Outliers in the spectrum for products of independent random matrices

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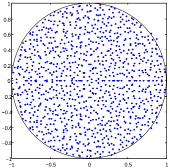
WISCONSIN
UNIVERSITY OF WISCONSIN–MADISON

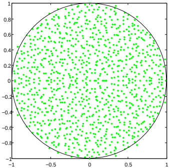
ICTS Conference: Random matrices and point processes
Joint work with Natalie Coston and Sean O'Rourke

1000 by 1000 random matrix with iid entries; scaled by $1/\sqrt{n}$.

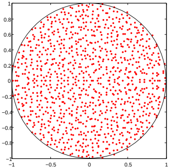
Rademacher ± 1



Gaussian normal



Exponential



Introduction: the circular law for i.i.d. random matrices

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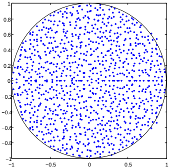

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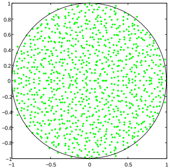

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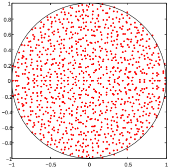

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M_n an n by n random matrix, entries are iid complex random variables with mean 0 and variance 1. Then eigenvalues of $\frac{1}{\sqrt{n}} M_n$ converges weakly to the uniform distribution on the unit disk.

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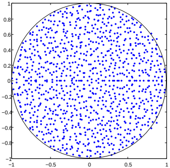

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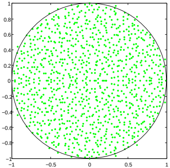

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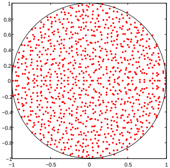

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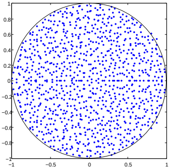

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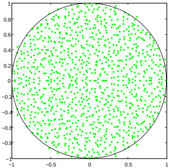

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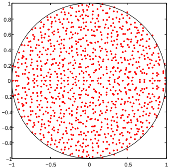

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No outliers: If $\mathbb{E}(|x|^4) < \infty$ for each entry x, then spectral radius converges to 1 a.s. as $n \to \infty$. (By truncation and moment method)

Outliers: fixed P and random M, where P low rank.

Constant rank and evals, e.g. $P = \mathrm{diag}(3i/2, 1 + i, 2, 0, \ldots, 0)$.

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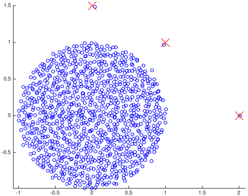

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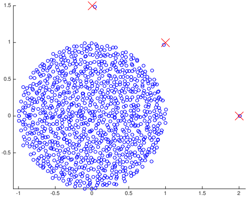
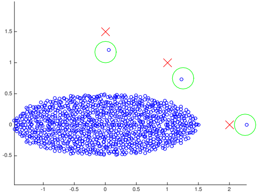


Elliptical, O'Rourke-Renfrew 2014

$\frac{1}{\sqrt{n}} M_{\underline{\underline{\mu}}} + P$, $\quad \lambda_j \mapsto \lambda_j + \frac{\mu}{\lambda_j}$



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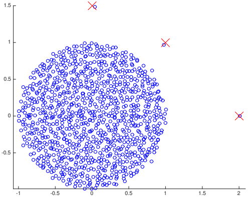

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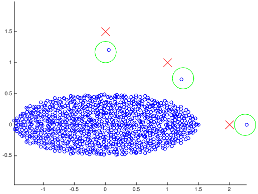

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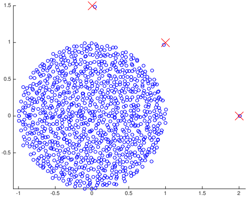

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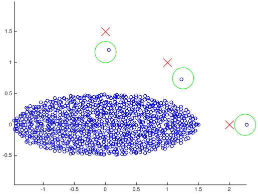
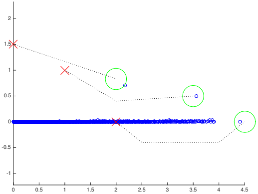


Sample Cov., O'Rourke-W. 2016

$\frac{1}{n}M^{\mathrm{T}}M(I+P)$

$\lambda_j \mapsto \lambda_j \left(1 + \frac{1}{\lambda_j}\right)^2$



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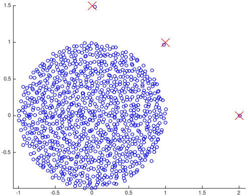

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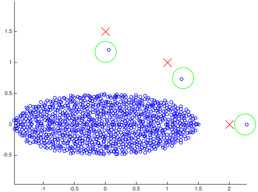

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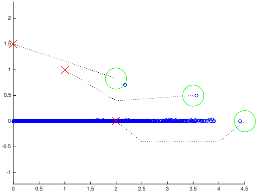

Bulk eigenvalue distributions maintain the same weak limit.

Products of independent elliptical matrices

Iid matrices, Götze-Tikhomirov (2011) and O'Rourke-Soshnikov (2011).

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Elliptic random matrix $Y = (y_{ij})$: Entries $\{y_{ii}, 1 \leq i \leq n\}$ are iid, and $\{(y_{ij}, y_{ji}), 1 \leq i < j \leq n\}$ are iid.
Also $\{y_{ii}, 1 \leq i \leq n\} \cup \{(y_{ij}, y_{ji}), 1 \leq i < j \leq n\}$ independent.

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Theorem (O'Rourke, Renfrew, Shoshnikov, and Vu, 2014)

Let $m > 1$ be an integer. For $1 \leq k \leq m$, let each $Y_{n,k}$ be a real elliptic random matrix, where all entries have mean zero, variance 1, and finite $2 + \epsilon$ moment. Also assume covariance $|\rho_k| < 1$ and that the $Y_{n,k}$ are independent. Then $P_n = n^{-m/2} Y_{n,1} \ldots Y_{n,m}$ has limiting measure converging a.s. to

$$f_m(z) = \begin{cases} \frac{1}{m\pi} |z|^{\frac{2}{m}-2}, & \text{for } |z| \leq 1 \\ 0, & \text{for } |z| > 1, \end{cases}$$

the m-th product of the circular law.

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$\boxed{\text{What can be said about outliers?}}$ We will focus on the iid case.

No outliers for products of indep. iid matrices

Nemish 2016: no outliers given sub-exponential decay condition.

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Theorem (Coston-O'Rourke, W., 2018)

Let $m \geq 1$. Let $X_{n,1}, \ldots, X_{n,m}$ be independent complex, iid random n by n matrices, where entries are mean 0, variance 1, have finite 4th moment, and have independent real and imaginary parts. Then, almost surely,

$$P_n = \prod_{k=1}^{m} \left(\frac{1}{\sqrt{n}} X_{n,k} \right)$$

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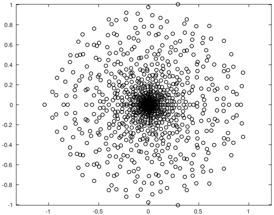

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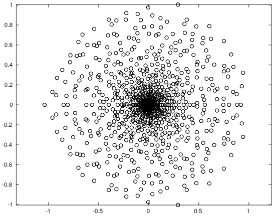

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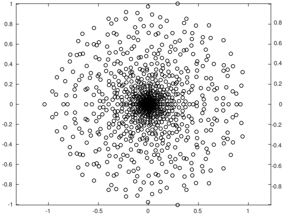

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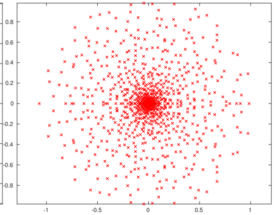

same, with $A_{n,k}$ rnk 5, op-norm 2

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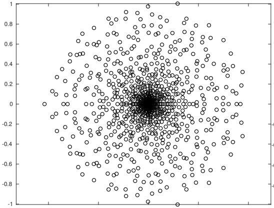

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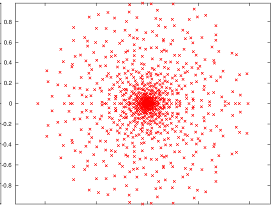

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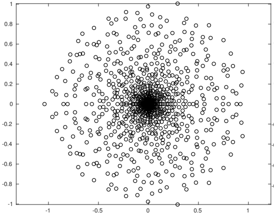

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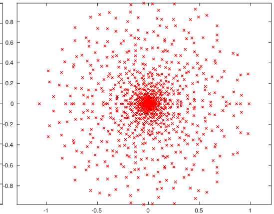

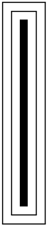

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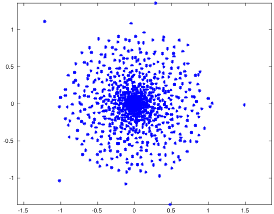

Different perturb <u>does</u> create outliers.

Outliers for perturbed products of iid matrices

Theorem (Coston-O'Rourke, W., 2018)

Let $m \geq 1$. Let $X_{n,1}, \ldots, X_{n,m}$ be independent complex, iid random matrices, where entries are mean 0, variance 1, have finite 4th moment, and have independent real and imaginary parts. Let $A_{n,k}$ be deterministic matrices with $O(1)$ rank and operator norm, and assume there is $\epsilon > 0$ so than no evals of $A_n = \prod_{k=1}^{m} A_{n,k}$ are within 3ϵ of edge of the unit disk. If A_n has j evals lying at least 3ϵ outside the unit disk, then, the product

$$P_n = \prod_{k=1}^{m} \left(\frac{1}{\sqrt{n}} X_{n,k} + A_{n,k} \right)$$

has exactly j evals lying at least 2ϵ outside the unit disk, each within $o(1)$ of the corresponding eval of A_n.

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Recover's Tao's outliers result when $m = 1$ (with additional assumption). Same approach works with other variations. General theme: the product has three parts: $\left(\prod_{k=1}^{m} \frac{1}{\sqrt{n}} X_{n,k} \right) + M_n + A_n$, and the mixed terms M_n do not substantially contribute.

Outline of the proof

1. Linearization: Let $\mathcal{M} = \begin{pmatrix} 0 & M_1 & & & 0 \\ 0 & 0 & M_2 & & 0 \\ & & \ddots & \ddots & \\ 0 & & & 0 & M_{m-1} \\ M_m & & & & 0 \end{pmatrix}$, an $mn \times mn$ matrix of $n \times n$ blocks; and let $P = M_1 \dots M_m$. Then $\det(\mathcal{M}^m - zI) = [\det(P - zI)]^m$ for every $z \in \mathbb{C}$.

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2. Sylvester's Determinant Formula: if A is $N \times k$ and B is $k \times N$, then

$$\det(I + AB) = \det(I + BA).$$

Low rank means that $k = O(1)$, so an $n \times n$ determinant becomes a $k \times k$ determinant.

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3. **Isotropic limit law**: Shows that the resolvent is a good approximation for $-1/z$ outside the unit disk, in *any* basis.

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3. Isotropic limit law: Shows that the resolvent is a good approximation for $-1/z$ outside the unit disk, in *any* basis. E.g., Isotropic limit laws known for Wigner matrices (KY2012), for sample covariance matrices (BEKYY2014), and for elliptical matrices (OR2014)

We prove a new isotropic local law for block matrices as above.

Isotropic limit law

Theorem (Coston, O'Rourke, W. 2018)

Let $\mathcal{Y}_n$ be a block matrix for the $X_{n,k}$, and let entries for the $X_{n,k}$ be mean zero, variance 1, with finite 4th moment, and with independent real and imaginary parts. Let $\mathcal{G}_n(z) := \left(\frac{1}{\sqrt{n}}\mathcal{Y}_n - zI\right)^{-1}$. Then for any $\delta > 0$, a.s. for n suff large, all evals of $\frac{1}{\sqrt{n}}\mathcal{Y}_n$ are within δ of the unit disk and $\sup_{|z|>1+\delta} \|\mathcal{G}_n\| = O_\delta(1)$. Also, if $u_n, v_n \in \mathbb{C}^{mn}$ are fixed unit vectors, then

$$\sup_{|z|>1+\delta} \left| u_n^* \mathcal{G}_n(z) v_n + \frac{1}{z} u_n^* v_n \right| \to 0 \text{ a.s. as } n \to \infty.$$

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Sketch proof of outliers result: Any $z \notin$ unit disk is eval for $\frac{1}{\sqrt{n}}\mathcal{Y}_n + \mathcal{A}_n$ iff $\det(I + \mathcal{G}_n(z)\mathcal{A}_n) = 0$. Let $\mathcal{A}_n = \mathcal{B}_{mn\times k}\mathcal{C}_{k\times mn}$, where $\mathcal{B}$ is $mn \times k$, etc.

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Now use Rouche's Theorem. $\square$

Outline for proof of isotropic limit law

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3. Concentration of bilinear forms involving the truncated resolvent $\tilde{\mathcal{G}}$, i.e., showing that $u_n^* \tilde{\mathcal{G}} v_n$ is close to its expectation on a relevant region. Thus it is sufficient to prove isotropic convergence for the *expectation*, i.e.,

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Further questions: Outliers result for products of elliptical matrices?
Products of Hermitian matrices? (bulk dist still open)