) = \Re \int (f(1+g^2), if(1-g^2), -2fg) d\tau, \tau \in D, D \subseteq \mathbb{C}$$

f is a holomorphic function on D, g is a meromorphic function on D, fg^2 is holomorphic on D and $|g(\tau)| \neq 1$ for $\tau \in D$.

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For Lorentzian catenoid $f(\tau) = \frac{-1}{2\tau^2}$, $g(\tau) = \tau$ and $M = \mathbb{C} - \{0\}$.

Next if the Gauss map (g) of given maximal surface is one-one, then, one can do a change of parametrisation atleast locally (inverse function theorem). Hence such maximal surface will have a local WE-representation

$$\Psi(\zeta) = \Re \int (M(\zeta)(1+\zeta^2), iM(\zeta)(1-\zeta^2), -2M(\zeta)\zeta) d\zeta \quad (2)$$

where $M(\zeta)$ is a meromorphic function.

Conjugate maximal graphs

We say that two maximal graphs

$$X_1(\tau,\bar{\tau}) = (x_1(\tau,\bar{\tau}), t_1(\tau,\bar{\tau}), f_1(\tau,\bar{\tau}))$$

and

$$X_2(\tau,\bar{\tau}) = (x_2(\tau,\bar{\tau}), t_2(\tau,\bar{\tau}), f_2(\tau,\bar{\tau}))$$

given in isothermal parametrization are *conjugate* if

$$X := X_1 + iX_2 : \Omega \subseteq \mathbb{C} \to \mathbb{C}^3$$

defines a *holomorphic* mapping, where $X_j(\tau,\bar{\tau}) : \Omega \subseteq \mathbb{C} \to \mathbb{L}^3, \tau = \tilde{u} + i\tilde{v} \in \Omega \;\; ; j = 1, 2$ and $\tilde{u}, \tilde{v}$ are isothermal parameters.

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Note that if a maximal graph in $\mathbb{L}^3$ is such that its Gauss map is one-one, then its conjugate maximal graph exists.

If $X_1(\tau,\bar{\tau}) = (x_1(\tau,\bar{\tau}), t_1(\tau,\bar{\tau}), f_1(\tau,\bar{\tau}))$ is a maximal surface and $X_2(\tau,\bar{\tau}) = (x_2(\tau,\bar{\tau}), t_2(\tau,\bar{\tau}), f_2(\tau,\bar{\tau}))$ its conjugate maximal surface, where $\tau = \tilde{u} + i\tilde{v}$ is an isothermal coordinate system.

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Then

$$X_\theta(\tau,\bar{\tau}) := X_1(\tau,\bar{\tau})\cos\theta + X_2(\tau,\bar{\tau})\sin\theta$$

defines a maximal surface for each θ.

Born-Infeld soliton

M. Born and L. Infeld [2] introduced in 1934 a geometric (nonlinear) theory of electromagnetism (known as Born-Infeld model) in order to overcome the infinity problem associated with a point charge source in the original Maxwell theory.

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Any smooth function $\varphi(x,t)$ which is a solution to the Born-Infeld equation

$$(1+\varphi_x^2)\varphi_{tt} - 2\varphi_x\varphi_t\varphi_{xt} + (\varphi_t^2 - 1)\varphi_{xx} = 0. \tag{3}$$

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Remark: We can see a Born-Infeld soliton as some kind of a surface in Lorentz-Minkowski space. This can be represented as a spacelike minimal graph or timelike minimal graph over a domain in timelike plane or a combination of both away from singular points.

If $(x, y, f(x, y))$ is a solution to maximal surface equation (1), then $(ix, y, \varphi(x, y) := f(ix, y))$ is a solution for Born-Infeld equation (3).

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If $X_1 = (x_1, y_1, f_1)$ and $X_2 = (x_2, y_2, f_2)$ are conjugate maximal surfaces, then we define

$$X_1^s = (ix_1, y_1, \varphi_1) \quad , \quad X_2^s = (ix_2, y_2, \varphi_2)$$

as conjugate Born-Infeld Soliton.

Define

$$X^s_\theta(\zeta,\bar{\zeta}) = X^s_1(\zeta,\bar{\zeta})\cos\theta + X^s_2(\zeta,\bar{\zeta})\sin\theta,$$

then

$$X^s_\theta = (ix_1, t_1, \varphi_1)\cos\theta + (ix_2, t_2, \varphi_2)\sin\theta,$$

we let

$$X^s_\theta = (i(x_1\cos\theta + x_2\sin\theta), (t_1\cos\theta + t_2\sin\theta), (\varphi_1\cos\theta + \varphi_2\sin\theta))$$

$$:= (x^s_\theta, t^s_\theta, \varphi^s_\theta).$$

Then X^s_θ gives us the required one parameter family of Born-Infeld solitons.

Ramanujan's identities

Let X and A be complex, where A is not an odd multiple of $\frac{\pi}{2}$. Then

$$\frac{\cos(X+A)}{\cos(A)} = \prod_{k=1}^{\infty}\left\{\left(1-\frac{X}{(k-\frac{1}{2})\pi - A}\right)\left(1+\frac{X}{(k-\frac{1}{2})\pi + A}\right)\right\}$$

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If X and A are real, then

$$\tan^{-1}(\tanh X \cot A)$$

$$= \tan^{-1}\left(\frac{X}{A}\right) + \sum_{k=1}^{\infty}\left(\tan^{-1}\left(\frac{X}{k\pi + A}\right) - \tan^{-1}\left(\frac{X}{k\pi - A}\right)\right).$$

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Using these Ramanujan's identities and Weierstrass-Enneper representation for maximal surfaces we get some further nontrivial identities.

Identity corresponding to Scherk's surface of first kind

Proposition

For $\zeta \in \Omega \subset \mathbb{C} - \{\pm 1, \pm i\}$, we have the following identity

$$\ln \left| \frac{\zeta^2 - 1}{\zeta^2 + 1} \right| = \sum_{k=1}^{\infty} \ln \left(\frac{(k - \frac{1}{2})\pi - i \ln \left| \frac{\zeta - i}{\zeta + i} \right|}{(k - \frac{1}{2})\pi - i \ln \left| \frac{\zeta + 1}{\zeta - 1} \right|} \right)$$

$$+ \sum_{k=1}^{\infty} \ln \left(\frac{(k - \frac{1}{2})\pi + i \ln \left| \frac{\zeta - i}{\zeta + i} \right|}{(k - \frac{1}{2})\pi + i \ln \left| \frac{\zeta + 1}{\zeta - 1} \right|} \right).$$

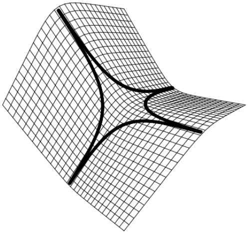

Figure: Scherk's surface of first kind[2]

[2] Image courtesy of Y. W. Kim, S-E Koh, H Shin, and S-D Yang.

Identity corresponding to helicoid of second kind

Proposition

For $\zeta \in \Omega \subset \mathbb{C} - \{0\}$, we have the following identity

$$\frac{Im\left(\zeta + \frac{1}{\zeta}\right)}{Im\left(\zeta - \frac{1}{\zeta}\right)} = \frac{1}{i}\prod_{k=1}^{\infty}\left(\frac{(k-1)\pi + i\ln|\zeta|}{(k-\frac{1}{2})\pi + i\ln|\zeta|}\right)$$

$$\times\left(\frac{k\pi - i\ln|\zeta|}{(k-\frac{1}{2})\pi - i\ln|\zeta|}\right).$$

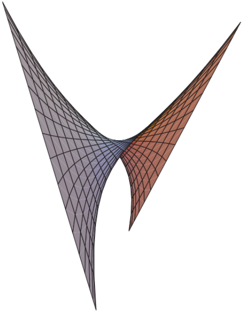

Figure: Helicoid of second kind[3]

[3]Image courtesy of L. J. Alías, R. M. B. Chaves and P. Mira.

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